Transient and Persistent Factor Structure in Equity Options

Abstract

We introduce a model for derivations of individual equity option prices that embeds persistent and transient variations of the market index in the dynamics of individual equities. We derive a closed-form pricing equation and show how transient and persistent factor loadings affect the instantaneous expected returns of equity options. For the firms listed on Dow Jones Index, we find that the option-implied transient betas are always greater than those of persistent betas, implying that large capitalization firms listed in the Dow Jones index co-move more with transient (larger) variations in the market index, which has a direct implication for a portfolio manager who hedges her portfolio exposure to transient versus persistent systematic market variations. In cross-sectional analysis, our model predicts that firms with higher transient betas have steeper term structures of implied volatility and steeper moneyness slopes. It also predicts that variances risk premiums have more significant effect on the equity option skew for firms with higher transient betas. On the empirical front, for the firms listed on the Dow Jones index, our model provides a good fit to the observed equity option prices, both in-sample and out-of-sample. At the market index level, we obtain negative prices for persistent and transient variance components, implying that investors are willing to pay for insurance against increases in volatility risk, even if those increases have little persistence.

JEL Classification: G10; G12; G13
Keywords: equity options; factor structure; skewness; variance risk premium; two-factor stochastic volatility; joint estimations;
1 Introduction

The dynamics of index return volatility and their role in pricing options have had a long history following the classic early works by Wiggins (1987) and Heston (1993), that recognized the volatility’s stochastic nature and managed to derive closed form expressions for the resulting European options. Related early contributions were also by Duan (1995), Duan et al. (1999), and Heston and Nandi (2000) under GARCH return dynamics, with option prices derived either by numerical methods or with closed form expressions. More recent studies, however, point out that a single factor stochastic volatility (SV) or GARCH is not sufficient to represent both the underlying ($P$) and the risk-neutral ($Q$) measures of the joint dynamics of returns and variances for the key S&P 500 index and its options.\footnote{See, for instance, Bollerslev and Zhou (2002), Alizadeh et al. (2002), and Chernov et al. (2003) for the underlying return distribution and Bates (2000), Christoffersen et al. (2008), and Christoffersen et al. (2009) for the option-based $Q$-distribution.} In particular, these studies show that one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility and also fitting level and slope of the volatility smirk in cross-section of option prices volatility smirk cannot. All of which emphasize that two volatility factors (one is highly persistent and the other one is highly mean-reverting) are needed to explain return volatility dynamics.

In this paper, we extend the insights of two factor stochastic volatility models into the pricing of equity options, formulates the simultaneous equilibrium of both equity underlying and option markets, and tests empirically the derived results. In particular, we examine how individual equity option prices respond to the existence of two volatility components and so multiple factor loadings.

First, we assume an affine two-factor SV process for the underlying market index returns where aggregate market volatility is decomposed into a more persistent volatility component, which has nearly a unit root, and a transitory volatility component, which has more rapid time decay. We derive the risk-neutral dynamics for the market index by introducing a u-shaped pricing kernel. Then, we extend the one-volatility-factor equity model in Christoffersen et al. (2017) and assume that individual equity returns are related to the market index with two distinct systematic components, and an idiosyncratic component which is stochastic and follows the standard square root process. We obtain risk-neutral dynamics for an individual equity by assuming a conventional stochastic discount factor. Given these dynamics, we derive a closed-form pricing equations for equity options in which equity returns are related to the market index returns with two distinct constant factor loadings, a transient beta and a persistent beta.

We estimate the structural parameters of the equity dynamics and filter spot idiosyncratic variance for the firms listed in the Dow Jones index. We find that option-implied transient beta is always higher than option-implied persistent beta, implying that for large capitalization firms listed in the Dow Jones index, transient and larger variations in the market index tend to be related to the proportionally larger systematic price reactions than persistent and
smaller variations in the market index. In our sample of 27 firms, the transient beta ranges from 1.01 to 1.35, while the persistent beta is about half the value, from 0.34 to 0.68. Our transient and persistent option-implied betas can be interpreted similar to the continuous beta and jump beta of Todorov and Bollerslev (2010), where they find that the average jump betas are larger than the continuous betas with few exceptions. Although we only use option data and estimate ad-hoc constant beta over the entire sample, we observe a similar pattern as theirs between our transient and persistent betas. Overall, we find that our option pricing model provides a good fit to the observed equity option prices across all of the 27 firms, both in-sample and out-of-sample relative to the one-factor structure of Christoffersen et al. (2017).

Our proposed factor structure has a number of important cross-sectional implications for equity options. Our model predicts that firms with higher transient betas have higher level of implied volatilities. It also predicts that firms with higher transient betas have steeper term structures of implied volatility while persistent beta has a marginal effect on the implied volatility term structure. Our model also predicts that the implied volatility moneyness slope is steeper for firms with higher transient betas. Further, we find that the transient variance risk premium has a more significant effect on the equity option skew, which mainly drives the slope of implied volatility smile for individual equities.

We also derive a closed-form expression for the instantaneous expected equity option returns and show how exposures to the level of market index and to market variance components can affect the expected equity option returns. In other words, our framework allow us to disentangle the effect of market risk premium from those of persistent and transient variance risk premiums on the expected equity option returns. Our models’ framework is especially important for a portfolio manager who hedges her portfolio’s exposure to the systematic risk factors in the portfolio of stocks and options. Our proposed factor structure and closed-form option pricing equation yields closed-form expressions for the exposure of equity options to the transient and persistent variations in the market returns in addition to the overall market returns. These results together support the importance of transient and persistent factor loadings in pricing equity options.

The dynamics for individual equities is introduced given the two-factor SV model of the market index. In order to examine the effect of systematic risk factors and variance components risk premiums on equity option prices and returns, we first obtain these premiums. We introduce an admissible pricing kernel by extending the one in Christoffersen et al. (2013) to accommodate multiple variance components and then derive the risk-neutral index dynamics and a closed-form pricing equation for European index options.

We calibrate our market index dynamics using a joint likelihood function that combine data

---

2 The proposed framework is equally important for risk managers and dispersion traders.
3 Note that the extracted risk-neutral index dynamics is not restricted to the proposed u-shaped pricing kernel, where investor’s variance risk preference is distinguished from her equity risk preference. We obtain the same risk-neutral dynamics by assuming a standard stochastic discount factor in Appendix F.
from S&P 500 index and option markets. The resulting parameter estimates are therefore characterizing the market index dynamics under both the \( P \) and \( Q \) measures. Further, joint estimation allow us to obtain two separate variance risk premiums; a transient variance risk premium and a persistent variance risk premium. To the best of our knowledge, this is the first study that estimates consistent \( P \)- and \( Q \)-parameters from underlying index return and option data and reports variance risk premium for persistent and transient components. In addition, we extract two vectors of daily spot variances using the Particle Filter (PF) method. We follow the conventional filtration procedure of similar studies but provide a methodologically important solution for the challenging issue of how to separate the two variance components’ paths.

In empirical analysis of the index model, we find that one of the volatility factors is highly persistent (persistent component) while the immediate impact of volatility shocks on the other volatility factor is bigger but short-lived (transient component). We obtain negative correlation between shocks to the market returns and each variance component, implying that both components are important in capturing the so-called leverage effect. We find the absolute value of transient correlation parameter is smaller (\( \rho_2 = -0.2173 \)) compared to that of the persistent correlation (\( \rho_1 = -0.6918 \)) and therefore has a less significant effect on the skewness and kurtosis dynamics of the market index and on the volatility smirk.

We find negative prices for both variance components, \( \lambda_1 = -1.0798 \) and \( \lambda_2 = -1.0355 \). Our finding implies that investors are willing to pay for an insurance against increases in volatility risk, even if those increases have little persistence. To the best of our knowledge none of the previous studies of two-factor stochastic volatility models reports the price of the variance risk factors as they either focused on the options market data or the underlying returns data. The negative variance risk premium for both transient and persistent variance components are consistent with the findings in Adrian and Rosenberg (2008). Using a large cross-section of stock returns data, they find negative and significant prices for both short-run and long-run volatility components.

Our proposed factor structure in equity options is motivated by the extensive empirical evidence that supports the presence of two variance components (mostly independent) in the

---

4 Joint estimations address the model’s ability to simultaneously fit the time-series of returns and cross-section of option prices. See, for instance, Bates (1996), Chernov and Ghysels (2000), Pan (2002), Eraker (2004), and Broadie et al. (2007) among others.

5 The main challenge in such an efficient joint estimation procedure is its heavy computational burden. To overcome this challenge, previous studies mostly focused on a very short time-series and/or weekly/monthly option dataset, See Pan (2002) and Eraker (2004). However, efficient programming and parallel computing techniques allow us to keep a large time-series of returns and the entire cross-section of daily option prices over the same time span.

6 According to Christoffersen et al. (2009, Section 6), “an integrated analysis of multifactor models using option data as well as underlying returns out to be done.”

7 For the application of PF in estimating the model parameters see Gordon et al. (1993), Johannes et al. (2009), Johannes and Polson (2009), and Christoffersen et al. (2010)

8 Note that unlike discrete time model of Adrian and Rosenberg (2008) we do not impose any restrictions on the variance dynamics other than independence of variance shocks.
dynamics of the market index. In the $P$-distribution domain, two volatility factors are needed to explain the volatility dynamics of the market index, since one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility. Chernov et al. (2003) suggest that the addition of a second volatility factor breaks the link between tail thickness and volatility persistence and leads to a significant improvement relative to a single SV model in capturing the return dynamics. Bollerslev and Zhou (2002) and Alizadeh et al. (2002) document the importance of two volatility components in capturing exchange rate dynamics. According to Dai and Singleton (2000; 2002), multifactor volatility models are needed to model the term structure of the interest rate.

Extensive empirical evidence in the $Q$-distribution domain also points toward the existence of two variance components. Egloff et al. (2010) and Mencía and Sentana (2013) show that two-factor SV models have more flexibility to fit the term structure of the volatility and to control the level and the slope of volatility smirk in cross-sections of option prices. Egloff et al. (2010, p 1289) show that the upward sloping autocorrelation term structure of variance swap rate quotes point to the existence of multiple variance risk factors. Christoffersen et al. (2009) show that two-factor SV models can better capture the time-varying nature of the smirk as it can generate sufficient amounts of conditional skewness and kurtosis. In a model free framework, they show that the first two principal components of the Black-Scholes implied variances on a sample of S&P 500 index options together explain more than 95% of the variation in the implied variances.

Similar inconsistencies in the joint estimation of the SV model are illustrated by Broadie et al. (2007). They note the failure of the SV model to reconcile the $P$- and $Q$-estimates of certain structural parameters, the correlation coefficient and volatility of volatility, and conclude that the SV model is basically misspecified. They also show that the joint restrictions on the returns and volatility dynamics under the $P$ and $Q$ measures leads to the poor performance of the SV model, measured by the high level of IVRMSE. In contrast, in our empirical analysis, we find that our two-factor SV model does not suffer from the joint restrictions on the $P$ and $Q$ dynamics.

Although our study is not the first one to examine multifactor SV models, it is the only one to present consistent $P$- and $Q$-parameter estimates both theoretically and empirically. For instance, Bates (2000) examine a multifactor specification in option pricing by relying on the $Q$-distribution only. Christoffersen et al. (2008) introduce a two-component GARCH model, which can generate more flexible skewness and volatility of volatility dynamics in capturing the dynamics of the S&P 500 index returns and in pricing European S&P 500 call options. Nonetheless, the absence of an explicit pricing kernel linking the $P$- and $Q$-distributions in that study necessitated either the use of an arbitrary price of volatility risk or the estimation of the risk-neutral parameters by relying on the $Q$-distribution only. Christoffersen et al. (2009) explore multiple variance factors model under $Q$-distribution only and find that it can generate stochastic correlation between total instantaneous volatility and stock returns.

The paper proceeds as follows. Section 2 presents the theoretical model for pricing index options and individual equity options. In Section 3, we discuss the properties and implications
of the proposed factor structure in equity options. Section 4 describes the data sets. Section
5 summarizes the estimation methodologies for both index and equity options. Section 6
contains the estimation results and parameter estimates for the market index and 27 indi-
vidual equities. Section 7 explores the performance of the models and reports goodness-of-fit
statistics. Section 8 measures the stability of the model and reports the out-of-sample per-
f ormance. Section 9 concludes. The appendix provides the proofs of the theoretical results
and further details on discretization of the model and the Particle Filter methodology.

2 Model Setup

We start by a two-factor stochastic volatility model that governs the market index dynam-
ics under the $P$-distributions and then introduce an admissible variance-dependent pricing
kernel to obtain the risk-neutral dynamics by imposing appropriate martingale’s restrictions
on the pricing kernel. We complete the index model setup by deriving a closed-form pricing
equation for index options. We then assume a dynamics for individual equity under $P$
distribution and introduce an appropriate stochastic discount factor (SDF) to obtain the equity
dynamics under $Q$ measure. Last, we derive a closed-form equation that gives the price of
individual equity options.

2.1 Two-factor Stochastic Volatility Model of the Market Index

We assume the following two-factor stochastic volatility process governs dynamics of the
market index returns and variance under the physical distributions.

\begin{equation}
\begin{align*}
\frac{dS_t}{S_t} &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) dt + \sqrt{v_{1,t}} dz_{1,t} + \sqrt{v_{2,t}} dz_{2,t} \\
v_{1,t} &= \kappa_1 (\theta_1 - v_{1,t}) dt + \sigma_1 \sqrt{v_{1,t}} dw_{1,t} \\
v_{2,t} &= \kappa_2 (\theta_2 - v_{2,t}) dt + \sigma_2 \sqrt{v_{2,t}} dw_{2,t}
\end{align*}
\end{equation}

with two independent variance components as described in the following stochastic structure.

\begin{equation}
\begin{align*}
\langle dw_{1,t}, dz_{1,t} \rangle &= \rho_1 dt, \quad -1 \leq \rho_1 \leq +1 \\
\langle dw_{2,t}, dz_{2,t} \rangle &= \rho_2 dt, \quad -1 \leq \rho_2 \leq +1 \\
\langle dw_{1,t}, dw_{2,t} \rangle &= 0 \\
\rho_1^2 + \rho_2^2 &\leq +1
\end{align*}
\end{equation}

As in the Heston (1993) SV model: $\theta_1$ and $\theta_2$ are the unconditional average variances of
persistent and transient components, $\kappa_1$ and $\kappa_2$ capture the speed of mean reversion of
each variance components, and $\sigma_1$ and $\sigma_2$ measure the volatility of variance components.
The market equity risk premiums are denoted by $\mu_1 v_{1,t}$ and $\mu_2 v_{2,t}$. Following Bollerslev and
Zhou (2006) we expect that $\mu_1$ and $\mu_2$ measure the persistent and transient “continuous-time”
volatility feedback effects or risk-return trade-offs. The instantaneous correlation between shocks to the market returns and shocks to the persistent variance component is measured by \( \rho_1 \) and the instantaneous correlation between shocks to market returns and shocks to the transient variance component is given by \( \rho_2 \). As in Bollerslev and Zhou (2006), we expect that \( \rho_1 \) and \( \rho_2 \) account for persistent and transient “continuous-time” leverage (asymmetry) effect.

Note that (2) implies that the total return variance and the correlation between return and total variance are as follows.

\[
\begin{align*}
\text{Var}_t[dS_t/S_t] &= v_{1,t} dt + v_{2,t} dt = \nu_1 dt \\
\text{Corr}_t[dS_t/S_t, dV_t] &= \frac{\rho_1 \sigma_1 v_{1,t} + \rho_2 \sigma_2 v_{2,t}}{\sqrt{\sigma_1^2 v_{1,t} + \sigma_2^2 v_{2,t}} \sqrt{v_{1,t} + v_{2,t}}} dt
\end{align*}
\]

We may then prove the following result.

**Proposition 1.** The market index has the following dynamics under the risk-neutral measure:

\[
\begin{align*}
\frac{dS_t}{S_t} &= rd Dt + \sqrt{v_{1,t}} dz_{1,t} + \sqrt{v_{2,t}} dz_{2,t}, \\
\frac{dv_{1,t}}{v_{1,t}} &= \tilde{\kappa}_1 (\tilde{\theta}_1 - v_{1,t}) dt + \sigma_1 \sqrt{v_{1,t}} d\tilde{w}_{1,t}, \\
\frac{dv_{2,t}}{v_{2,t}} &= \tilde{\kappa}_2 (\tilde{\theta}_2 - v_{2,t}) dt + \sigma_2 \sqrt{v_{2,t}} d\tilde{w}_{2,t},
\end{align*}
\]

where, \( \tilde{\kappa}_1 = \kappa_1 + \lambda_1, \tilde{\kappa}_2 = \kappa_2 + \lambda_2, \tilde{\theta}_1 = \frac{\kappa_1 \theta_1}{\tilde{\kappa}_1 + \lambda_1}, \tilde{\theta}_2 = \frac{\kappa_2 \theta_2}{\tilde{\kappa}_2 + \lambda_2} \). The market prices of risk factors are

\[
\begin{align*}
\psi_{1,t} &= \frac{\sigma_1 \mu_1 - \rho_1 \lambda_1}{\sigma_1 (1 - \rho_1^2)} \sqrt{v_{1,t}}, \quad \psi_{2,t} = \frac{\sigma_2 \mu_2 - \rho_2 \lambda_2}{\sigma_2 (1 - \rho_2^2)} \sqrt{v_{2,t}}, \\
\psi_{3,t} &= \frac{\lambda_1 - \rho_1 \sigma_1 \mu_1}{\sigma_1 (1 - \rho_1^2)} \sqrt{v_{1,t}}, \quad \psi_{4,t} = \frac{\lambda_2 - \rho_2 \sigma_2 \mu_2}{\sigma_2 (1 - \rho_2^2)} \sqrt{v_{2,t}}.
\end{align*}
\]

One admissible pricing kernel that links the physical dynamics in (1) to the risk-neutral dynamics in (4) takes the following exponential affine form.

\[
\frac{M_t}{M_0} = \left( \frac{S_t}{S_0} \right)^{\phi} \exp \left[ \delta t + \eta_1 \int_0^t v_{1,s} ds + \eta_2 \int_0^t v_{2,s} ds + \zeta_1 (v_{1,t} - v_{1,0}) + \zeta_2 (v_{2,t} - v_{2,0}) \right]
\]

As in the Christoffersen et al. (2013), \{\delta, \eta_1, \eta_2\} govern the time-preferences, while \{\phi, \zeta_1, \zeta_2\} govern the respected risk aversion to the index and variance risk factors, all of which are defined in the appendix.

**Proof.** See Appendix A. □
We note that the introduced nonlinear log pricing kernel in (6) is one way of “completing
the market” and linking $P$- to $Q$- dynamics, where $\zeta_1$, $\zeta_2$ capture the nonlinearity of the log
pricing kernel. Transforming the physical dynamics in (1) into the risk-neutral dynamics
in (4) can also be done by assuming the following standard stochastic discount factor and
without explicit assumptions about the investor’s variance preferences. The proof of such a
transformation together with a more general SDF that also includes the price of risk factors
for individual equities are provided in appendix F.

\[
\frac{dM_t}{M_t} = -rdt - \psi'_t dW_t ,
\]

where $\psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}]$ is the vector of market price of risk factors and $W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}]$ is the vector of innovations in return and variance.

To embed the options market data into the estimation of structural parameters of the index
dynamics, we determine a closed-from expression for the price of the European call op-
tions, with strike price $K$ and time to maturity $\tau$, by inverting the conditional characteristic
function of the log spot index prices, $x_t = \ln(S_t)$.

\[
C_t(S_t, K, v_{1,t}, v_{2,t}, \tau) = S_t P_1 - K e^{-r\tau} P_2 ,
\]

where,

\[
P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi - i)}{i\phi} \right] d\phi,
\]

\[
P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi)}{i\phi} \right] d\phi ,
\]

and where the risk-neutral conditional characteristic function of the natural logarithm of the
index price at expiration, $x_{t+\tau}$, is

\[
\tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) \equiv \mathbb{E}_t^Q \left[ \exp(i\phi x_{t+\tau}) | x_t \right].
\]

Since the two-factor SV model in (4) is an affine process, following Heston (1993), the
conditional risk-neutral characteristic function in (10) has the following affine exponential
form.\(^9\)

\(^9\) Note that $\zeta_1$, $\zeta_2$ affect a wedge between physical and risk-neutral structural parameters of volatility
dynamics.

\(^{10}\) Note that the conditional risk-neutral characteristic function of the natural logarithm of the market
index return, $x_{t+\tau} - x_t = \ln(S_{t+\tau}/S_t)$, can be defined with the same expression as (11) but without the first
component, $i\phi x_t$. 

8
\[ \tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) = \exp \left[ i\phi x_t + i\phi r\tau + A_1(\tau, \phi) + A_2(\tau, \phi) + B_1(\tau, \phi)v_{1,t} + B_2(\tau, \phi)v_{2,t} \right], \] (11)

where\(^1\) for every \( j = \{1, 2\} \)

\[ A_j(\tau, \phi) = \frac{\tilde{\kappa}_j \tilde{\theta}_j}{\sigma_j^2} \left[ (\tilde{\kappa}_j - \rho_j \sigma_j i\phi - d_j)\tau - 2\ln \left[ \frac{1 - c_j e^{-d_j\tau}}{1 - c_j} \right] \right] \]

\[ B_j = \frac{\tilde{\kappa}_j - \rho_j \sigma_j i\phi - d_j}{\sigma_j^2} \left[ \frac{1 - e^{-d_j\tau}}{1 - c_j e^{-d_j\tau}} \right] \]

\[ c_j = \frac{\tilde{\kappa}_j - \rho_j \sigma_j i\phi - d_j}{\tilde{\kappa}_j - \rho_j \sigma_j i\phi + d_j} \]

\[ d_j = \sqrt{ (\tilde{\kappa}_j - \rho_j \sigma_j i\phi)^2 + \sigma_j^2 (\phi + i)} \]. (12)

2.2 The Individual Equity Model

For individual equities, we assume that equity returns are related to the market returns with two distinct systematic risk factors, two constant factor loadings \( \beta_1^i \) and \( \beta_2^i \), and an idiosyncratic component. Following Bakshi et al. (2003) we assume idiosyncratic shocks to equity returns \( \xi_t^i \) follow a standard square-root process. This assumption allows us to characterize the differences in the moments’ dynamics of individual equity and index options.\(^2\)

\[ dS_t^i/S_t^i = \mu^i dt + \beta_1^i (\mu_1 v_{1,t} dt + \sqrt{v_{1,t}} dz_{1,t}) + \beta_2^i (\mu_2 v_{2,t} dt + \sqrt{v_{2,t}} dz_{2,t}) + \sqrt{\xi_t^i} dz_t^i \]

\[ d\xi_t^i = \kappa^i (\theta_t^i - \xi_t^i) dt + \sigma^i \sqrt{\xi_t^i} dw_t^i \]

\[ \langle dz_t^i, dw_t^i \rangle = \rho^i dt \] (13)

where \( \kappa^i, \theta^i, \) and \( \sigma^i \) can be defined as for their market counterparts. \( \rho^i \) is the correlation coefficient between idiosyncratic return innovations and idiosyncratic variance innovations for every individual equity \( i \). This parameter defines an asymmetry in the relation between idiosyncratic volatility and returns for individual equities.\(^3\) Given the dynamics in (13), the total instantaneous variance for stock \( i \) at time \( t \) under physical measure is given by

\(^1\) Following Duffie et al. (2000), the coefficients \( A_1, A_2, B_1, \) and \( B_2 \) are the solutions of a system of Riccati equations subject to appropriate boundary conditions. For the ease of computation we modify these solutions based on the little Heston trap formulation of Albrecher et al. (2006).

\(^2\) Our model can be extended to examine the idiosyncratic variance risk premium while incorporating two factor structure in the dynamics of equity returns. We discuss the implications of priced idiosyncratic variance in the following section.

\(^3\) Following Andersen et al. (2001) we expect that the observed asymmetry should be weaker but still present for individual equities.
\[ v_i^t \equiv (\beta_1^i)^2 v_{1,t} + (\beta_2^i)^2 v_{2,t} + \xi_i^t \]  \hspace{1cm} (14)

Proposition (2) gives the risk-neutral dynamics of an individual equity by assuming a conventional stochastic discount factor, given the physical dynamics of the market index (1) and individual equity \( i \) in (13). As in the index model, we also assume that the prices of market variance components are proportional to the spot volatility components.\footnote{We can simply extend our model and consider the priced idiosyncratic variance risk by assuming that idiosyncratic variance risk is also proportional to the spot idiosyncratic volatility. In this case, \( \tilde{\kappa}^i = \kappa^i + \lambda^i \), \( \tilde{\theta}^i = \kappa^i + \lambda^i \). Further details are provided in the proof of the Proposition (1).}

**Proposition 2.** Using a conventional stochastic discount factor similar to (11) and given the dynamics of the individual equity returns under \( P \)-measure (13), the following dynamics governs its \( Q \)-measure counterparts.

\[
\begin{align*}
\frac{dS_i^t}{S_i^t} &= r dt + \beta_1^i \sqrt{v_{1,t}} d\tilde{z}_{1,t} + \beta_2^i \sqrt{v_{2,t}} d\tilde{z}_{2,t} + \sqrt{\xi_i^t} d\tilde{z}_i^t \\
\frac{d\xi_i^t}{\xi_i^t} &= \kappa^i (\theta^i - \xi_i^t) dt + \sigma^i \sqrt{\xi_i^t} dw_i^t 
\end{align*}
\]  \hspace{1cm} (15)

The market prices of risk factors are

\[ \psi_{1,t} = \frac{\mu^i - r}{\sqrt{\xi_i^t (1 - (\rho^i)^2)}} \, , \quad \psi_{2,t} = -\frac{\mu^i - r}{\sqrt{\xi_i^t (1 - (\rho^i)^2)}} \]  \hspace{1cm} (16)

**Proof.** See Appendix B.

As the dynamics of individual equity is affine, the conditional risk-neutral characteristic function of the natural logarithm of the equity price \( i \) is derived analytically in the following proposition. We may then compute a closed-from pricing equation for European equity call options with strike price \( K \) and time to maturity \( \tau \). See also Appendix C.

**Proposition 3.** Given the dynamics of individual equity returns under the \( Q \)-measure (15), the risk-neutral conditional characteristic function of the natural logarithm of individual equity price \( i \), \( x_{i,t+\tau} = \ln(S_{i,t+\tau}) \), is:

\[
\tilde{f}^i(x_{i,t}^t, v_{1,t}, v_{2,t}, \xi_t, \beta_1^i, \beta_2^i, \tau, \phi) \equiv \mathbb{E}_Q^t \left[ \exp(i\phi x_{i,t+\tau}^i) \mid x_{i,t}^i \right] = \exp \left[ i\phi x_{i,t}^i + i\phi r\tau - A_1(\tau, \phi) - A_2(\tau, \phi) - B(\tau, \phi) + C_1(\tau, \phi) v_{1,t} + C_2(\tau, \phi) v_{2,t} + D(\tau, \phi) \xi_t^i \right], \hspace{1cm} (17)
\]

where, the expressions for \( A_1(\tau, \phi), \ A_2(\tau, \phi), \ B(\tau, \phi), \ C_1(\tau, \phi), \ C_2(\tau, \phi), \) and \( D(\tau, \phi) \) are provided within the proof. Then, individual equity option prices may be found as follows.
\[ C_t^i(S_t^i, K, \tau) = S_t^i P_1^i - K e^{-r\tau} P_2^i, \] (18)

where,

\[ P_1^i = \frac{1}{2} + \frac{1}{\pi} S_t^i e^{r\tau} \int_0^\infty \Re \left[ \frac{e^{-i\phi \ln K f_i(v_1, t, v_2, t, \xi_t^i, \tau, \phi - i) \cos \phi}}{i\phi} \right] d\phi, \]
\[ P_2^i = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-i\phi \ln K f_i(v_1, t, v_2, t, \xi_t^i, \tau, \phi) \cos \phi}}{i\phi} \right] d\phi. \] (19)

Proof. See Appendix C. \qed

3 Model Properties and Implications

This section explores, both theoretically and numerically, some of the implications of the proposed two-factor structure in the dynamics of equity returns. In particular, we examine the relative importance of the transient and persistent factors on the sensitivity of the equity option prices with respect to the level of the market index and with respect to each variance components. We also investigate the effects of factor loadings \( \beta_1^i \) and \( \beta_2^i \) and their importance on the instantaneous expected returns of individual equity options. We close this section by exploring a number of important cross-sectional implications of two-factor structure in equity options, some of which shed some lights on the relations between the systematic risk factors and moments of the conditional distribution of equity returns.

In the numerical analysis, we fix parameters as follows; structural parameters for the market index model are from Christoffersen et al. (2009), for individual equities the parameters are set to replicate the observed patterns in the one-factor model of Christoffersen et al. (2017). Further, these parameter values highlight the importance of two-factor structure relative to one-factor structure in examining the properties and cross-sectional implications of factor structure in equity options. Since we are interested in the role of the persistent beta, \( \beta_1^i \), and transient beta, \( \beta_2^i \), we explore the model properties for different sets of betas while keeping the total unconditional risk-neutral equity variance constant.

The total unconditional risk-neutral equity variance is evaluated at its mean reverting value equal to \( \bar{\sigma}^i \equiv (\beta_1^i)^2 \bar{\theta}_1 + (\beta_2^i)^2 \bar{\theta}_2 + \theta^i = 0.11 \). Note that we fix the total unconditional risk-neutral market variance to 0.05, with its persistent component \( \bar{\theta}_1 = 0.006 \) and transient component \( \bar{\theta}_2 = 0.044 \). Therefore, for every set of betas, the unconditional idiosyncratic equity variance can be defined by \( \theta^i = \bar{\sigma}^i - (\beta_1^i)^2 \bar{\theta}_1 - (\beta_2^i)^2 \bar{\theta}_2 \). The spot market persistent and transient variance components are set to \( v_{1,t} = 0.012 \) and \( v_{2,t} = 0.048 \) respectively and the total spot equity variance is set to \( v_i^t = 0.05 \). Consequently, for different sets of betas, the spot idiosyncratic variance under the physical measure can be defined as \( \xi_i^t = v_i^t - (\beta_1^i)^2 v_{1,t} - (\beta_2^i)^2 v_{2,t} \). We choose the remaining structural parameters of the market and equity.
dynamics as follows: \( \{ \tilde{\kappa}_1 = 0.18, \tilde{\kappa}_2 = 2.8, \sigma_1 = 3.6, \sigma_2 = 0.29, \rho_1 = -0.96, \rho_2 = -0.83 \} \) and \( \{ \tilde{\kappa}^i = 0.8, \sigma^i = 0.2, \rho^i = 0 \} \). We fix the risk-free rate at 4% per year and examine at-the-money equity options with 3 months to maturity. We explore the model properties and its cross-sectional implications by assuming the ratio of spot index price over spot equity price as \( S^i_t / S_t = 0.1 \).

The proposed two-factor structure explicitly shows how changes in the level of the spot market index are translated into the equivalent changes in the equity option prices. It also allows us to examine how equity option prices respond to variations in the persistent and transient market variance components. The following proposition establishes these relations and creates a basis for further sensitivity analysis.

**Proposition 4.** Given the closed-form equity option pricing expression in Proposition (3), the sensitivity of the individual equity call option prices \( C^i_t \) with respect to the level of the market index \( S_t \) may be given by:

\[
\frac{\partial C^i_t}{\partial S_t} = \frac{\partial C^i_t}{\partial S^i_t} \frac{S^i_t}{S_t} (\beta^i_1 + \beta^i_2).
\]

Further, the sensitivity of the individual equity call option prices \( C^i_t \) with respect to the market variance components \( v_{1,t} \) and \( v_{2,t} \) are:

\[
\begin{align*}
\frac{\partial C^i_t}{\partial v_{1,t}} &= \frac{\partial C^i_t}{\partial v^i_t} (\beta^i_1)^2, \\
\frac{\partial C^i_t}{\partial v_{2,t}} &= \frac{\partial C^i_t}{\partial v^i_t} (\beta^i_2)^2.
\end{align*}
\]

where the total spot variance for equity \( i \) is \( v^i_t = (\beta^i_1)^2 v_{1,t} + (\beta^i_2)^2 v_{2,t} + \xi^i_t \).

**Proof.** See Appendix D.

We interpret the expression (20) as the “market delta” and the expressions (21) as the “persistent market vega” and “transient market vega” for call options on equity \( i \). Figure (1) shows the market sensitivity (market delta) of the model-implied equity call option prices, given the structural parameter values defined above. We plot the market delta for different sets of betas to examine the relative importance of transient and persistent factors. Consistent with Christoffersen et al. (2017), we find that firms with different sets of betas have different sensitivities to changes in the level of the market index. Consistent with Proposition (4), we observe that firm’s with higher transient (persistent) beta are more sensitive to the changes in the level of the market index when we keep persistent (transient) beta constant. The same is also true for firms with higher average beta. Although, we cannot distinguish between the effect of transient and persistent betas on market delta per se, we observe that at-the-money equity call option prices are relatively more sensitive to the
transient beta. Note that the top panel of figure (1) replicates the market delta following the calibration in the one-factor model of Christoffersen et al. (2017).

Figures (2) and (3) plot the sensitivity of the model-implied equity call option prices with respect to the persistent and transient market variance components using the parameter values described above. Christoffersen et al. (2017) find that firms with higher betas are more sensitive to changes in the market volatility. Our model predicts the same pattern with respect to the total market volatility. In particular, we find that firms with higher persistent betas are more sensitive to changes in the persistent variance component while the effect of the transient beta on the persistent market vega is marginal but reverse. Further, firms with higher transient betas are more sensitive to changes in the transient variance components while the effect of the persistent beta on the transient market vega is reverse but significant. In other words, persistent beta has an important effect on the transient market vega across different levels of moneyness, see Figure (3). This distinctive property of our model allows a portfolio manager to better examine the exposure of her portfolio to the variations in market returns, a feature that is absent in the single factor structure of Christoffersen et al. (2017). Comparing the level of transient market vega and persistent market vega, our model predicts that equity call option prices are more sensitive to the transient volatility component compared to the persistent volatility component.

Our two-factor structure and closed-form equity option pricing expression allow us to shed some light on the relation between the expected returns of individual equity options and the characteristics of market returns and variance components as expressed in Proposition (5) below. This result allows us to disentangle the effect of the market risk premium from those of variance component risk premiums on the equity option returns. It also shows how equity betas play a direct role on the equity option returns. In particular, the second component in the right-hand-side (RHS) of equation (22), which is related to the market risk premium, affects the equity option returns through the market delta by an adjustment factor which includes the persistent and transient betas. Moreover, the third component in the RHS of (22), which is related to variance risk premiums, shows how equity betas affect the equity option returns through the total market vega of equity options. Note that $\frac{\partial C_i}{\partial v}$ measures the total market vega of equity options.

Remember that option market vega is the amount of money per underlying share that the option value will gain or loose as market volatilities rise or fall by 1%. It is also important as value of some option strategies are partially sensitive to changes in volatility.
Proposition 5. Given the closed-form equity option pricing expression (18)-(19), the dynamics of the market index (1) and individual equity returns (13), the instantaneous expected excess returns on individual equity call options under the physical measure can be characterized as follows.

\[
\frac{1}{dt} E_t^P \left[ \frac{dC_i}{C_i} - r dt \right] = \left( \mu^i - r \right) \frac{S_i}{C_i} \frac{\partial C_i}{\partial S_i} \\
+ \left[ \beta_{i1} \mu_1 v_{1,t} + \beta_{i2} \mu_2 v_{1,t} \frac{S_i}{C_i} \right] \frac{\partial C_i}{\partial S_t} \\
+ \left[ (\beta_{i1})^2 \lambda_1 v_{1,t} + (\beta_{i2})^2 \lambda_2 v_{2,t} \frac{1}{C_i} \right] \frac{\partial C_i}{\partial v_t} 
\]

(22)

Proof. See Appendix D.

Our proposed two-factor structure has also important cross-sectional implications for equity options. Christoffersen et al. (2017) document that firms with higher betas have a steeper term structure of implied volatility. However, our model moves further and provides a novel term structure effect. In particular, we show how the term structure of implied volatility responds differently to the transient and persistent variations in market returns. Using the parameter values introduced at the beginning of this section, we show how \( \beta_{i1} \) and \( \beta_{i2} \) have different and non-trivial effects on the implied volatility term structures of individual equity options. Figure (4) plots the model implied volatility for at-the-money equity call options with respect to time-to-maturity for different sets of betas. Consistent with the finding in Christoffersen et al. (2017) (the top LHS panel), the higher the average betas the steeper the term structure of the implied volatility of equity options (the top RHS panel). In particular, our model predicts that the term structure of implied volatility of equity options is more sensitive to the Transient beta (the bottom LHS panel) while the impact of the persistent beta on the term structures of implied volatility of equity options is marginal (the bottom RHS panel). In other words, firms with higher transient betas have a term structure of implied volatility that co-moves more with the market term structure of IV.

[Figure (4) about here]

We close this section by discussing the implications of two-factor structure on the relation between the market variance risk premiums and the equity option skew. Figure (6) plots the difference between the model implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is computed as the difference between equity call option IV when we increase variance component risk premiums from \( \lambda_1 = \lambda_2 = -0.5 \) to \( \lambda_1 = \lambda_2 = 0 \). As expected, the variance

\[ \text{Note that in all the graphs the total unconditional equity variance under the risk-neutral measure is fixed at } \bar{\sigma} = (\beta_{i1})^2 \bar{\theta}_1 + (\beta_{i2})^2 \bar{\theta}_2 + \bar{\sigma} = 0.11. \]
risk premiums have a more significant effect on the implied volatility of equity call options when the beta is higher (the top RHS panel). In particular, we observe that the transient beta has a more significant effect on the slope of equity implied volatility smile (the bottom LHS panel) compared to the persistent beta (the bottom RHS panel). In other words, in-the-money equity call options are getting relatively more expensive for firms with higher transient betas when we increase variance risk premiums. Note that for all the graphs the total unconditional equity variance is fixed \( \tilde{\nu}^i = (\beta_1^i)^2 \tilde{\theta}_1 + (\beta_2^i)^2 \tilde{\theta}_2 + \theta^i = 0.11 \). Note also that the top LHS panel replicates the same pattern following the calibration in the one-factor model of Christoffersen et al. (2017).

We close this section by discussing the implications of two-factor structure on the relation between the market variance risk premiums and the equity option skew. Figure (6) plots the difference between the model implied volatility for three-month equity call options with respect to the moneyness \((S/K)\) for different sets of betas. The implied volatility difference is computed as the difference between equity call option IV when we increase variance component risk premiums from \( \lambda_1 = \lambda_2 = -0.5 \) to \( \lambda_1 = \lambda_2 = 0 \). As expected, the variances risk premiums have more significant effect on the implied volatility of equity call option when the beta is higher (the top RHS panel). In particular, we observe that the transient beta has more significant effect on the slope of equity implied volatility smile (the bottom LHS panel) compared to the persistent beta (the bottom RHS panel). In other words, in-the-money equity call options are getting relatively more expensive for firms with higher transient beta when we increase variance risk premiums. Note that for all the graphs the total unconditional equity variance is fixed \( \tilde{\nu}^i = (\beta_1^i)^2 \tilde{\theta}_1 + (\beta_2^i)^2 \tilde{\theta}_2 + \theta^i = 0.11 \). Note also that the top LHS panel replicates the same pattern following the calibration in one-factor model of Christoffersen et al. (2017).

4 Data

We obtain daily prices of S&P 500 index options from the OptionMetrics volatility surface data set. Our sample of S&P 500 index options is from January 4, 1996 through December 29, 2011. We follow the data cleaning routine commonly used in the empirical option pricing literature: we remove options with implied volatility less than 5% and greater than 150%; we also follow the filtering rules in Bakshi et al. (1997) to remove options that violate various no-arbitrage conditions. We focus on out-of-the-money (OTM) options with maturity up to and
including one-year and with 10% moneyness (spot price over strike price).\textsuperscript{17,18} Our option-based optimization function minimizes the squared deviations between model and market option prices and therefore may put greater weight on expensive in-the-money (ITM) and long-maturity options.\textsuperscript{19} Moreover, ITM S&P 500 call options are less liquid than OTM call options. To prevent such biases in our optimization, we discard all ITM options and use OTM S&P 500 put options and convert them into ITM call options. After cleaning, we have 345,710 S&P 500 index option quotes together with daily underlying returns. This is the data set that we use to filter daily spot variances and to estimate a set of structural parameters.

For individual equities, we choose all the firms listed in the Dow Jones Industrial Average index and collect equity options data from OptionMetrics.\textsuperscript{20} We keep all options up to 10% moneyness and with maturity up to and including 1 year. Note that options on individual equities are American, the price of which could be affected by early exercise premium. To reduce any possible bias in the estimation of the structural parameters of equity dynamics and daily spot idiosyncratic variance, the loss function needs to be defined based on the implied volatility as implied volatilities and deltas for equity options reported in OptionMetrics are computed by the Cox et al. (1979) binomial tree model. Otherwise, if the loss function is based on mean-squared option pricing errors, we either need to restrict our sample to out-of-the-money equity options that are less sensitive to early exercise premium or have to covert the American-style equity options into European-style equity options by taking into account the early exercise premium. Due to the computational burden of such adjustments and considering the closed-from European option pricing equation in Proposition (3), we focus on OTM equity options.\textsuperscript{21}

The data for daily equity prices, equity returns, daily index level, index returns, and dividend yields are from CRSP. In the empirical analysis, we first adjust daily equity prices and index level with dividend yields and then compute option prices using the dividend-adjusted returns. Risk-free interest rates for all maturities are estimated by linear interpolation between the closest zero-coupon rates of the Zero Coupon Yield Curve from OptionMetrics.

Table (1) presents the descriptive statistics of the S&P 500 index call option contracts in our sample sorted by moneyness (stock price over strike price) and day-to-maturity (DTM). Note

\textsuperscript{17} This range of moneyness implies that we keep OTM call options with moneyness less than 1.1 and OTM put options with moneyness greater than 0.9.

\textsuperscript{18} As discussed in previous section, multiple-factor SV models could better capture the slope and the level of smirk compare to single-factor SV models. Therefore, unlike similar analysis, we undertake a more extensive calibration exercise by incorporating the information content of options on longer maturity horizons and wider moneyness ranges. For instance, Ait-Sahalia and Kimmel (2007, Section 7) only include short-maturity at-the-money S&P 500 Index Options; Eraker (2004) use 3,270 call options contracts recorded over 1,006 trading days; Jones (2003) models are estimated using a sample of 3537 S&P 100 index options from January 1986 to June 2000.

\textsuperscript{19} See Huang and Wu (2004).

\textsuperscript{20} Note that we drop the Bank of America, the Kraft Foods Incorporation, and the Travelers Companies Incorporation.

\textsuperscript{21} See Bakshi et al. (2003) and Christoffersen et al. (2017).
that we focus on OTM option contracts, which means \( S/K \) is below 1 for OTM call contracts. After cleaning, we have 208,098 out-of-the-money call option contracts with an average day-to-maturity of 143 days, an average price of $35.59, an average implied volatility of 20.64\%, and an average delta of 0.37. Table (2) reports the descriptive statistics of the S&P 500 index put option contracts in our sample sorted by moneyness and day-to-maturity. After cleaning, we use 137,612 out-of-the-money (\( S/K \) is above 1) put option contracts with an average day-to-maturity of 136 days, an average price of $32.11, an average implied volatility of 24.34\%, and an average delta of -0.29. Note that Panel C in Tables (1) and (2) reflect the well-known volatility smirk in index options, as implied volatility is larger for OTM put options (Table (2), Panel C) compared to the OTM call options (Table (1), Panel C).

Table (3) presents the descriptive statistics of the equity option contracts that are used to filter daily spot idiosyncratic variance, and to estimate the structural parameters for individual equities and market index. This table reports the number of available call and put option contracts for each firm after data cleaning. For every firm, we also report the average number of days-to-maturity and average implied volatility of option contracts in our sample. Overall, we have 4,241,990 equity call options and 3,209,990 equity put options with an average days-to-maturity of 135 days. On average, for every firm we have 275,999 option contracts with an average implied volatility of 28.52\%.

Tables (4) and (5) provide further details regarding equity call options and put options. On average we observe that equity call options in our sample are more expensive (2.688 for calls versus 2.344 for puts), more sensitive to underlying equity prices and volatilities, have lower implied volatility (27.32\% for calls and 29.73\% for puts), and have a greater number of days-to-maturity (137 days for calls and 134 days for puts.)
5 Estimation Methodology

Our estimation methodology is twofold. At the market index level, we do a joint-estimation to filter vectors of daily spot variance components and to obtain a consistent set of structural parameters for the market index dynamics. Then, for every individual equity, we filter spot idiosyncratic variance and estimate structural parameters of the equity dynamics, given the dynamics of the market index and its filtered spot variance components.

5.1 Estimation of the Index Model

We follow the approach in the Appendix (E) and combine information from underlying index and option markets (joint estimation). We use a two-component likelihood function, a return-based component and an option-based component, to impose consistency between structural parameters under $P$ and $Q$ distributions. To filter unobserved transient and persistent spot variance components, we use the sampling-importance-resampling (SIR) implementation of the Particle Filter (PF) methods.\footnote{See Appendix E for implantation of PF in the context of two-factor stochastic volatility model. See Pitt (2002) for the detailed description of the PF algorithm.}

Our optimization function is as follows.

$$\max_{\Theta, \tilde{\Theta}} (LLR + LLO)$$  \hspace{1cm} (23)

where $LLR$ is the return-based and $LLO$ is the option-based likelihood functions and $\Theta$ is the set of structural parameters of the market index model under $P$-measure and $\tilde{\Theta}$ is the equivalent set under $Q$-measure.

$$LLR \propto \sum_{t=1}^{T} \ln \left( \frac{1}{N} \sum_{j=1}^{N} \tilde{W}_t^j(\Theta) \right)$$  \hspace{1cm} (24)

where $\tilde{W}_t^j$ is the normalized weight of particle $j$ at time $t$, $N$ is the number of daily particles, and $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \lambda_1, \lambda_2\}$, and $T$ is the total number of days in our sample.

$$LLO \propto -\frac{1}{2} \left( M \ln(2\pi) + \sum_{n=1}^{M} \left( \ln(s^2) + \eta_n^2/s^2 \right) \right),$$  \hspace{1cm} (25)

where $M$ is the total number of index option contracts and $\eta_n$ is the Vega-weighted loss function for option $n$. 

\footnotetext{22 See Appendix E for implantation of PF in the context of two-factor stochastic volatility model. See Pitt (2002) for the detailed description of the PF algorithm.}
\[ \eta_n = (C^O_n - C^M_n(\hat{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau))/\text{Vega}_n, \quad n = 1, \ldots, M \] (26)

where \( C^O_n \) is the observed price of call option \( n \) and \( C^M_n(\hat{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau) \) is the model price of call option \( n \). \( \text{Vega}_n \) is the Black and Scholes (1973) option Vega for the same option contract. Note that we obtain daily persistent (\( \hat{v}^Q_{1,t} \)) and transient (\( \hat{v}^Q_{2,t} \)) spot variance components under \( Q \) measure as the average of smoothly re-sampled particles of daily variance components.

\[ \hat{v}^Q_{1,t} = \frac{1}{N} \sum_{j=1}^{N} v^j_{1,t}, \quad \hat{v}^Q_{2,t} = \frac{1}{N} \sum_{j=1}^{N} v^j_{2,t} \] (27)

Our index optimization algorithm is iterative. Each iteration starts with an initial set of structural parameters, \( \Theta_0 \), which then will be used to filter transient and persistent daily spot variance components using the information content of index returns. Then, given spot variance components, structural parameters of the market index, and observed option prices, the next set of optimal parameters, \( \hat{\Theta} \), can be reached by minimizing the option pricing errors over the entire sample. The procedure iterates until an optimal set of structural parameters is reached and thereby we obtain the final vectors of transient and persistent spot variance components, \( \{\hat{v}^Q_{1,t}, \hat{v}^Q_{2,t}\} \).

### 5.2 Estimation of the Individual Equity Model

We estimate a set of structural parameters \( \hat{\Theta}^i \equiv \{\kappa^i, \theta^i, \sigma^i, \rho^i, \beta_1^i, \beta_2^i\} \) and a vector of daily spot idiosyncratic variance \( \{\xi^i_t\} \) for each individual equity in our sample following the two-step iterative approach of Bates (2000) and Huang and Wu (2004). In the first step, given a set of initial structural parameters for each equity, \( \hat{\Theta}_0^i \), we estimate a vector of daily spot idiosyncratic variance conditional on a set of risk-neutral structural parameters of the market model, \( \hat{\Theta} \), and filtered daily risk-neutral spot variance components, \( \{\hat{v}^Q_{1,t}, \hat{v}^Q_{2,t}\} \). Using a Vega-weighted loss function, a set of daily spot idiosyncratic variance \( \hat{\xi}^i_t \) for every firm \( i \) can be obtained as the solution to the following optimization problem, which minimizes the Vega-weighted daily mean-squared option pricing errors.

\[ \hat{\xi}^i_t = \arg \min_{\xi^i_t} \sum_{n=1}^{M^i_t} (C^{i,O}_{n,t} - C^{i,M^i_t}_{n,t}(\Theta^i_0, \hat{\Theta}, \hat{v}^Q_{1,t}, \hat{v}^Q_{2,t}, \xi^i_t))^2/(\text{Vega}_{n,t}^i)^2, \quad t = 1, \ldots, T, \] (28)

where \( M^i_t \) is the total number of available option contracts for the equity \( i \) on day \( t \), \( C^{i,O}_{n,t} \) is the observed price of equity call option \( n \) for stock \( i \) on day \( t \), \( C^{i,M^i_t}_{n,t} \) is the model price for the same option obtained from equity pricing equation (18), and \( \text{Vega}_{n,t}^i \) is the Black-Scholes
option Vega for the same equity option contract. We repeat the optimization in (28) every
day and for every equity to estimate a vector of spot idiosyncratic variances over the entire
sample.

The second step estimates the structural parameters $\tilde{\Theta}^i$ for firm $i$, by minimizing sum of daily
Vega-weighted mean-squared option pricing errors over the entire sample, given filtered daily
spot idiosyncratic variance obtained in the first step, the dynamics of the market index and
filtered daily spot variance components. We may then solve the the following optimization
problem.

$$
\hat{\Theta}^i = \arg \min_{\xi^i_t} \sum_{n=1}^{M^i} \left( C_{n,O}^i - C_{n,M}^i (\tilde{\Theta}^i, v_{1,t}, v_{2,t}, \xi^i_t) \right)^2 / \left( \text{Vega}_n^i \right)^2,
$$

where $M^i \equiv \sum_{t=1}^{T} M^i_t$ is the total number of available option contracts for equity $i$. For
every equity, the procedure iterates between the optimizations in (28) and (29) to minimize
the pricing error until the change in the RMSE of the estimation in the second step is no
longer significant. Note that every new iteration starts based on the structural parameters
of the previous iteration, $\hat{\Theta}_0^i = \hat{\Theta}^i$.

6 Parameter Estimation Results

This section first reports the filtered daily spot variance components together with the struc-
tural parameter estimates for our two-factor SV model. We use a long time-series of daily
S&P 500 index returns and the entire cross-section of S&P 500 option prices that span the
period from January 4, 1996 to December 29, 2011. Given the slow mean-reversion in the
dynamic of market volatility, it is important to let the data set span a long time series. This
is in particular important in our analysis as we decompose the overall market volatility into
two independent components and would like to characterize the dynamics of transient and
persistent variance components. We also report structural parameters of the equity dynamics
and daily spot idiosyncratic variance for 27 firms listed in the Dow Jones Industrial Average
Index. The parameter estimates and latent idiosyncratic variances are conditional on the
transient and persistent spot variance components $v_{1,t}$ and $v_{2,t}$ and structural parameters $\hat{\Theta}$
of the index model.

To provide a basis for further comparison and to examine the model fit under the joint-
estimation, we also report the structural parameters of the market model, estimated only
from option data.
6.1 Parameter Estimates, Market Index Model

We set the market risk premium $\mu$ to the sample average daily index returns. We use OTM index options with up to 10% moneyness and then convert the OTM puts into ITM calls through put-call parity. Table (6) reports structural parameter estimates (under $P$ measure) that characterize the dynamics of index returns and its persistent and transient variance components. Panel A provides result of the joint estimation; a consistent set of parameters under $P$ and $Q$ measures. Therefore, the speeds of mean reversion and the unconditional mean of the persistent and transient variance components under $Q$-measure are linked to their $P$-measure equivalents through the market prices of the volatility risk factors ($\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1 \theta_1}{k_1 + \lambda_1}$, $\tilde{\theta}_2 = \frac{k_2 \theta_2}{k_2 + \lambda_2}$).\(^{23}\) To provide a basis for further comparison and to examine the goodness of fit of the two-factor SV model under the joint-estimation, we also estimate structural parameters using only option data. This result is provided in Panel C. Note that we assume that the transient and persistent beta coefficients are the same under $P$ and $Q$ measures following Serban et al. (2008).

As discussed, the purpose of two-factor stochastic volatility model is to capture independent movements in the underlying returns and option prices over time. Consistent with previous studies in both discrete time GARCH models and continuous time stochastic volatility models, we find that one of the volatility factors is highly persistent and the other one is highly mean-reverting. In joint-estimation, we find that the first variance component is slowly mean-reverting with $\kappa_1 = 1.4271$ under physical measure while the rate of mean reversion in the second variance component is much higher with $\kappa_2 = 3.5874$ under the physical measure.\(^{24}\) The point estimate of mean reversion parameters from option-based estimation is similar to those from joint estimation. Using options data only, we find that $\tilde{\kappa}_1 = 0.2267$ and $\tilde{\kappa}_2 = 2.9137$, which is consistent with the speed of mean reversion from joint estimation where under $Q$-measure $\tilde{\kappa}_1 = 0.3473$ and $\tilde{\kappa}_2 = 2.5520$.

To obtain a better intuition about persistent and transient variance components we define the half-life ($T_{1/2}$) of a variance component as the number of weeks that it takes for autocorrelation of a variance component to decay to half of its weekly autocorrelation level. Half-life can be computed as $T_{1/2} = \ln(\phi/2)/\ln(\phi)$ where $\Delta t = 7/365$ and $\phi = \exp(-\kappa \Delta t)$, denoting weekly autocorrelation of time-series of a variance component. The risk-neutral point estimate of mean reversion speed in transient variance component implies a half-life around 15 weeks while it is 105 weeks in the persistent variance component, almost 7 times larger than its transient counterpart. These values confirm that first variance component is highly persistent while the second one is highly auto-correlated and thus the immediate impact of variance shocks on this component is larger but short-lived.

\(^{23}\) See Proposition (1).

\(^{24}\) These value correspond to a daily variance persistence of $1 - 1.4271/365 = 0.9961$ for the first component and $1 - 3.5874/365 = 0.9901$ for the second component.
We observe that the unconditional persistent variance under \( P \)-measure is \( \theta_1 = 0.0026 \), which is much less than the unconditional transient variance \( \theta_2 = 0.0171 \). The unconditional risk-neutral persistent and transient variance components are \( \hat{\theta}_1 = 0.0106 \) and \( \hat{\theta}_2 = 0.0240 \) which correspond to 10.30% and 15.49% volatility per year. Note that the unconditional variance of both components are consistent with the average filtered daily spot persistent variance and daily spot transient variance over the entire sample.

Consistent with our intuition, we observe a wide spread between the volatility of variance in persistent and transient variance components of the market index. In our joint estimation exercise, we find that \( \sigma_1 = 0.0855 \) and \( \sigma_2 = 0.3496 \). This result is consistent with the option-based estimation where we find that transient variance component is much more volatile with \( \sigma_2 = 0.5678 \) compared to persistent variance component with \( \sigma_1 = 0.0958 \). The higher level of volatility of variance in option-based estimation compared to the joint estimation is consistent with previous studies.\(^{25}\)

We find negative prices for both variance components where \( \lambda_1 = -1.0798 \) and \( \lambda_2 = -1.0355 \). These negative prices imply that investors are willing to pay for an insurance against an increase in volatility risk, even if that increase has little persistence. To the best of our knowledge none of the previous studies of two-factor stochastic volatility models in option market reports the prices of the variance risk factors as they either focused on the options market data or the underlying index returns data. Our negative prices for both variance components is consistent with asset pricing studies where the short-run and the long-run volatility components are priced cross-sectional asset pricing factors. Adrian and Rosenberg (2008) use a large cross-section of individual stocks over a very long period and find that prices of both short-run and long-run variance components are negative and highly significant. Therefore, our join estimation result may confirm that there is a consensus of opinions about the price of transient and persistent variance components among option traders and equity traders.

Our joint estimation results show that correlation between shocks to the market index returns and shocks to the persistent variance component is \( \rho_1 = -0.6918 \). The correlation between shocks to the index returns and shocks to the transient variance component is \( \rho_2 = -0.2173 \). \( \rho_1 \) and \( \rho_2 \) captures asymmetry in the response of persistent and transient variance components to positive versus negative return shocks and can be considered as the persistent and transient continuous time leverage (asymmetry) effect. The leverage effect induces negative skewness in index returns and thus yields a volatility smirk. Our results show that that leverage effect is more significant in the persistent variance component compared to the transient variance component. Therefore, persistent variance component has more significant effect on the dynamic of index skewness. Using the data from option market only, we find that \( \rho_1 = -0.91 \) and \( \rho_1 = -0.49 \). The higher absolute level of option implied correlation coefficients compared to those of joint estimation is partly related to the well documented fact that risk-neutral distribution is more negatively skewed.

\(^{25}\) For instance, Bates (2000) reports that option-based estimates of volatility of variance is larger than the one obtained from time-series-based estimates.
Our persistent and transient correlation coefficients are almost consistent with those of previous studies in option market. The average correlation coefficients in Christoffersen et al. (2009, Table 3) are $\rho_1 = -0.96$ for the first variance component and $\rho_2 = -0.83$ for the second variance component. Bates (2000) also reports the structural parameter estimates of a two-factor SV model using 1988-1993 S&P 500 futures option prices. He obtains one set of structural parameters over the entire sample where $\rho_1 = -0.78$ and $\rho_2 = -0.38$. To provide a basis for comparison, we also estimate structural parameters using options data only over the same sample period and find $\rho_1 = -0.91$ and $\rho_2 = -0.49$. There are potential explanations for differences between the reported estimates of the correlation coefficients in these studies, not in the least, the very different data set and the very different time span. Despite differences in the magnitude of the coefficients, the point estimates for the correlation coefficients are negative for both persistent and transient variance components across all these studies. Further, the transient variance component exhibits a lower (in absolute value) level of correlation to the market index returns in all these studies.

To provide some empirical evidence on the difference between persistent and transient variance components over time, we plot the paths of filtered variance components. Figure (7) plots filtered time series of risk-neutral spot variance component of S&P 500 index based on our two-factor stochastic volatility model. Panel A shows time series of persistent variance component and Panel B shows time series of transient variance component. The blue plots are based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation) and the red plots are filtered spot variances using only S&P 500 options data.

Naturally, the overall patterns of persistent and transient variance components filtered in our joint estimation exercise are consistent with those filtered from options data only. However, option implied variance components are more volatile in the sense that when variance increases, it tends to do more sharply compared to those filtered based on joint estimation and thus exhibit more spikes. In particular, this pattern in more pronounced in the transient variance component (Panel B). The sharper spikes in option-based filtered variance in the two-factor SV model is consistent with previous studies of one-factor SV model. The smoother variance paths in joint-estimation is partly due to smooth resampling procedure in SIR PF method and partly due to imposed consistency between parameter estimates under $P$ and $Q$ measures.

To provide more intuition about the total risk-neutral variance in our two-factor SV model, Figure (8) combines persistent and transient variance components and plots time series of total spot variance versus model-free option-implied VIX volatility index. As we expect, the time series of option implied total spot variance is closely related to the VIX volatility index.

---

26 Christoffersen et al. (2009) use data on European S&P 500 call option quotes over the period 1990-2004. They estimate a correlation coefficient for every year in their sample.
Further, the time series of total spot variance from joint estimation follow the same pattern as the VIX volatility index. However, due to joint restrictions and smooth resampling procedure in SIR method, the total spot variance from joint estimation do not exhibits volatility spikes as large as those observed in the VIX volatility index.

[Figure (8) about here]

6.2 Parameter Estimates, Individual Equity Model

The data for individual equities starts from June 1, 1996 rather than January 1, 1996. We drop the first 5 months of each equity’s data set to prevent any estimation bias, as the filtered spot market variance components are noisy in the first months of the estimation period. Note also that S&P 500 Index options are European style while the individual equity options are American style, the price of which might be affected by early exercise premium. To reduce the bias in the calculation of equity option prices using the closed-form pricing equation in Proposition (3) we focus on OTM options.\textsuperscript{27,28}

Table (7) reports the structural parameter estimates that characterize the dynamics of individual equity returns and idiosyncratic variance under the $Q$ measure. The table also contains the point estimates of the persistent and transient betas for 27 firms in our sample.

[Table (7) about here]

The speed of mean reversion for risk-neutral idiosyncratic variance ranges from $\tilde{\kappa}^i = 0.3920$ for Coca Cola to $\tilde{\kappa}^i = 1.7078$ for 3M. This range of $\tilde{\kappa}^i$ implies that most of the firms in our sample have highly persistent idiosyncratic variance with average speed of mean reversion 0.8055. In other words, the average half-life of idiosyncratic variance for the firms in our sample is almost 46 weeks, implying that it takes 46 weeks for the idiosyncratic variance autocorrelation to decay to half of its weekly autocorrelation. We also find that most of the firms in our sample have an idiosyncratic variance that is more persistent than the overall market variance.

\textsuperscript{27} Bakshi et al. (2003) show that for OTM S&P 100 American options the early exercise premium is negligible. They estimate two separate implied volatilities: the implied volatility that equates the option price to the American option price from binomial tree model, and the implied volatility that equates the option price to the Black-Scholes price where the discounted dividends are subtracted from the spot price. They find that although American option implied volatility is smaller than its Black-Scholes counterparts, the difference is negligible and within the bid-ask spread.

\textsuperscript{28} Using the data of the firms listed on Dow Jones Index, Christoffersen et al. (2017) show that the early exercise premium is negligible for equity call options. As a robustness test, we also estimate the equity model by using only the equity call options rather than OTM calls and puts. We find that the point estimates of structural parameters are quite similar to our base case estimation where we use OTM put and call option contracts. This result is available from the author upon request.
The unconditional risk-neutral idiosyncratic variance of the firms in our sample starts from \( \tilde{\theta}_i = 0.0093 \) for General Electric and increases up to \( \tilde{\theta}_i = 0.0756 \) for Hewlett-Packard. The point estimates for the volatility of the idiosyncratic variance range from \( \sigma^i = 0.0670 \) for General Electric to \( \sigma^i = 0.3967 \) for Hewlett-Packard. For all the firms in our sample, the average point estimates for the volatility of the idiosyncratic variance is 0.1823. The correlation between shocks to equity returns and shocks to idiosyncratic variance is negative for all the equities (except for Verizon) and ranges from \( \rho^i = -0.99 \) for JP Morgan to \( \rho^i = 0.512 \) for Verizon.

The betas estimates are novel and to the best of our knowledge this is the first study that reports the option-implied persistent beta and transient beta for individual equities and thus there is no benchmark for further comparisons. We find that firms respond differently to transient and persistent variations in market index returns. The persistent beta ranges from \( \beta_1^i = 0.3430 \) for American Express to \( \beta_1^i = 0.6798 \) for IBM. The transient beta starts from \( \beta_2^i = 1.0125 \) for Procter & Gamble and increases to \( \beta_2^i = 1.3466 \) for JP Morgan. The average persistent beta is 0.4899 and the average transient beta is 1.2284. Across all 27 firms in our sample the transient beta is always greater than the persistent beta, implying that for the large capitalization firms listed in the Dow Jones index, transient and larger variations in the market index tend to be related to the proportionally larger systematic price reactions across equities than persistent and smaller variations in the market index.

Our point estimates of the transient and persistent option-implied betas are similar to the continuous beta and jump beta of Todorov and Bollerslev (2010) who introduce a framework to separate and identify continuous and discontinuous systematic risks. Using high frequency data from a large cross-section of forty large-capitalized individual stocks, they find that the average jump betas are larger than the continuous betas with few exceptions. Although we only use option data and estimate ad-hoc constant beta over the entire sample, we observe a similar pattern as theirs between our transient and persistent betas.

We close this section by providing more intuition about the idiosyncratic variance across the firms in our sample by presenting the distributional properties of the filtered spot idiosyncratic variance. Table (8) reports the mean, median, standard deviation, and the maximum value of the filtered spot idiosyncratic variances for every firm conditional on the dynamics of the market index. We observe that for all the firms the median is significantly lower than the mean, implying that the mean estimates of the filtered spot idiosyncratic volatilities are driven by outliers that may be common to all firms.

[Table (8) about here]

\(^{29}\) The assumption of constant transient and persistent betas allow us to keep the affine specification of the dynamics of individual equity and derive a closed-form equity option pricing equation. We can, however, estimate time-varying betas by modifying our estimation procedure. We can fix the structural parameters of the market and individual equities and estimate conditional betas and spot idiosyncratic variance on a daily basis, given the transient and persistent spot variance components using a loss function very similar to \(^{29}\).
7 Model Performance and In-Sample Fit

We measure the goodness of fit of the market index model using the following Vega-weighted root mean squared option pricing errors (Vega RMSE) as it is consistent with the loss function that we used in the optimization routine.

\[
\text{Vega RMSE} = \sqrt{\frac{1}{N} \sum_{n,t} \left( \frac{C_{n,t} - C_{n,t}^{M}(\hat{\Theta}, \hat{\nu}_{1,t}, \hat{\nu}_{2,t})}{Vega_{n,t}} \right)^2},
\]

(30)

where, \(C_{n,t}^{O}\) is the observed price of index option \(n\) on day \(t\), \(C_{n,t}^{M}\) is the model price for the same index option on the same day, and \(Vega_{n,t}\) is the Black-Scholes option Vega for the same option contract on the same day. To provide a reference for comparison, we also report the implied volatility root mean squared error (IVRMSE) of option pricing model.

\[
\text{IVRMSE} = \sqrt{\frac{1}{N} \sum_{n,t} \left( IV_{n,t}^{O} - IV(C_{n,t}^{M}(\hat{\Theta}, \hat{\nu}_{1,t}, \hat{\nu}_{2,t})) \right)^2},
\]

(31)

where, \(IV_{n,t}^{O}\) is the Black-Scholes implied volatility of index option \(n\) on day \(t\) given the observed call option price \(C_{n,t}^{O}\) and \(IV(C_{n,t}^{M}(\hat{\Theta}, \hat{\nu}_{1,t}, \hat{\nu}_{2,t}))\) is the Black-Scholes implied volatility given the model option price \(C_{n,t}^{M}\) for the same index option on the same day.

For individual equities, Vega RMSEs and IVRMSEs are computed with equations similar to (30) and (31) while replacing \(C_{n,t}^{O}\) with \(C_{n,t}^{O,i}\), \(C_{n,t}^{M}(\hat{\Theta}, \hat{\nu}_{1,t}, \hat{\nu}_{2,t}, \hat{\xi})\) with \(C_{n,t}^{M,i}(\hat{\Theta}, \hat{\nu}_{1,t}, \hat{\nu}_{2,t}, \hat{\xi})\), \(Vega_{n,t}\) with \(Vega_{n,t}^{i}\), \(IV_{n,t}^{O}\) with \(IV_{n,t}^{O,i}\), and \(IV(C_{n,t}^{M}(\hat{\Theta}, \hat{\nu}_{1,t}, \hat{\nu}_{2,t}))\) with \(IV(C_{n,t}^{M,i}(\hat{\Theta}, \hat{\nu}_{1,t}, \hat{\nu}_{2,t}^{i}, \hat{\xi}))\).

Table (9) reports in-sample goodness-of-fit statistics for the two-factor stochastic volatility model over the entire sample, 1996 through 2011 for various maturities. Panels A and B report in-sample fit statistics for calls and puts separately. The left panel reports model fit based on the joint estimation while the right panel reports those statistics for option-based estimation. We find that the Vega-weighted RMSE of joint estimation and option-based estimation are 2.56% and 0.98% respectively. Note that the IVRMSE are 2.59% and 0.99% respectively, which means that Vega-weighted RMSE could be used as an approximation of IVRMSE. Overall, our two-factor SV model provides a better fit to call option contracts compared to put option contracts, which is consistent with the findings in one-factor stochastic volatility model.

Note that joint estimation imposes a consistency between physical and risk-neutral parameters which are otherwise not identical. Such a restriction is not required in option-based estimation, which could partly explain the better in-sample fit of option-based estimation compared to joint estimation. However, the reported RMSEs confirm that unlike stochastic
volatility model, joint restrictions on return and variance dynamics under \( P \) and \( Q \) measures
does not lead to the poor performance of the two-factor SV model.

Broadie et al. (2007) show the inconsistency between the option-based estimates of certain
structural parameters in SV model and the parameter estimates from underlying time-series
of returns and indicate that the SV model is basically misspecified. In particular, they
state that the point estimates of the correlation coefficient and volatility of volatility are
incompatible under the \( P \) and \( Q \) measures. They also show that the joint restrictions on the
returns and volatility dynamics under the \( P \) and \( Q \) measures lead to the poor performance
of the stochastic volatility model, measured by high level of RMSE. Using S&P 500 returns
and futures options data over the period of 1987 through 2003, they find IVRMSE of 1.1%
for the option-based estimation and 8.73% while imposing time-series consistency.

They note that this poor performance of SV model indicates the inability of the SV models to
generate sufficient amounts of conditional skewness and kurtosis. This drawback in standard
SV models is mainly attributed to the fact that the estimated conditional higher moments
are highly correlated with the estimated conditional variance. By contrast, in-sample fit of
our two-factor SV model is significantly improved relative to the Heston SV model. Further,
the wedge between Vega-weighted RMSE of joint estimation and option-based estimation is
reduced significantly, from 7.63% to 1.58%, in our two-factor SV model versus a standard
Heston SV model. The better performance of two-factor SV model is due to the fact that it
can generate stochastic correlation between volatility and stock returns. This feature enables
the two-factor SV model to better capture the conditional skewness and kurtosis of the index
dynamics.

Table (14) provides goodness-of-fit statistics for 27 the firms in our sample, both in-sample
and out-of-sample. Using option data over the period 1996-2011, we find that all the firms
in our sample has a Vega RMSE below 2 except for Cisco and Chevron. We find similar
in-sample performance when the goodness-of-fit is measure by IVRMSE. The average Vega
RMSEs and IVRMSEs across all the firms are 1.61% and 1.59% respectively. The average
relative IVRMSE, measured as the ratio of IVRMSE over the average Black-Scholes IV, is
5.66%. We find that Boeing has the best fit with IVRMSE of 1.35% and Cisco has the worst
fit with IVRMSE of 2.12%; however, the fit is quite similar across the firms. Overall we
conclude that the model provides a reasonably good fit for all 27 firms.

We find that our model has a relatively better in-sample fit compared to the one-factor
structure model. For the firms listed on Dow Jones index, Christoffersen et al. (2017, Table
4) find that the average IVRMSE is 1.66%. Further, comparing goodness-of-fits in our
model with those of one-factor model for the same firms, reported in Christoffersen et al.
(2017, Table A.2), also supports the performance of our model. Overall, the in-sample
performance of our model over the one-factor structure together with its cross-sectional
implications regarding IV term-structure, moneyness slope, and equity option skew support
the importance of transient and persistent factor loadings in pricing equity options.

\[30\] Note that their sample span the period 1996 to 2010.
8 Model Stability and Out-of-Sample Performance

In order to examine the stability of the two-factor SV model of the market index and its out-of-sample performance, we divide the dataset into two subsample periods. The first subsample is from January 1996 through December 2003 and contains 169,800 daily option contracts. The second one is from January 2004 to December 2011 which contains 175,910 daily option contracts. Using both daily returns and option data we filter spot daily persistent variance path and transient variance path and repeat the joint estimation routine within each subsample. Table (10) reports the parameter estimates within each subsample (Panels A and B). For the sake of comparison, Panels C and D also report the parameter estimates from option-based estimation. The main results of the subsample tests are as follows.

First, we find that the PF method is a reliable filtering technique even within shorter sample period of 8 years. We observe that the time series of total spot daily variances under the risk-neutral measure is largely consistent with the time series of the VIX option implied volatility index within each subsample period and also consistent with the filtered spot daily variances over the entire sample.

Second, the parameter estimates within each subsample period is largely inline with those obtained from the entire sample. Moreover, within each subsample period, the joint estimation results is also consistent with option-based parameter estimates. We find that point estimate for the transient mean reversion parameter is higher in the second subsample period while the opposite is true for the persistent mean reversion speed. Overall, the level and the order of parameter estimates are almost consistent within both subsample periods and also across both estimation methods (joint estimation and option-based estimation).  

Third, the correlation coefficients between shocks to variance components and shocks to the market index returns within subsample periods remain consistent with those estimated over the entire sample period and those reported in previous studies in the sense that the magnitude of persistent correlation coefficient is higher than its transient counterpart. Further, persistent and transient variance risk premiums remain negative with the same relative order within two subsample periods, confirming our previous findings that investors are willing to pay to avoid transient and highly mean reverting volatility shocks.

Fourth, we evaluate our model fit within both subsample periods and report Vega RMSEs and IVRMSEs separately for calls and puts and for different maturities. Entries in Table (11) and Table (12) are inline with model fit over the entire sample period, reported in Table (9). Our joint estimation results show a better in-sample fit over the second subsample period as

---

31 Christoffersen et al. (2009, Table 3) report annual risk-neutral parameter estimates for the two-factor SV model over the period 1990 through 2004 using data from S&P 500 index option data. Our option-based subsample parameter estimates are mostly consistent with their average annual result except for the volatility of volatility parameter. Apart from differences in the size of sample, this difference in point estimates may partly be explained by the fact that the annual parameter estimates in Christoffersen et al. (2009) does not satisfy the Feller condition. Feller (1951) shows that a square root process is strictly positive if $2\kappa \theta > \sigma^2$.

32 See Section 6.
Vega RMSEs and IVRMSEs are reduced.

Fifth, in order to measure the out-of-sample performance of the two-factor SV model in capturing the behaviour of S&P 500 index options, we use the parameter estimates form the first subsample (1996-2003). Given the parameter estimates from the first subsample period, we use Particle Filter methods to filter risk-neutral spot daily persistent and transient variance components over the second subsample period and then compute the IVRMSEs and Vega RMSE over the second subsample (2004-2011). Table (13) reports the summary statistics of the out-of-sample performance for different maturities and for calls and puts separately. Comparing out-of-sample entries in (13) with those of in-sample in (12) over the same period supports the stable performance of our two-factor SV model either in joint-estimation or in option-based estimation.

Entries in the last column of Table (14) reports out-of-sample performance of our equity option pricing model. We divide the data set into two subsample periods, using data from 1996 to 2003. We estimate structural parameters for the index model, for every individual equity, and filter daily spot index variance components of the market index and spot idiosyncratic variance for all the firms. In the next step we filter spot idiosyncratic variance for all the firms over the period 2004 to 2011, given spot variance components and structural parameters in the first subsample period. Note that we use an optimization function similar to (28). We find that the model provides good out-of-sample fit. For most of the firms, the out-of-sample Vega RMSEs are consistent with their in-sample Vega RMSEs. Overall, the average Vega RMSE is 1.81% across all 27 firms.\textsuperscript{33}

9 Concluding Remarks

Motivated by the extensive empirical evidence that supports the existence of two volatility components in the dynamics of the market index, we explore how individual equity option prices respond to transient and persistent factor loadings. We adopt a two-factor stochastic volatility model where aggregate market volatility is decomposed into two independent volatility components, a transient component and a persistent component. We extend the model in Christoffersen et al. (2017) and assume that individual equity returns are related to market index returns with two distinct systematic components and an idiosyncratic component, which is stochastic and follows a standard square root process. We derive a closed form pricing equation for individual equity options where equity option prices depend on two constant factor loadings, a transient beta and a persistent beta.

\textsuperscript{33}The out-of-sample performance can also be examined with spot idiosyncratic variance obtained from one-day ahead \((t+1)\) forecast of idiosyncratic variance for individual equity \(i\) given the in-sample structural parameter estimates and time \(t\) spot idiosyncratic variance. One-day ahead \((t+1)\) forecast of idiosyncratic variance may be computed as \(\hat{\xi}_{i,t+1|t} = E_t[\xi_{i,t+1}] = \theta^i + (\xi_{i,t} - \theta^i)(1 - \exp(-\frac{\kappa_{i}t}{252}))\). However, this approach may be more suitable for instance if in-sample fit is based on a Wednesday options and then out-of-sample fit can be examined based on the Thursday options.
Our equity option pricing model sheds some lights on the impact of systematic changes in the market index returns on the equity option prices. We find closed-form expressions for the sensitivity of the equity option prices to the changes in the market index level (market delta) and changes in the persistent and transient variance components (persistent and transient market vega) and show how transient and persistent betas affect the expected returns on individual equity options. Our closed-form pricing equation and proposed factor structure allow a portfolio manager to hedge her portfolio exposure to the level of the market index, and to the persistent and transient variations in the market index.

We show that the proposed two-factor structure has important cross-sectional implications for equity options. Consistent with the findings of Duan and Wei (2009), our model predicts that firms with a higher beta have a higher implied volatility. In particular, we find that firms with higher transient betas have steeper term structures of implied volatility and steeper implied volatility moneyness slopes. We also observe that the variance risk premiums have more significant effect on the implied volatility smile of equity options (equity option skew) when transient beta is higher.

For the firms listed on Dow Jones Index, we estimate structural parameters of the equity dynamics and filter spot idiosyncratic variances, which together characterize the dynamics of the individual equity under the risk-neutral measure. Given the level of IVRMSEs, we find that our model provides a good-fit both in-sample and out-of-sample. We also report the point estimates of transient and persistent betas for 27 firms. In particular, we find that for all the firms in our sample, the transient beta is always greater than the persistent beta, implying that for large capitalization firms listed in the Dow Jones index, transient and larger variations in the market tends to be related to the proportionally larger systematic price reactions across equities than persistent and smaller variations in the market index.

Overall, the in-sample performance of our model over the one-factor structure, its out-of-sample performance, different sensitivities to the transient and persistent market variations, and its cross-sectional implications regarding IV term structure, moneyness slope, and equity option skew support the importance of transient and persistent factor structure in pricing equity options.

To obtain consistent sets of $P$ and $Q$ parameters for the market index dynamics, we extend the variance-dependent pricing kernel in Christoffersen et al. (2013), and introduce an admissible pricing kernel that supports two stochastic volatility factors. We use a long time-series of daily S&P 500 index returns and the entire cross-section of S&P 500 option prices over the same time span. We filter time series of persistent and transient spot variance components and simultaneously estimate a set of structural parameters that characterizes the dynamics of index return and variance components.

In empirical analysis, we show that the proposed decomposition of volatility can be characterized by different sensitivity of the variance components to the volatility shocks and different persistence in variance components. We obtain negative risk premium for both variance components, implying that investors are willing to pay for insurance against in-
creases in volatility risk, even if such increases have little persistence. The negative risk premiums of both variance components are consistent with the findings in equity market where Adrian and Rosenberg (2008) find that short-run and long-run variance components are priced factors with negative risk premium. We also obtain negative correlations between shocks to the index returns and shocks to the transient and persistent variance components. In particular, we observe that the persistent correlation coefficient has more significant effect on the dynamics of index skewness.

Our market index model also provides good fit to observed option prices both in- and out-of-sample, measured by Vega-weighted root mean squared option pricing errors and implied volatility root mean squared errors. In particular, we find that unlike stochastic volatility model, joint restrictions on return and variance dynamics under $P$ and $Q$ measures does not lead to the poor performance of our two-factor SV model.
Appendix

A  Proof of Proposition 1

We impose the condition that the product of the price of any traded asset and the pricing kernel under physical measure is a martingale. We also impose the condition that the discounted price of any traded asset under risk-neutral measure is also a martingale. We show that the two-factor stochastic volatility process under physical measure in (1) are linked to its risk-neutral counterpart in (4) by the unique arbitrage free pricing kernel introduced in (6) and deduce restrictions on the time-preference parameters, \( \{\delta, \eta_1, \eta_2\} \), risk-aversion (equity aversion) parameter, \( \phi \), and variance preference parameters (variance aversion), \( \{\zeta_1, \zeta_2\} \). We close this proof by showing how physical Wiener processes \( \{z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}\} \) are linked to risk-neutral Wiener processes \( \{\tilde{z}_{1,t}, \tilde{z}_{2,t}, \tilde{w}_{1,t}, \tilde{w}_{2,t}\} \) by equity premium \( \{\mu_1, \mu_2\} \) and variance premium \( \{\lambda_1, \lambda_2\} \) parameters.

Consider that index return under physical and risk-neutral measures follows the dynamics (A.1) and (A.2).

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
v_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1 \sqrt{v_{1,t}}(\rho_1 dz_{1,t} + \sqrt{1 - \rho_1^2} dB_{1,t}) \\
v_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2 \sqrt{v_{2,t}}(\rho_2 dz_{2,t} + \sqrt{1 - \rho_2^2} dB_{2,t}) \\
\frac{dS_t}{S_t} &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\
v_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1 \sqrt{v_{1,t}}(\rho_1 d\tilde{z}_{1,t} + \sqrt{1 - \rho_1^2} d\tilde{B}_{1,t}) \\
v_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2 \sqrt{v_{2,t}}(\rho_2 d\tilde{z}_{2,t} + \sqrt{1 - \rho_2^2} d\tilde{B}_{2,t})
\end{align*}
\]  

(A.1)  

(A.2)

Then, following Christoffersen et al. (2013), we show that the pricing kernel links the physical and risk-neutral measures has the following exponential affine form.

\[
\frac{M_t}{M_0} = \left(\frac{S_t}{S_0}\right)^\phi \exp \left[ \delta t + \eta_1 \int_0^t v_{1,s} ds + \eta_2 \int_0^t v_{2,s} ds + \zeta_1(v_{1,t} - v_{1,0}) + \zeta_2(v_{2,t} - v_{2,0}) \right] 
\]  

(A.3)

Note that in the sprite of Cox et al. (1985) and Heston (1993) we assume that the market price of each variance risk factor is proportional to spot variance. Therefore, the risk-neutral process in (A.2) can be defined as follows.
Replacing (A.5) and (A.1) into (A.6) we have:

\[\begin{align*}
\frac{dS_t}{S_t} &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\
\frac{dv_{1,t}}{v_{1,t}} &= (\kappa_1(\theta_1 - v_{1,t}) - \lambda_1 v_{1,t})dt + \sigma_1 \sqrt{v_{1,t}}d\tilde{w}_{1,t} \\
\frac{dv_{2,t}}{v_{2,t}} &= (\kappa_2(\theta_2 - v_{2,t}) - \lambda_2 v_{2,t})dt + \sigma_2 \sqrt{v_{2,t}}d\tilde{w}_{2,t}
\end{align*}\]  

(A.6)

The log stock price process under physical measure and log pricing kernel process have the following dynamics respectively.

\[\begin{align*}
d(\log(S_t)) &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
\end{align*}\]  

(A.5)

\[\begin{align*}
d(\log(M_t)) &= \phi \cdot d(\log(S_t)) + (\delta + \eta_1 v_{1,t} + \eta_2 v_{2,t})dt + \zeta_1 dv_{1,t} + \zeta_2 dv_{2,t} \\
\end{align*}\]  

(A.6)

Replacing (A.5) and (A.1) into (A.6) we have:

\[\begin{align*}
d(\log(M_t)) &= \left[\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \\
&\quad + \zeta_1 \kappa_1(\theta_1 - v_{1,t}) + \zeta_2 \kappa_2(\theta_2 - v_{2,t})\right]dt \\
&\quad + \left[\phi \sqrt{v_{1,t}} + \zeta_1 \rho_1 \sigma_1 \sqrt{v_{1,t}}\right]dz_{1,t} + \left[\phi \sqrt{v_{2,t}} + \zeta_2 \rho_2 \sigma_2 \sqrt{v_{2,t}}\right]dz_{2,t} \\
&\quad + \left[\zeta_1 \sigma_1 \sqrt{v_{1,t}} \sqrt{1 - \rho_1^2}\right]dB_{1,t} + \left[\zeta_2 \sigma_2 \sqrt{v_{2,t}} \sqrt{1 - \rho_2^2}\right]dB_{2,t} \\
\end{align*}\]  

(A.7)

As \(dM_t/M_t = d(\log(M_t)) + \frac{1}{2} [d(\log(M_t))]^2\) we have

\[\begin{align*}
dM_t/M_t &= \left[\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \\
&\quad + \zeta_1 \kappa_1(\theta_1 - v_{1,t}) + \zeta_2 \kappa_2(\theta_2 - v_{2,t}) + \frac{1}{2} \phi^2(v_{1,t} + v_{2,t}) \\
&\quad + \phi(\zeta_1 \rho_1 \sigma_1 v_{1,t} + \zeta_2 \rho_2 \sigma_2 v_{2,t}) + \frac{1}{2} \zeta_1^2 \sigma_1^2 v_{1,t} + \frac{1}{2} \zeta_2^2 \sigma_2^2 v_{2,t}\right]dt \\
&\quad + \left[\phi \sqrt{v_{1,t}} + \zeta_1 \rho_1 \sigma_1 \sqrt{v_{1,t}}\right]dz_{1,t} + \left[\phi \sqrt{v_{2,t}} + \zeta_2 \rho_2 \sigma_2 \sqrt{v_{2,t}}\right]dz_{2,t} \\
&\quad + \left[\zeta_1 \sigma_1 \sqrt{v_{1,t}} \sqrt{1 - \rho_1^2}\right]dB_{1,t} + \left[\zeta_2 \sigma_2 \sqrt{v_{2,t}} \sqrt{1 - \rho_2^2}\right]dB_{2,t} \\
\end{align*}\]  

(A.8)

The first restriction on the pricing kernel is that the product of the money market account, \(B_t = B_0 \exp(rt)\), and the pricing kernel, \(M_t\), should be a martingale under physical measure. Therefore, \(E[d(B_t \cdot M_t)] = 0\) or \(E[dM_t/M_t] = -rdt\).
\[ \begin{align*}
\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2} v_{1,t}^2 - \frac{1}{2} v_{2,t}^2) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} + \zeta_1 \kappa_1 (\theta_1 - v_{1,t}) + \zeta_2 \kappa_2 (\theta_2 - v_{2,t}) \\
+ \frac{1}{2} \phi^2 (v_{1,t} + v_{2,t}) + \phi (\zeta_1 \rho_1 v_{1,t} + \zeta_2 \rho_2 v_{2,t}) + \frac{1}{2} \zeta_1^2 \sigma_1^2 v_{1,t}^2 + \frac{1}{2} \zeta_2^2 \sigma_2^2 v_{2,t}^2 \right] dt = -rdt
\end{align*} \]

(A.9)

As (A.9) holds for \( v_{1,t} = v_{2,t} = 0 \),

\[ \delta = -r(\phi + 1) - \zeta_1 \kappa_1 \theta_1 - \zeta_2 \kappa_2 \theta_2. \]  

(A.10)

(A.9) also holds for \( v_{1,t} = v_{2,t} = \infty \).

\[ \begin{align*}
\eta_1 &= -\phi \mu_1 + 1/2 \phi + \zeta_1 \kappa_1 - 1/2 (\phi^2 + \zeta_1^2 \sigma_1^2 + 2 \phi \zeta_1 \sigma_1 \rho_1) \\
\eta_2 &= -\phi \mu_2 + 1/2 \phi + \zeta_2 \kappa_2 - 1/2 (\phi^2 + \zeta_2^2 \sigma_2^2 + 2 \phi \zeta_2 \sigma_2 \rho_2)
\end{align*} \]

(A.11)

The second restriction on the pricing kernel is based on the fact that \([S_t \cdot M_t] \) is also a martingale under physical measure. Therefore, \( E[d(S_t \cdot M_t)] = 0 \). As a result of this restriction we have

\[ \begin{align*}
v_{1,t} (\mu_1 + \phi + \zeta_1 \sigma_1 \rho_1) + v_{2,t} (\mu_2 + \phi + \zeta_2 \sigma_2 \rho_2) = 0,
\phi = -\frac{1}{v_{1,t} + v_{2,t}} [(\mu_1 + \zeta_1 \sigma_1 \rho_1) v_{1,t} + (\mu_2 + \zeta_2 \sigma_2 \rho_2) v_{2,t}].
\end{align*} \]

(A.12)

If we impose the restriction that \( \mu_1 + \zeta_1 \sigma_1 \rho_1 \equiv \mu_2 + \zeta_2 \sigma_2 \rho_2 \), then (A.12) can be simplified as follows.

\[ \phi = -(\mu_1 + \zeta_1 \sigma_1 \rho_1) = -(\mu_2 + \zeta_2 \sigma_2 \rho_2) \]

(A.13)

We impose the third restriction on pricing kernel so that for any asset \( U \equiv U(S,v_1,v_2,t) \), \([U(t) \cdot M_t] \) is also a martingale under \( P \)-distribution. Therefore, \( E[d(U \cdot M_t)] = E[dU.M_t + U.dM_t + dU.dM_t] = 0 \). Replacing \( M_t \) and \( dM_t \) into this equation we have the following restriction where \( U_S = \partial U(S,v_1,v_2,t)/\partial S \), \( U_{v_1} = \partial U(S,v_1,v_2,t)/\partial v_1 \), and \( U_{v_2} = \partial U(S,v_1,v_2,t)/\partial v_2 \).
\[-rU + U_t + U_S(r + \mu_1v_{1,t} + \mu_2v_{2,t})S + U_{v_{1,t}}\kappa_1(\theta_1 - v_{1,t}) + U_{v_{2,t}}\kappa_2(\theta_2 - v_{2,t}) + \frac{1}{2}U_{SS}(v_{1,t} + v_{2,t}) + \left(U_{v_{1,t}}^2\sigma_1^2 + \frac{1}{2}U_{v_{2,t}}\sigma_2^2 + U_{v_{1,t}}\rho_1v_{1,t} + U_{v_{2,t}}\rho_2v_{2,t}ight)
+ (U_S\sqrt{v_{1,t}} + U_{v_{1,t}}\rho_1\sqrt{v_{1,t}})(\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sqrt{v_{1,t}})
+ (U_S\sqrt{v_{2,t}} + U_{v_{2,t}}\rho_2\sqrt{v_{2,t}})(\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sqrt{v_{2,t}})
+ U_{v_{1,t}}\rho_2(1 - \rho_1^2) + U_{v_{2,t}}\rho_2(1 - \rho_2^2) = 0\] (A.14)

The last restriction is based on the fact that discounted price process should be a martingale under risk-neutral measure. Therefore, for any asset, \(U(S, v_{1}, v_{2}, t)\), whose payoff depends on the state variables \(\{S, v_{1}, v_{2}\}\), \(U/B_t\) is a \(Q\)-martingale. This restriction implies that \(E^Q[d(U/B_t)] = 0\) or equivalently \(E^Q[d(U(S, v_{1}, v_{2}, t))] = rU(S, v_{1}, v_{2}, t)\).

\[U_t + rSU_S + U_{v_{1,t}}(\kappa_1(\theta_1 - v_{1,t}) - \lambda_1v_{1,t}) + U_{v_{2,t}}(\kappa_1(\theta_1 - v_{1,t}) - \lambda_2v_{2,t}) + \frac{1}{2}U_{SS}(v_{1,t} + v_{1,t})
+ \frac{1}{2}U_{v_{1,t}}\sigma_1^2 + \frac{1}{2}U_{v_{2,t}}\sigma_2^2 + U_{v_{1,t}}\rho_1v_{1,t} + U_{v_{2,t}}\rho_2v_{2,t} = rU.\] (A.15)

Replace (A.15) from the last restriction into (A.14) from the third restriction.

\[U_S(\mu_1v_{1,t} + \mu_2v_{2,t})S + U_{v_{1,t}}\lambda_1v_{1,t} + U_{v_{2,t}}\lambda_2v_{2,t}
+ (U_S\sqrt{v_{1,t}} + U_{v_{1,t}}\rho_1\sqrt{v_{1,t}})(\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sqrt{v_{1,t}})
+ (U_S\sqrt{v_{2,t}} + U_{v_{2,t}}\rho_2\sqrt{v_{2,t}})(\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sqrt{v_{2,t}})
+ U_{v_{1,t}}\zeta_1(1 - \rho_1^2) + U_{v_{2,t}}\zeta_2(1 - \rho_2^2) = 0\]

\[U_S(\mu_1v_{1,t} + \mu_2v_{2,t})S + U_{v_{1,t}}\lambda_1v_{1,t} + U_{v_{2,t}}\lambda_2v_{2,t}
+ U_S\phi v_{1,t} + U_S\zeta_1\rho_1v_{1,t} + U_{v_{1,t}}\rho_1\phi v_{1,t} + U_{v_{1,t}}\zeta_1v_{1,t}
+ U_S\phi v_{2,t} + U_S\zeta_2\rho_2v_{2,t} + U_{v_{2,t}}\rho_2\phi v_{2,t} + U_{v_{2,t}}\zeta_2v_{2,t} = 0\] (A.16)

From the second restriction in (A.12) we know that \(\mu_1v_{1,t} + \mu_2v_{2,t} = -\phi v_{1,t} - \zeta_1\rho_1v_{1,t} - \phi v_{2,t} - \zeta_2\rho_2v_{2,t}\). Therefore, we can further simplify (A.16).

\[U_{v_{1,t}}(\rho_1\phi + \lambda_1 + \zeta_1v_{1,t}) + U_{v_{2,t}}(\rho_2\phi + \lambda_2 + \zeta_2v_{2,t}) = 0\] (A.17)

One admissible solution for (A.17) would be:
\[ \rho_1 \sigma_1 \phi + \lambda_1 + \zeta_1 \sigma_1^2 = 0 \]  
\[ \rho_2 \sigma_2 \phi + \lambda_2 + \zeta_2 \sigma_2^2 = 0 \]  
(A.18)

If we combine restrictions in (A.18) with those introduced in (A.13) and replace them back into (A.13) we have \( \phi, \zeta_1, \) and \( \zeta_2. \)

\[ \zeta_1 = \frac{\rho_1 \sigma_1 \mu_1 - \lambda_1}{\sigma_1^2 (1 - \rho_1^2)} \]  
\[ \zeta_2 = \frac{\rho_2 \sigma_2 \mu_2 - \lambda_2}{\sigma_2^2 (1 - \rho_2^2)} \]  
(A.19)

\[ \phi = -\mu_1 - \frac{\rho_1^2 \sigma_1^2 \mu_1 - \lambda_1 \rho_1 \sigma_1}{\sigma_1^2 (1 - \rho_1^2)} = -\mu_2 - \frac{\rho_2^2 \sigma_2^2 \mu_2 - \lambda_2 \rho_2 \sigma_2}{\sigma_2^2 (1 - \rho_2^2)} \]  
(A.20)

Therefore, an admissible pricing kernel linking the \( P \) and \( Q \) dynamics in (A.1) and (A.2) is as follows.

\[ \frac{dM_t}{M_t} = -r dt - \mu_1 \sqrt{v_{1,t}} dz_{1,t} - \mu_2 \sqrt{v_{2,t}} dz_{2,t} + \frac{\rho_1 \sigma_1 \mu_1 - \lambda_1}{\sigma_1^2 (1 - \rho_1^2)} dB_{1,t} + \frac{\rho_2 \sigma_2 \mu_2 - \lambda_2}{\sigma_2^2 (1 - \rho_2^2)} dB_{2,t} \]  
(A.21)

This is the pricing kernel introduced in (1).

Now, we show that how physical shocks are linked to risk-neutral shocks through equity premium \( \{\mu_1, \mu_2\} \) and variance premium \( \{\lambda_1, \lambda_2\} \) parameters.

\[ d\tilde{z}_{1,t} = dz_{1,t} + (\psi_{1,t} + \rho_1 \psi_{3,t}) dt \]  
\[ d\tilde{z}_{2,t} = dz_{2,t} + (\psi_{2,t} + \rho_2 \psi_{4,t}) dt \]  
\[ d\tilde{w}_{1,t} = dw_{1,t} + (\psi_{3,t} + \rho_1 \psi_{1,t}) dt \]  
\[ d\tilde{w}_{2,t} = dw_{2,t} + (\psi_{4,t} + \rho_2 \psi_{2,t}) dt \]  
(A.22)

Replace physical shocks in return dynamics (1) by risk-neutral shocks introduced in (A.22).

\[ dS_t/S_t = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) dt \]  
\[ + \sqrt{v_{1,t}} d\tilde{z}_{1,t} - (\psi_{1,t} + \rho_1 \psi_{3,t}) \sqrt{v_{1,t}} dt + \sqrt{v_{2,t}} d\tilde{z}_{2,t} - (\psi_{2,t} + \rho_2 \psi_{4,t}) \sqrt{v_{2,t}} dt \]  
(A.23)
As a result of risk neutralization in (A.23), the expected stock returns in (A.23) should be equal to the risk free rate of returns. Therefore, we have the following restriction.

\[
(\mu_1 v_{1,t} + \mu_2 v_{2,t})dt = (\psi_{1,t} + \rho_1 \psi_{3,t})\sqrt{v_{1,t}}dt + (\psi_{2,t} + \rho_2 \psi_{4,t})\sqrt{v_{2,t}}dt
\]  
(A.24)

One possible solution of (A.24) is as follows.

\[
\mu_1 \sqrt{v_{1,t}} = \psi_{1,t} + \rho_1 \psi_{3,t}
\]  
(A.25)

\[
\mu_2 \sqrt{v_{2,t}} = \psi_{2,t} + \rho_2 \psi_{4,t}
\]  
(A.25)

Similarly, we replace the proposed transformation in (A.22) into the dynamics of volatilities in (1).

\[
dv_{1,t} = \kappa_1 (\theta_1 - v_{1,t})dt + \sigma_1 \sqrt{v_{1,t}}d\tilde{w}_{1,t} - \sigma_1 \sqrt{v_{1,t}}(\psi_{3,t} + \rho_1 \psi_{1,t})dt
\]
\[
dv_{2,t} = \kappa_2 (\theta_2 - v_{2,t})dt + \sigma_2 \sqrt{v_{2,t}}d\tilde{w}_{2,t} - \sigma_2 \sqrt{v_{2,t}}(\psi_{4,t} + \rho_2 \psi_{2,t})dt
\]  
(A.26)

The risk-neutral variance dynamics in (A.26) should be equivalent to those in (A.4), where the market price of variance risk factors is proportional to spot variance. Therefore, we have following restrictions:

\[
\sigma_1 \sqrt{v_{1,t}}(\psi_{3,t} + \rho_1 \psi_{1,t}) = \lambda_1 v_{1,t}
\]
\[
\sigma_2 \sqrt{v_{2,t}}(\psi_{4,t} + \rho_2 \psi_{2,t}) = \lambda_2 v_{2,t}
\]  
(A.27)

Combining the restrictions in (A.25) and (A.27), we have the following results, which link the physical distribution (1) to the risk-neutral distribution (4).

\[
\psi_{1,t} = \frac{\sigma_1 \mu_1 - \rho_1 \lambda_1}{\sigma_1 (1 - \rho_1^2)} \sqrt{v_{1,t}}
\]
\[
\psi_{2,t} = \frac{\sigma_2 \mu_2 - \rho_2 \lambda_2}{\sigma_2 (1 - \rho_2^2)} \sqrt{v_{2,t}}
\]
\[
\psi_{3,t} = \frac{\lambda_1 - \rho_1 \sigma_1 \mu_1}{\sigma_1 (1 - \rho_1^2)} \sqrt{v_{1,t}}
\]
\[
\psi_{4,t} = \frac{\lambda_2 - \rho_2 \sigma_2 \mu_2}{\sigma_2 (1 - \rho_2^2)} \sqrt{v_{2,t}}
\]  
(A.28)
B Proof of Proposition 2

We transform the physical dynamics of individual equity returns (B.1) to its risk-neutral counterparts (B.2) by assuming an appropriate stochastic discount factor (SDF).

\[
\begin{align*}
\frac{dS_t^i}{S_t^i} &= \mu^i dt + \beta_1^i (\mu_1 v_{1,t} dt + \sqrt{\nu_1^i} dz_{1,t}) + \beta_2^i (\mu_2 v_{2,t} dt + \sqrt{\nu_2^i} dz_{2,t}) + \sqrt{\xi_t^i} d\xi_t^i \\
\frac{d\xi_t^i}{\xi_t^i} &= \kappa^i (\theta^i - \xi_t^i) dt + \sigma^i \sqrt{\xi_t^i} dw_t^i \quad (B.1)
\end{align*}
\]

\[
\begin{align*}
\frac{dS_t^i}{S_t^i} &= r dt + \beta_1^i \sqrt{\nu_1^i} dz_{1,t} + \beta_2^i \sqrt{\nu_2^i} dz_{2,t} + \sqrt{\xi_t^i} d\xi_t^i \\
\frac{d\xi_t^i}{\xi_t^i} &= \kappa^i (\theta^i - \xi_t^i) dt + \sigma^i \sqrt{\xi_t^i} dw_t^i \quad (B.2)
\end{align*}
\]

where

\[
\begin{align*}
\langle dz_t^i, dw_t^j \rangle &= \rho_{ij} dt \\
\langle dz_t^i, dw_t^i \rangle &= 0 \quad \forall (i \neq j) \quad (B.3)
\end{align*}
\]

As individual equity returns are linked to the market index returns with a two-factor model and two constant factor loadings $\beta_1$ and $\beta_2$, the proposed SDF should jointly specify the risk-neutral distributions of the market index and individual equity returns. Remember that the dynamics of market index returns under the $P$- and $Q$-measure are as follows.

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) dt + \sqrt{\nu_1} dz_{1,t} + \sqrt{\nu_2} dz_{2,t} \\
dv_{1,t} &= \kappa_1 (\theta_1 - v_{1,t}) dt + \sigma_1 \sqrt{\nu_1} (\rho_1 dz_{1,t} + \sqrt{1 - \rho_1^2} dB_{1,t}) \quad (B.4) \\
dv_{2,t} &= \kappa_2 (\theta_2 - v_{2,t}) dt + \sigma_2 \sqrt{\nu_2} (\rho_2 dz_{2,t} + \sqrt{1 - \rho_2^2} dB_{2,t}) \\
\frac{dS_t}{S_t} &= rd_t + \sqrt{\nu_1} dz_{1,t} + \sqrt{\nu_2} dz_{2,t} \\
dv_{1,t} &= \tilde{\kappa}_1 (\tilde{\theta}_1 - v_{1,t}) dt + \sigma_1 \sqrt{\nu_1} (\rho_1 dz_{1,t} + \sqrt{1 - \rho_1^2} d\tilde{B}_{1,t}) \quad (B.5) \\
dv_{2,t} &= \tilde{\kappa}_2 (\tilde{\theta}_2 - v_{2,t}) dt + \sigma_2 \sqrt{\nu_2} (\rho_2 dz_{2,t} + \sqrt{1 - \rho_2^2} d\tilde{B}_{2,t}) \\
\end{align*}
\]

where

\[
\begin{align*}
\langle dv_{1,t}, dz_{1,t} \rangle &= \rho_1 dt, -1 \leq \rho_1 \leq +1 \\
\langle dv_{2,t}, dz_{2,t} \rangle &= \rho_2 dt, -1 \leq \rho_2 \leq +1 \\
\langle dv_{1,t}, dw_{2,t} \rangle &= 0 \\
\rho_1^2 + \rho_2^2 &\leq +1 \quad (B.6)
\end{align*}
\]

We assume the following standard SDF.
\[
\frac{dM_t}{M_t} = -rdt - \psi'_t dW_t, \quad (B.7)
\]

where \( \psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}, \psi_{i,1,t}, \psi_{i,2,t}] \) with \( i = \{1, 2, \cdots, n\} \) is the vector of market price of risk factors and \( W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}, z_{i,t}, w_{i,t}] \) with \( i = \{1, 2, \cdots, n\} \) is the vector of innovations in market return, market variance components, equity \( i \) return, and equity \( i \) idiosyncratic variance. Given the SDF in (B.7), the change-of-measure from \( P^- \) to \( Q^- \)-distribution has the following exponential form.

\[
\frac{dQ}{dP}(t) \equiv M_t \exp(rt) = \exp [ - \int_0^t \psi'_u dW_u - \frac{1}{2} \int_0^t \psi'_u d\langle W, W' \rangle_u \psi_u ] \quad (B.8)
\]

where \( \langle W, W' \rangle \) is the covariance operator.

We follow the notion of Doléans-Dade exponential (stochastic exponential) and define the stochastic exponential \( \varepsilon(\cdot) \) as follow.

\[
\varepsilon \left( \int_0^t \vartheta'_u dW_u \right) \equiv \exp \left[ \int_0^t \vartheta'_u dW_u - \frac{1}{2} \int_0^t \vartheta'_u d\langle W, W' \rangle_u \vartheta_u \right] \quad (B.9)
\]

Therefore, the change-of-measure (B.8) can be expressed in term of stochastic exponential as

\[
\frac{dQ}{dP}(t) = \varepsilon \left( \int_0^t -\psi'_u dW_u \right) \quad (B.10)
\]

Applying Ito's lemma, for every individual equity \( i \), we have the following dynamic under the physical measure.

\[
\log \left( \frac{S^i_t}{S^i_0} \right) = [\mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t} - \frac{1}{2} (\beta_1^i)^2 v_{1,t}^2 - \frac{1}{2} (\beta_2^i)^2 v_{2,t}^2 - \frac{1}{2} \xi^i_t] t
\]

\[
+ \beta_1^i \int_0^t \sqrt{v_{1,u}} dz_{1,u} + \beta_2^i \int_0^t \sqrt{v_{2,u}} dz_{2,u} + \int_0^t \sqrt{\xi^i_u} dz^i_u \quad (B.11)
\]

Given (B.11) and definition of stochastic exponential (B.9) we have

\[
\frac{S^i_t}{S^i_0} = \exp \left[ (\mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t}) t \right] \varepsilon \left( \int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} + \int_0^t \beta_2^i \sqrt{v_{2,u}} dz_{2,u} + \int_0^t \sqrt{\xi^i_u} dz^i_u \right) \quad (B.12)
\]

Note that
that we know that \( \varepsilon \left( \int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} \right) = \exp \left[ \int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} - \frac{1}{2} \int_0^t (\beta_1^i)^2 v_{1,u} du \right] \) \( \text{(B.13)} \)

To find the market prices of risk we impose the restriction that the product of the price of any individual equity and the pricing kernel under physical measure is a \( P \)-martingale. Given the change-of-measure \( \text{(B.10)} \), for every individual equity \( i \), the following process \( N(t) \) should be a \( P \)-martingale.

\[
N(t) \equiv \frac{S_t^i}{S_0^i} dQ(t) \exp (-rt)
\]

where

\[
N(t) = \exp \left[ (-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t \right]
\]

\[
\varepsilon \left( \int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} \right) \varepsilon \left( - \int_0^t \psi_{1,u} dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u} \right) = \exp \left( \int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} - \frac{1}{2} \int_0^t (\beta_1^i)^2 v_{1,u} du \right) \quad \text{(B.15)}
\]

We decompose \( N(t) \) into two orthogonal components \( N(t) \equiv I(t)L(t) \) and then make sure that \( I(t) \) and \( L(t) \) are a \( P \)-martingale.

\[
I(t) = \exp \left[ (-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t \right]
\]

\[
\varepsilon \left( \int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} \right) \varepsilon \left( - \int_0^t \psi_{1,u} dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u} \right) = \exp \left( \int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u} - \frac{1}{2} \int_0^t (\beta_1^i)^2 v_{1,u} du \right) \quad \text{(B.16)}
\]

\[
L(t) = \varepsilon \left( - \sum_{j \neq i} \int_0^t \psi_{1,u}^j dz_{1,u} - \sum_{j \neq i} \int_0^t \psi_{2,u}^j dw_{1,u} \right)
\]

\( \text{(B.17)} \)

From the definition of a stochastic exponential we know that \( \varepsilon(\cdot) \) are \( P \)-martingales and so does \( L(t) \). Therefore, we only need to make sure that \( I(t) \) is also a \( P \)-martingale. Using
the properties of a stochastic exponential $\varepsilon(\cdot), \varepsilon(X_t)\varepsilon(Y_t) = \varepsilon(X_t + Y_t)\exp((X, Y)_t)$ and the correlation structure (B.3) and (B.6) we can rewrite the process of $I(t)$ as follows.

\[
I(t) = \exp\left[(-r + \mu^t + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right]
\]

\[
\varepsilon\left(\int_0^t (\beta_1^i \sqrt{v_{1,u}} - \psi_{1,u})dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\right) \exp\left[\int_0^t \beta_1^i \sqrt{v_{1,u}} (\psi_{1,u} + \rho_1 \psi_{3,u})du\right]
\]

\[
\varepsilon\left(\int_0^t (\beta_2^i \sqrt{v_{2,u}} - \psi_{2,u})dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\right) \exp\left[\int_0^t \beta_2^i \sqrt{v_{2,u}} (\psi_{2,u} + \rho_2 \psi_{4,u})du\right]
\]

\[
\varepsilon\left(\int_0^t (\sqrt{\xi^t_{i,u}} - \psi_{1,u})dz_{1,u} - \int_0^t \psi_{2,u} dw_{1,u}\right) \exp\left[\int_0^t \sqrt{\xi^t_{i,u}} (\psi_{1,u} + \rho_1 \psi_{2,u})du\right]
\]

(B.18)

Thus, given $\varepsilon(\cdot)$ are $P$-martingales, the process $I(t)$ is a $P$-martingale when the following restriction holds.

\[
\exp\left[(-r + \mu^t + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right]
\]

\[
\exp\left[-\int_0^t \beta_1^i \sqrt{v_{1,u}} (\psi_{1,u} + \rho_1 \psi_{3,u})du\right] \exp\left[-\int_0^t \beta_2^i \sqrt{v_{2,u}} (\psi_{2,u} + \rho_2 \psi_{4,u})du\right]
\]

\[
\exp\left[-\int_0^t \sqrt{\xi^t_{i,u}} (\psi_{1,u} + \rho_1 \psi_{2,u})du\right] = 1
\]

(B.19)

The restriction (B.19) holds if the following conditions for the market index, (B.20), and for every individual equity $i$, (B.21), hold.

\[
\mu_1 v_{1,t} t - \sqrt{v_{1,t}} (\psi_{1,t} + \rho_1 \psi_{3,t}) t = 0
\]

\[
\mu_2 v_{2,t} t - \sqrt{v_{2,t}} (\psi_{3,t} + \rho_2 \psi_{4,t}) t = 0
\]

\[
-r t + \mu^t t - \sqrt{\xi^t_{i}} (\psi_{1,t} + \rho_1 \psi_{2,t}) t = 0
\]

(B.20)

(B.21)

To fully specify the market prices of risk we assume that market price of variance risk factors are proportional to spot volatility components, following Heston (1993).

\[
(\psi_{3,t} + \rho_1 \psi_{1,t}) = \frac{v_{1,t}}{\sigma_1 \sqrt{v_{1,t}}} \lambda_1
\]

\[
(\psi_{4,t} + \rho_2 \psi_{2,t}) = \frac{v_{2,t}}{\sigma_2 \sqrt{v_{2,t}}} \lambda_2
\]

(B.22)

If we assume that the idiosyncratic variance is also a priced risk factor, then its price is also proportional to the spot idiosyncratic volatility for every individual equity $i$. Otherwise, $\lambda^i = 0$. 41
\[(\psi_{2,t}^i + \rho^i \psi_{1,t}^i) = \frac{\xi_i^i}{\sigma^i \sqrt{\xi_i^i}} \lambda^i \] (B.23)

Combining the restrictions in (B.20) and (B.22), we have the following market price of risk factors.

\[
\begin{align*}
\psi_{1,t} &= \frac{\sigma_1 \mu_1 - \rho_1 \lambda_1 \sqrt{\nu_{1,t}}}{(1 - \rho_1^2) \sigma_1} \\
\psi_{2,t} &= \frac{\sigma_2 \mu_2 - \rho_2 \lambda_2 \sqrt{\nu_{2,t}}}{(1 - \rho_2^2) \sigma_2} \\
\psi_{3,t} &= \frac{\lambda_1 - \rho_1 \sigma_1 \mu_1 \sqrt{\nu_{1,t}}}{(1 - \rho_1^2) \sigma_1} \\
\psi_{4,t} &= \frac{\lambda_2 - \rho_2 \sigma_2 \mu_2 \sqrt{\nu_{2,t}}}{(1 - \rho_2^2) \sigma_2}
\end{align*}
\] (B.24)

Combining the restrictions in (B.21) and (B.23) and given that idiosyncratic variance is not priced, we have the following results for every individual equity.

\[
\begin{align*}
\psi_{1,t}^i &= \mu_i - r \sqrt{\xi_i^i (1 - (\rho^i)^2)} \\
\psi_{2,t}^i &= (\mu_i^i - r) + \psi_{1,t}^i (1 - \rho^i) \\
\psi_{3,t}^i &= (\mu_i^i - r) + \psi_{1,t}^i (1 - \rho^i) \xi_i^i \lambda_i^i \sigma_i^i \\
\psi_{4,t}^i &= \psi_{2,t}^i (1 - \rho^i)
\end{align*}
\] (B.25)

Given the market prices of risk factors (B.24) (B.25), we apply the Girsanov’s theorem to transform physical innovations of the market index dynamics (B.4) and individual equity dynamics (B.1) to their risk-neutral counterparts in (B.5) and (B.2). Note that we assume idiosyncratic variance is not priced and thus \(\lambda^i = 0\).

\[
\begin{align*}
&d\tilde{z}_t^i = dz_t^i + \psi_{1,t}^i dt + \rho^i \psi_{2,t}^i dt \\
&d\tilde{z}_{1,t} = dz_{1,t} + \psi_{1,t} dt + \rho_1 \psi_{3,t} dt \\
&d\tilde{z}_{2,t} = dz_{2,t} + \psi_{2,t} dt + \rho_2 \psi_{4,t} dt \\
&d\tilde{w}_{1,t} = dw_{1,t} + \psi_{3,t} dt + \rho_1 \psi_{1,t} dt \\
&d\tilde{w}_{2,t} = dw_{2,t} + \psi_{4,t} dt + \rho_2 \psi_{2,t} dt
\end{align*}
\] (B.26)

With some algebra we have the following transformations.
Replacing $dz^i_t, dw^i_t, dz_{1,t}, dz_{2,t}, dw_{1,t}, dw_{2,t}$ from (B.27) into the physical dynamics in (B.1) and (B.4) and knowing that $\tilde{\kappa}_1 = \kappa_1 + \lambda_1, \tilde{\kappa}_2 = \kappa_2 + \lambda_2, \tilde{\theta}_1 = \frac{k_1 \theta_1}{\kappa_1 + \lambda_1}, \tilde{\theta}_2 = \frac{k_2 \theta_2}{\kappa_2 + \lambda_2}$ we obtain risk-neutral return and variance components dynamics.

\begin{align}
    dS^i_t/S^i_t &= \mu^i dt + \beta^i_1 (\mu_1 v_{1,t} dt + \sqrt{\bar{v}_{1,t}} dz_{1,t}) + \beta^i_2 (\mu_2 v_{2,t} dt + \sqrt{\bar{v}_{2,t}} dz_{2,t}) + \sqrt{\xi^i_t} dz^i_t \\
    &= \mu^i dt + \beta^i_1 (\mu_1 v_{1,t} dt + \sqrt{\bar{v}_{1,t}} (dz_{1,t} - \mu_1 \sqrt{\bar{v}_{1,t}} dt)) \\
    &\quad + \beta^i_2 (\mu_2 v_{2,t} dt + \sqrt{\bar{v}_{2,t}} (dz_{2,t} - \mu_2 \sqrt{\bar{v}_{2,t}} dt)) + \sqrt{\xi^i_t} (dz^i_t - (\mu^i - r) dt/\sqrt{\xi^i_t}) \\
    &= rd^t + \beta^i_1 \sqrt{\bar{v}_{1,t}} dz_{1,t} + \beta^i_2 \sqrt{\bar{v}_{2,t}} dz_{2,t} + \sqrt{\xi^i_t} dz^i_t \\
    \end{align}

\begin{align}
    dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) dt + \sqrt{\bar{v}_{1,t}} dz_{1,t} + \sqrt{\bar{v}_{2,t}} dz_{2,t} \\
    &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) dt + \sqrt{\bar{v}_{1,t}} (dz_{1,t} - \mu_1 \sqrt{\bar{v}_{1,t}} dt) + \sqrt{\bar{v}_{2,t}} (dz_{2,t} - \mu_2 \sqrt{\bar{v}_{2,t}} dt) \\
    &= rd^t + \sqrt{\bar{v}_{1,t}} dz_{1,t} + \sqrt{\bar{v}_{2,t}} dz_{2,t} \\
    \end{align}

\begin{align}
    dv_{1,t} &= \kappa_1 (\theta_1 - v_{1,t}) dt + \sigma_1 \sqrt{\bar{v}_{1,t}} (dw_{1,t} - (\lambda_1/\sigma_1) \sqrt{\bar{v}_{1,t}} dt) \\
    &= (\kappa_1 \theta_1 - (\kappa_1 + \lambda_1) v_{1,t}) dt + \sigma_1 \sqrt{\bar{v}_{1,t}} dw_{1,t} \\
    \end{align}

\begin{align}
    dv_{2,t} &= \kappa_2 (\theta_2 - v_{2,t}) dt + \sigma_2 \sqrt{\bar{v}_{2,t}} (dw_{2,t} - (\lambda_2/\sigma_2) \sqrt{\bar{v}_{2,t}} dt) \\
    &= (\kappa_2 \theta_2 - (\kappa_2 + \lambda_2) v_{2,t}) dt + \sigma_2 \sqrt{\bar{v}_{2,t}} dw_{2,t} \\
    \end{align}

\section{Proof of Proposition 3}

Given the $Q$ dynamics of index returns and individual equities returns in (4) and (15), applying Ito’s lemma on $x^i_t$, delivers the following expression.

\begin{align}
    x^i_{t+\tau} - x^i_t &= r \tau - \frac{1}{2} \left[ \beta^2_1 v_{1,t+\tau} + \beta^2_2 v_{2,t+\tau} + \xi^i_{t+\tau} \right] \tau \\
    &\quad + \beta^1_1 \int_t^{t+\tau} \sqrt{\bar{v}_{1,u}} d\bar{z}_{1,u} + \beta^2_2 \int_t^{t+\tau} \sqrt{\bar{v}_{2,u}} d\bar{z}_{2,u} + \int_t^{t+\tau} \sqrt{\xi^i_u} d\bar{z}^i_u \\
    \end{align}
For the ease of notations we define:

\[ \tilde{z}_{v_1, \tau} \equiv \int_{t}^{t+\tau} \sqrt{v_{1,u}} d\tilde{z}_{1,u}, \]
\[ \tilde{z}_{v_2, \tau} \equiv \int_{t}^{t+\tau} \sqrt{v_{2,u}} d\tilde{z}_{2,u}, \]
\[ \tilde{z}_{\xi, \tau} \equiv \int_{t}^{t+\tau} \sqrt{\xi_{u}} d\tilde{z}_{u}. \]

By the definition of risk-neutral conditional characteristic function of log-returns in (17) we have:

\[ \tilde{f}_i(\tau, \phi) = E_t^Q \left[ \exp \left[ i\phi (r_\tau - \frac{1}{2} (\beta_{1}^{2} v_{1,t:t+\tau} + \beta_{2}^{2} v_{2,t:t+\tau} + \xi_{t:t+\tau}) \tau + \beta_{1}^i \tilde{z}_{v_1, \tau} + \beta_{2}^i \tilde{z}_{v_2, \tau} + \tilde{z}_{\xi, \tau}) \right] \right]. \] (C.2)

Define the stochastic exponential \( \zeta(\cdot) \) as follows.

\[ \zeta\left( \int_{0}^{t} w_u' dW_u \right) \equiv \exp \left[ \int_{0}^{t} w_u' dW_u - \frac{1}{2} \int_{0}^{t} w_u' d\langle W, W \rangle u \right] \] (C.3)

Therefore,

\[ \zeta\left( i\phi \beta_{1}^{i} \tilde{z}_{v_1, \tau} \right) = \exp \left[ i\phi \beta_{1}^{i} \tilde{z}_{v_1, \tau} - \frac{1}{2} (i\phi \beta_{1}^{i})^{2} \langle \tilde{z}_{v_1, \tau}, \tilde{z}_{v_1, \tau} \rangle \right] \]
\[ = \exp \left[ i\phi \beta_{1}^{i} \tilde{z}_{v_1, \tau} + \frac{1}{2} \phi^2 \beta_{1}^{2} v_{1,t:1+\tau} \right]. \] (C.4)

Similar to (C.4), define \( \zeta\left( i\phi \beta_{2}^{i} \tilde{z}_{v_2, \tau} \right) \) and \( \zeta\left( i\phi \tilde{z}_{\xi, \tau} \right) \) and then combine these three stochastic exponential with (C.2) to get the following risk-neutral conditional characteristic function.

\[ \tilde{f}_i(\tau, \phi) = e^{i\phi r_\tau} E_t^Q \left[ \zeta\left( i\phi \beta_{1}^{i} \tilde{z}_{v_1, \tau} \right) \zeta\left( i\phi \beta_{2}^{i} \tilde{z}_{v_2, \tau} \right) \zeta\left( i\phi \tilde{z}_{\xi, \tau} \right) \exp \left[ - g_1 v_{1,t:t+\tau} - g_2 v_{2,t:t+\tau} - g_3 \xi_{t:t+\tau} \right] \right] \] (C.5)

where, \( g_1 = \frac{1}{2} i\phi \beta_{1}^{2} (1 - i\phi), g_2 = \frac{1}{2} i\phi \beta_{2}^{2} (1 - i\phi), \) and \( g_3 = \frac{1}{2} i\phi (1 - i\phi). \) Following Carr and Wu (2004), we define a new change-of-measure from \( Q \)-measure to \( C \)-measure as follows.\(^{35}\)

\(^{34}\) For compactness, the dependence of risk-neutral conditional characteristic function to \( x_i^t, v_{1,t}, v_{2,t}, \xi_{t}, \beta_{1}^{i}, \) and \( \beta_{2}^{i} \) is suppressed in (C.2).

\(^{35}\) As the Radon-Nikodym derivatives in (C.6) is defined based on the stochastic exponential \( \zeta(\cdot), \) it is Martingale by definition.
\[
\frac{dC}{dQ}(t) \equiv \zeta(i\phi\beta_1^i\tilde{z}_{t,1})\zeta(i\phi\beta_2^i\tilde{z}_{t,2})\zeta(i\phi\tilde{z}_{t,3}) \tag{C.6}
\]

The Radon-Nikodym derivatives of \(C\) with respect to \(Q\) in (C.6) allows to write (C.5) as

\[
f^i(\tau, \phi) = e^{i\phi\tau}E_t^C\left[\frac{dC}{dQ}(T)\frac{dC}{dQ}(t)\exp\left[ -g_1v_{1,t:t+\tau} - g_2v_{2,t:t+\tau} - g_3\xi_{t,t+\tau} \right]\right] = e^{i\phi\tau}E_t^C\left[\exp\left[ -g_1v_{1,t:t+\tau} - g_2v_{2,t:t+\tau} - g_3\xi_{t,t+\tau} \right]\right]. \tag{C.7}
\]

Accordingly, we transform the risk-neutral shocks to index returns volatilities and to the idiosyncratic returns volatility to their \(C\)-measure counterparts by applying the extension of Grisanov’s theorem within the complex plane.

\[
\begin{align*}
\dot{\tilde{w}}_{1,t} &= dw^1_{t,t} + (i\phi\rho_1\beta_1^i\sqrt{v_{1,t}})dt \\
\dot{\tilde{w}}_{2,t} &= dw^2_{t,t} + (i\phi\rho_2\beta_2^i\sqrt{v_{2,t}})dt \\
\dot{\tilde{w}}^i_t &= dw^{i,C}_{t,t} + (i\phi\rho^i\sqrt{\xi^i_t})dt
\end{align*} \tag{C.8}
\]

As a result, the index volatilities dynamics and idiosyncratic volatility dynamics of individual equity under the \(C\)-measure are

\[
\begin{align*}
dv_{1,t} &= \kappa_1^C(\theta_1^C - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw^1_{1,t} \\
dv_{2,t} &= \kappa_2^C(\theta_2^C - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw^2_{2,t} \\
d\xi^i_t &= \kappa^i,C(\theta^i,C - \xi^i_t)dt + \sigma^i\sqrt{\xi^i_t}dw^{i,C}_{t,t}
\end{align*} \tag{C.9}
\]

where,

\[
\begin{align*}
\kappa^C_1 &= \tilde{\kappa}_1 - i\phi\rho_1\beta_1^i\sigma_1 \\
\theta^C_1 &= \tilde{\kappa}_1\tilde{\theta}_1/\kappa^C_1 \\
\kappa^C_2 &= \tilde{\kappa}_2 - i\phi\rho_2\beta_2^i\sigma_2 \\
\theta^C_2 &= \tilde{\kappa}_2\tilde{\theta}_2/\kappa^C_2 \\
\kappa^{i,C} &= \tilde{\kappa}^i - i\phi\rho^i\sigma^i \\
\theta^{i,C} &= \tilde{\kappa}^i\tilde{\theta}^i/\kappa^{i,C}
\end{align*}
\]

Using the closed-form solution of the moment generating functions of \(E_t^C[\exp(-g_1v_{1,t:t+\tau})]\), \(E_t^C[\exp(-g_2v_{2,t:t+\tau})]\), and \(E_t^C[\exp(-g_3\xi_{t,t+\tau})]\), the risk-neutral conditional characteristic function of log individual equity prices has the following affine form.

\[
f^i(v_{1,t}, v_{2,t}, \xi^i_t, \tau, \phi) = \exp\left[ i\phi x^i_t + i\phi r\tau - A_1(\tau, \phi) - A_2(\tau, \phi) - B(\tau, \phi) \right] - C_1(\tau, \phi)v_{1,t} - C_2(\tau, \phi)v_{2,t} - D(\tau, \phi)\xi^i_t, \tag{C.10}
\]

45
\[ A_1(\tau, \phi) = \frac{\tilde{k}_1 \tilde{\theta}_1}{\sigma_1^2} \left[ 2 \ln \left[ 1 - \frac{d_1 - \kappa^C_1}{2d_1} (1 - e^{-d_1 \tau}) \right] + (d_1 - \kappa^C_1) \tau \right], \]

\[ A_2(\tau, \phi) = \frac{\tilde{k}_2 \tilde{\theta}_2}{\sigma_2^2} \left[ 2 \ln \left[ 1 - \frac{d_2 - \kappa^C_2}{2d_2} (1 - e^{-d_2 \tau}) \right] + (d_2 - \kappa^C_2) \tau \right], \]

\[ B(\tau, \phi) = \frac{\tilde{k}_i \tilde{\theta}_i}{\sigma_i^2} \left[ 2 \ln \left[ 1 - \frac{d_i - \kappa^{i, C}_i}{2d_i} (1 - e^{-d_i \tau}) \right] + (d_i - \kappa^{i, C}_i) \tau \right], \]

\[ C_1(\tau, \phi) = \frac{2}{d_1} - (d_1 - \kappa^C_1)(1 - e^{-d_1 \tau}), \]

\[ C_2(\tau, \phi) = \frac{2}{d_2} - (d_2 - \kappa^C_2)(1 - e^{-d_2 \tau}), \]

\[ D(\tau, \phi) = \frac{2}{d_i} - (d_i - \kappa^{i, C}_i)(1 - e^{-d_i \tau}), \]

\[ d_1 = \sqrt{(\kappa^C_1)^2 + 2\sigma_1^2 g_1}, \]

\[ d_2 = \sqrt{(\kappa^C_2)^2 + 2\sigma_2^2 g_2}, \]

\[ d_i = \sqrt{(\kappa^{i, C}_i)^2 + 2\sigma_i^2 g_i}, \]

\[ g_1 = 1 - i\phi \beta_1^2 (1 - i \phi), \]

\[ g_2 = 1 - i\phi \beta_2^2 (1 - i \phi), \]

\[ g_i = 1 - i\phi (1 - i \phi). \]

We determine the price of a European call option on an individual equity with the strike price \( K \) and the time-to-maturity \( \tau \) by inverting the risk-neutral conditional characteristic function of log-returns.\(^{36}\)

\[ C^i(S^i_t, K, \tau) = S^i_t P^i_1 - K e^{-r \tau} P^i_2, \tag{C.12} \]

where,

\[ P^i_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ e^{-i \phi \ln K} \tilde{f}_i (v_{1, t}, v_{2, t}, \xi^i_t, \tau, \phi - i \phi) \right] d\phi, \]

\[ P^i_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ e^{-i \phi \ln K} \tilde{f}_i (v_{1, t}, v_{2, t}, \xi^i_t, \tau, \phi) \right] d\phi. \tag{C.13} \]

\(^{36}\)Note that the risk-neutral conditional characteristic function of the logarithm of individual equity returns, \( x_{t+\tau}^i - x_t^i = \ln(S_{t+\tau}^i/S_t^i) \), can be defined with the same expression as \((C.10)\) but without the first component, \( i\phi x_t^i \).
Proofs of Proposition (4) and Proposition (5) are available upon request.

E Estimation of the Index Model - Discretization and Particle Filter Methods

To estimate the parameters of two-factor stochastic volatility model of the index we follow the literature on the estimation of stochastic volatility models, where the main challenge is the estimation of unobserved latent volatilities. There are several approaches to estimate stochastic volatility model. Our own approach combines the information from underlying index and option markets to impose consistency between structural parameters under $P$ and $Q$ distributions, known as joint estimation. Therefore, we use a likelihood function that contains a return-based component and an option-based component, as in Santa-Clara and Yan (2010) and Christoffersen et al. (2013). Here we do a joint-estimation by filtering the two vectors of daily spot variances, $\{v_{1,t}, v_{2,t}\}$, and simultaneously estimating a set of structural parameters of the dynamics of index returns and variances, including the market price of each variance component, $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \lambda_1, \lambda_2\}$. Note that joint estimation allow us to have reliable prices of variance risk factors, as we can get a consistent set of structural parameters between the $P$ and $Q$ distributions.

Since the market variances are unobserved state variables, we first extract daily instantaneous persistent and transient variance components using the Particle Filter (PF) method. This optimal filtering methodology provides a tool for learning about unobserved shocks and states from discretely observed prices generated by continuous-time models. Although we generally follow the conventional filtration procedure in the literature, we provide a novel approach to the challenge of filtering the two separate variance paths. Our proposed solution is not trivial and to the best of our knowledge is novel and constitutes a methodological contribution to the option pricing literature.

To define the return-based likelihood function and filter spot variances, we start by discretizing the returns dynamics (1). Applying Ito’s lemma to equation (1), gives the dynamics of logarithm of stock prices as follows.

---

37 Consistency can also be imposed through moment-based and simulation-based methods; see Ait-Sahalia and Kimmel (2007), Eraker (2004), Jones (2003), Chernov and Ghysels (2000), and Pan (2002). Other approaches use only option-based data to estimate only the $Q$ distribution; Bakshi et al. (1997), Bates (2000), Huang and Wu (2004), and Christoffersen et al. (2009).

38 For the application of PF in estimating the model parameters see Gordon et al. (1993), Johannes et al. (2009), Johannes and Polson (2009), Christoffersen et al. (2010), and Boloorforoosh (2014).
\[ d \ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t}, \]
\[ dv_{1,t} = \kappa_{1}(\theta_{1} - v_{1,t})dt + \sigma_{1}\sqrt{v_{1,t}}dw_{1,t}, \]
\[ dv_{2,t} = \kappa_{2}(\theta_{2} - v_{2,t})dt + \sigma_{2}\sqrt{v_{2,t}}dw_{2,t}, \]  

(E.1)

where, \( \mu \equiv r + \mu_{1}v_{1,t} + \mu_{2}v_{2,t} \). We discretize (E.1) using the Euler scheme.\(^{39}\) Equation (E.2) models the relation between observed index prices and unobserved variances at time \( t + \Delta t \) conditional on the time \( t \) variances.

\[ \ln(S_{t+\Delta t}) - \ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))\Delta t + \sqrt{v_{1,t}\Delta t}z_{1,t+\Delta t} + \sqrt{v_{2,t}\Delta t}z_{2,t+\Delta t}, \]
\[ v_{1,t+\Delta t} = v_{1,t} + \kappa_{1}(\theta_{1} - v_{1,t})\Delta t + \sigma_{1}\sqrt{v_{1,t}\Delta t}w_{1,t+\Delta t}, \]
\[ v_{2,t+\Delta t} = v_{2,t} + \kappa_{2}(\theta_{2} - v_{2,t})\Delta t + \sigma_{2}\sqrt{v_{2,t}\Delta t}w_{2,t+\Delta t}. \]  

(E.2)

Brownian shocks \( z_{1,t+\Delta t}, z_{2,t+\Delta t}, w_{1,t+\Delta t}, \) and \( w_{2,t+\Delta t} \) are normal random variables with mean zero and variance one. From the first equation in (E.2) we use the observed daily index log-prices \( (\ln(S_t), \ln(S_{t+\Delta t})) \) to first filter the daily return’s shocks \( (z_{1,t+\Delta t}, z_{2,t+\Delta t}) \) and then, using the filtered shocks in returns and the last two equation in (E.2), we filter daily spot variances \( (v_{1,t+\Delta t}, v_{2,t+\Delta t}) \). Note that we filter filter the summation of return shocks \( z_{1,t+\Delta t} + z_{2,t+\Delta t} \) as we cannot separate the daily observed shocks into two components, \( z_{1,t+\Delta t} \) and \( z_{2,t+\Delta t} \). Therefore, we rewrite the underlying dynamics as (E.3), given that the return shocks are uncorrelated and then discretize this dynamics.

\[ d \ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t} + v_{2,t}}dz_t, \]
\[ dv_{1,t} = \kappa_{1}(\theta_{1} - v_{1,t})dt + \sigma_{1}\sqrt{v_{1,t}}dw_{1,t}, \]
\[ dv_{2,t} = \kappa_{2}(\theta_{2} - v_{2,t})dt + \sigma_{2}\sqrt{v_{2,t}}dw_{2,t}, \]  

(E.3)

with the correlation structure:

\[ \langle dw_{1,t},dz_{1,t} \rangle = \rho_{1}dt, -1 \leq \rho_{1} \leq +1 \]
\[ \langle dw_{2,t},dz_{2,t} \rangle = \rho_{2}dt, -1 \leq \rho_{2} \leq +1 \]
\[ \langle dw_{1,t},dw_{2,t} \rangle = 0 \]
\[ \rho_{1}^{2} + \rho_{2}^{2} \leq +1 \]
\[ \rho_{1}^{2} + \rho_{2}^{2} \leq +1. \]  

(E.4)

We decompose the variance shocks into orthogonal components as in (E.5) and then discretize the return dynamics (E.3) using the Euler scheme and shock’s decomposition (E.5).\(^{40}\)

\(^{39}\) According to Eraker (2004) and Li et al. (2008) the discretization bias of the Euler scheme is negligible for daily data.

\(^{40}\) Note that the quadratic variations of the transformed using the proposed shocks decomposition (E.5) should remain the same as \( \sqrt{dt} \).
\[ dw_{1,t} = \rho_1 dz_t + \sqrt{1 - \rho_1^2} dB_{1,t} \]
\[ dw_{2,t} = \rho_2 dz_t - \frac{\rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} dB_{1,t} + \sqrt{\frac{1 - \rho_1^2 - \rho_2^2}{1 - \rho_1^2}} dB_{2,t} \]  
(E.5)

\[ \ln(S_{t+\Delta t}) - \ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t})) \Delta t + \sqrt{(v_{1,t} + v_{2,t}) \Delta t} \ z_{t+\Delta t}, \]
\[ v_{1,t+\Delta t} = v_{1,t} + \kappa_1 (\theta_1 - v_{1,t}) \Delta t + \sigma_1 \sqrt{v_{1,t} \Delta t} \ w_{1,t+\Delta t}, \]
\[ v_{2,t+\Delta t} = v_{2,t} + \kappa_2 (\theta_2 - v_{2,t}) \Delta t + \sigma_2 \sqrt{v_{2,t} \Delta t} \ w_{2,t+\Delta t}, \]  
(E.6)

where, \( z_{t+\Delta t} \), \( w_{1,t+\Delta t} \), and \( w_{2,t+\Delta t} \) are all \( N(0,1) \). Now, using daily index log-returns, we proceed to filter the spot variances from the discretized model in (E.6) given the correlation structure in (E.5).

We follow Pitt (2002)\(^{41}\) and adopt a particular implementation of the PF, which is referred to as the sampling-importance-resampling (SIR) PF. This implementation of PF method allow us to approximate the true density of the persistent variance component \( (v_{1,t}) \) and the transient variance component \( (v_{2,t}) \) using two sets of particles that are updated recursively through equations (E.6). In other words, we recursively simulate next period particles of each variance component until we have the empirical distributions of each variance factor over the entire sample. That is, given \( N \) particles of \( \{v_{1,j,t}^j\}_{j=1}^N \), \( N \) particles of \( \{v_{2,t}^j\}_{j=1}^N \), simulated return shocks, and \( w_{1,t+\Delta t} \) and \( w_{2,t+\Delta t} \), we generate the next period particles, \( N \) particles \( \{v_{1,t+\Delta t}^j\}_{j=1}^N \) and another \( N \) particles \( \{v_{2,t+\Delta t}^j\}_{j=1}^N \) at any time \( t + \Delta t \).

We start by simulating return’s shocks \( z_{t+\Delta t}^j \) given the initial value of structural parameters \( \Theta_0 \) and current variance particles \( \{v_{1,t}^j, v_{2,t}^j\} \), on every day \( t \) and for every particle \( j = 1,2,...,N \), according to (E.7). Then using (E.8) we simulate volatility shocks \( w_{1,t+\Delta t}^j \) and \( w_{2,t+\Delta t}^j \). Note that \( e_{1,t+\Delta t}^j \) and \( e_{2,t+\Delta t}^j \) are independent standard normal random variables.

\[ z_{t+\Delta t}^j = [\ln(S_{t+\Delta t}/S_t) - (\mu - \frac{1}{2}(v_{1,t}^j + v_{2,t}^j)) \Delta t] / \sqrt{(v_{1,t}^j + v_{2,t}^j) \Delta t} \]  
(E.7)

\[ w_{1,t+\Delta t}^j = \rho_1 z_{t+\Delta t}^j + \sqrt{1 - \rho_1^2} e_{1,t+\Delta t}^j \]
\[ w_{2,t+\Delta t}^j = \rho_2 z_{t+\Delta t}^j - \frac{\rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} e_{1,t+\Delta t}^j + \sqrt{\frac{1 - \rho_1^2 - \rho_2^2}{1 - \rho_1^2}} e_{2,t+\Delta t}^j \]  
(E.8)

\(^{41}\) See Pitt (2002), Christoffersen et al. (2010), and Boloorforoosh (2014) for a detailed description of the PF algorithm.
Then, given the simulated return’s shocks \( \{z^j_{t+\Delta t}\}_{j=1}^N \) and simulated shocks to the persistent and transient variance components \( \{w^j_{1,t+\Delta t}\}_{j=1}^N \) and \( \{w^j_{2,t+\Delta t}\}_{j=1}^N \), we simulate next period variance particles \( \{\tilde{v}^j_{1,t+\Delta t}\} \) and \( \{\tilde{v}^j_{2,t+\Delta t}\} \), for every day \( t \) according to (E.9).

\[
\begin{align*}
\tilde{v}^j_{1,t+\Delta t} &= v^j_{1,t} + \kappa_1 (\theta_1 - v^j_{1,t}) \Delta t + \sigma_1 \sqrt{v^j_{1,t+\Delta t}} w^j_{1,t+\Delta t} \\
\tilde{v}^j_{2,t+\Delta t} &= v^j_{2,t} + \kappa_2 (\theta_2 - v^j_{2,t}) \Delta t + \sigma_2 \sqrt{v^j_{2,t+\Delta t}} w^j_{2,t+\Delta t}
\end{align*}
\]

(E.9)

This is the “Sampling Step,” at the end of which we generate \( N \) possible daily values for the persistent variance component \( v^j_{1,t+\Delta t} \) and another \( N \) possible daily values for the transient variance component \( v^j_{2,t+\Delta t} \) over the entire sample. In the next step, “Importance Step,” we evaluate importance of the sampled daily particles by assigning appropriate weights \( \tilde{W}^j_{t+\Delta t} \) to the simulated daily particles using a multivariate normal distribution. Intuitively, these weights, \( \tilde{W}^j_{t+\Delta t} \), are likelihood that the next day return at \( t + 2\Delta t \) is generated by this set of particles. Then, the probability of each daily particle can be defined by normalizing the weights within each day according to (E.12). Note that these weights are the basis of our likelihood function under the \( P \) distribution.

\[
(r_{t+2\Delta t} | \{\tilde{v}^j_{1,t+\Delta t}, \tilde{v}^j_{2,t+\Delta t}\}) \sim N\left( \mu - \frac{1}{2} (\tilde{v}^j_{1,t+\Delta t} + \tilde{v}^j_{2,t+\Delta t}) \Delta t, (\tilde{v}^j_{1,t+\Delta t} + \tilde{v}^j_{2,t+\Delta t}) \Delta t \right)
\]

(E.10)

\[
\tilde{W}^j_{t+\Delta t} = \frac{1}{\sqrt{2\pi(\tilde{v}^j_{1,t+\Delta t} + \tilde{v}^j_{2,t+\Delta t})\Delta t}} \cdot \exp \left( -\frac{1}{2} \left( \ln \left( \frac{S_{t+2\Delta t}}{S_{t+\Delta t}} \right) - (\mu - \frac{1}{2} (\tilde{v}^j_{1,t+\Delta t} + \tilde{v}^j_{2,t+\Delta t}) \Delta t) \right)^2 \right)
\]

(E.11)

\[
\tilde{W}^j_{t+\Delta t} = \frac{\tilde{W}^j_{t+\Delta t}}{\sum_{j=1}^N \tilde{W}^j_{t+\Delta t}}
\]

(E.12)

Note that combining independent shocks \( z_{1,t} \) and \( z_{2,t} \) in (E.3) imposes a restriction on the weights of daily variance particles. Therefore, the importance probability is assigned to the summation of return’s shocks. However, estimation results show that the path of filtered spot persistent variance component and transient variance component in our two-factor SV model are not sensitive to this assumption. We investigate the sensitivity of our result to this weighting assumption by estimating daily spot variances using the two-step iterative approach, following Huang and Wu (2004). We do not observe significant difference between filtered spot variances in two-step iterative approach and those filtered with particle filter method.

In the last step, “Resampling Step,” we find the empirical distribution of smoothly resampled daily particles. Following the Pitt (2002) algorithm, we draw smoothed daily particles by
assigning uniform distributions to the raw daily particles for persistent and transient variance components. As in the sampling step, we start from the beginning of the sample period and recursively simulate the next period daily particles using the smoothly resampled daily particles. The procedure continues until we have the empirical distributions of the persistent and transient variance components over the entire sample.

Given the appropriate weights (E.12), we define the return-based likelihood function as follows.

\[
LLR \propto \sum_{t=1}^{T} \ln \left( \frac{1}{N} \sum_{j=1}^{N} \tilde{W}_t^j(\Theta) \right)
\]  
(E.13)

Our implementation uses the maximum likelihood importance sampling (MLIS) methodology to maximize \( LLR \) criterion. Note that return-based likelihood function (E.13) is a function of the structural parameters of the market model under \( P \) measure, \( \Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_1, \rho_1, \rho_2\} \). Note also that the filtered daily spot persistent variance component \( v_{P,1,t} \) and transient variance component \( v_{P,2,t} \) can be defined as the average of the smoothly resampled particles.

\[
\hat{v}_{P,1,t} = \frac{1}{N} \sum_{j=1}^{N} v_{j,1,t}, \quad \hat{v}_{P,2,t} = \frac{1}{N} \sum_{j=1}^{N} v_{j,2,t}
\]  
(E.14)

### F Risk Neutral Distribution

Risk-neutral distribution in (4) can also be extracted by assuming the following standard stochastic discount factor, without explicit assumptions about the investor’s variance preferences.

\[
\frac{dM_t}{M_t} = -rdt - \psi_t^t dW_t,
\]  
(F.1)

where \( \psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}] \) is the vector of market price of risk factors and \( W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}] \) is the vector of innovations in market index return and variance components. Given the SDF in (F.1), the change-of-measure from \( P \) to \( Q \) distribution has the following exponential form.

\[
\frac{dQ}{dP}(t) \equiv M_t \exp(rt) = \exp \left[ - \int_0^t \psi_u^u dW_u - \frac{1}{2} \int_0^t \psi_u^u d\langle W, W' \rangle_u \psi_u \right]
\]  
(F.2)

where \( \langle W, W' \rangle \) is the covariance operator.
We follow the notion of Doléans-Dade exponential (stochastic exponential) and define the stochastic exponential $\varepsilon(\cdot)$ as follow.

$$
\varepsilon\left(\int_0^t \vartheta_u' dW_u\right) \equiv \exp\left[ \int_0^t \vartheta_u' dW_u - \frac{1}{2} \int_0^t \vartheta_u' d\langle W, W \rangle_u \right] \quad \text{(F.3)}
$$

Therefore, the change-of-measure (F.2) can be expressed in term of stochastic exponential as

$$
\frac{dQ}{dP}(t) = \varepsilon\left(\int_0^t -\psi_u' dW_u\right) \quad \text{(F.4)}
$$

Applying Ito’s lemma, we get the following dynamic for the log stock price process under physical measure.

$$
\log\left(\frac{S_t}{S_0}\right) = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) t - \frac{1}{2} v_{1,t} t + \int_0^t \sqrt{v_{1,u}} dz_{1,u} - \frac{1}{2} v_{2,t} t + \int_0^t \sqrt{v_{2,u}} dz_{2,u} \quad \text{(F.5)}
$$

Given (F.5) and definition of stochastic exponential (F.3) we have

$$
\frac{S_t}{S_0} = \exp\left[(r + \mu_1 v_{1,t} + \mu_2 v_{2,t}) t\right] \varepsilon\left(\int_0^t \sqrt{v_{1,u}} dz_{1,u}\right) \varepsilon\left(\int_0^t \sqrt{v_{2,u}} dz_{2,u}\right) \quad \text{(F.6)}
$$

To find the market prices of risk we impose the restriction that the product of the price of any traded asset and the pricing kernel under physical measure is a $P$-martingale. Given the change-of-measure (F.2), the following process, $N(t)$, should be a $P$-martingale.

$$
N(t) \equiv \frac{S_t}{S_0} \frac{dQ}{dP}(t) \exp(-rt) \quad \text{(F.7)}
$$

where

$$
N(t) = \exp\left[(\mu_1 v_{1,t} + \mu_2 v_{2,t}) t\right] \varepsilon\left(\int_0^t \sqrt{v_{1,u}} dz_{1,u}\right) \varepsilon\left(- \int_0^t \psi_{1,u} dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\right) \varepsilon\left(\int_0^t \sqrt{v_{2,u}} dz_{2,u}\right) \varepsilon\left(- \int_0^t \psi_{2,u} dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\right) \quad \text{(F.8)}
$$

Using the properties of a stochastic exponential $\varepsilon(\cdot)$, $\varepsilon(X_t)\varepsilon(Y_t) = \varepsilon(X_t + Y_t)\exp(\langle X, Y \rangle_t)$ we can rewrite the process of $N(t)$ as follows.
\[ N(t) = \exp \left[ (\mu_1 v_{1,t} + \mu_2 v_{2,t}) t \right] \]

\[
\varepsilon \left( \int_0^t (\sqrt{v_{1,u}} - \psi_{1,u}) \, dz_{1,u} - \int_0^t \psi_{3,u} \, dw_{1,u} \right) \, \exp \left[ - \int_0^t \sqrt{v_{1,u}} (\psi_{1,u} + \rho_1 \psi_{3,u}) \, du \right] \\
\varepsilon \left( \int_0^t (\sqrt{v_{2,u}} - \psi_{2,u}) \, dz_{2,u} - \int_0^t \psi_{4,u} \, dw_{2,u} \right) \, \exp \left[ - \int_0^t \sqrt{v_{2,u}} (\psi_{2,u} + \rho_2 \psi_{4,u}) \, du \right] 
\]

(F.9)

From the definition of a stochastic exponential we know that \( \varepsilon(\cdot) \) are \( P \)-martingales. Thus, the process \( N(t) \) is a \( P \)-martingale when the following restriction holds.

\[
\exp \left[ (\mu_1 v_{1,t} + \mu_2 v_{2,t}) t \right] \, \exp \left[ - \int_0^t \sqrt{v_{1,u}} (\psi_{1,u} + \rho_1 \psi_{3,u}) \, du \right] \, \exp \left[ - \int_0^t \sqrt{v_{2,u}} (\psi_{2,u} + \rho_2 \psi_{4,u}) \, du \right] = 1 
\]

(F.10)

The restriction in (F.10) can be satisfied if

\[
\mu_1 v_{1,t} t - \sqrt{v_{1,t}} (\psi_{1,t} + \rho_1 \psi_{3,t}) t = 0 \\
\mu_2 v_{2,t} t - \sqrt{v_{2,t}} (\psi_{2,t} + \rho_2 \psi_{4,t}) t = 0 
\]

(F.11)

To fully specify the market prices of risk we assume that market price of variance risk factors are proportional to spot volatilities, following Heston (1993).

\[
(\psi_{3,t} + \rho_1 \psi_{1,t}) = \frac{v_{1,t}}{\sigma_1 \sqrt{v_{1,t}}} \lambda_1 \\
(\psi_{4,t} + \rho_2 \psi_{2,t}) = \frac{v_{2,t}}{\sigma_2 \sqrt{v_{2,t}}} \lambda_2 
\]

(F.12)

Combining the restrictions in (F.11) and (F.12), we have the following market price of risk factors. Note that these prices are the same as those we find in Proposition (1).

\[
\psi_{1,t} = \frac{\sigma_1 \mu_1 - \rho_1 \lambda_1 \sqrt{v_{1,t}}}{(1 - \rho_1^2) \sigma_1} \\
\psi_{2,t} = \frac{\sigma_2 \mu_2 - \rho_2 \lambda_2 \sqrt{v_{2,t}}}{(1 - \rho_2^2) \sigma_2} \\
\psi_{3,t} = \frac{\lambda_1 - \rho_1 \lambda_1}{(1 - \rho_1^2) \sigma_1} \\
\psi_{4,t} = \frac{\lambda_2 - \rho_2 \lambda_2}{(1 - \rho_2^2) \sigma_2} 
\]

(F.13)
Given the market price of risk factors (F.13), we can apply Girsanov’s theorem to find transform physical innovations in (1) to its risk-neutral counterpart in (4).

\[
d\tilde{z}_{1,t} = dz_{1,t} + \psi_{1,1}dt + \rho_1\psi_{3,1}dt \\
d\tilde{z}_{2,t} = dz_{2,t} + \psi_{2,1}dt + \rho_2\psi_{4,1}dt \\
d\tilde{\omega}_{1,t} = dw_{1,t} + \psi_{3,1}dt + \rho_1\psi_{1,1}dt \\
d\tilde{\omega}_{2,t} = dw_{2,t} + \psi_{4,1}dt + \rho_2\psi_{2,1}dt
\]  
\text{(F.14)}

With some algebra we have the following transformations.

\[
d\tilde{z}_{1,t} = dz_{1,t} + \mu_1\sqrt{v_{1,t}}dt \\
d\tilde{z}_{2,t} = dz_{2,t} + \mu_2\sqrt{v_{2,t}}dt \\
d\tilde{\omega}_{1,t} = dw_{1,t} + (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt \\
d\tilde{\omega}_{2,t} = dw_{2,t} + (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt
\]  
\text{(F.15)}

Replacing \(dz_{1,t}, dz_{2,t}, dw_{1,t}, dw_{2,t}\) from (F.15) into the physical dynamics in (1) and knowing that \(\tilde{\kappa}_1 = \kappa_1 + \lambda_1, \tilde{\kappa}_2 = \kappa_2 + \lambda_2, \tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}, \tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}\) we obtain risk-neutral return and variance dynamics.

\[
dS_t/S_t = (r + \mu_1v_{1,t} + \mu_2v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
= (r + \mu_1v_{1,t} + \mu_2v_{2,t})dt + \sqrt{v_{1,t}}(d\tilde{z}_{1,t} - \mu_1\sqrt{v_{1,t}}dt) + \sqrt{v_{2,t}}(d\tilde{z}_{2,t} - \mu_2\sqrt{v_{2,t}}dt) \\
= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t}
\]
\text{(F.16)}

\[
dv_{1,t} = \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(d\tilde{\omega}_{1,t} - (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt) \\
= (\kappa_1\theta_1 - (\kappa_1 + \lambda_1)v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{\omega}_{1,t} \\
= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{\omega}_{1,t}
\]
\text{(F.17)}

\[
dv_{2,t} = \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(d\tilde{\omega}_{2,t} - (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt) \\
= (\kappa_2\theta_2 - (\kappa_2 + \lambda_2)v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{\omega}_{2,t} \\
= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{\omega}_{2,t}
\]
\text{(F.18)}
References


Table 1: S&P 500 Index Call Option Data Characteristics by Moneyness and Maturity

<table>
<thead>
<tr>
<th>Panel A: Number of call option contracts</th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤0.92</td>
<td>152</td>
<td>3,371</td>
<td>12,690</td>
<td>8,782</td>
<td>24,995</td>
</tr>
<tr>
<td>0.92&lt;S/K≤0.94</td>
<td>642</td>
<td>8,220</td>
<td>17,345</td>
<td>8,342</td>
<td>34,549</td>
</tr>
<tr>
<td>0.94&lt;S/K≤0.96</td>
<td>4,033</td>
<td>14,436</td>
<td>18,557</td>
<td>8,096</td>
<td>45,122</td>
</tr>
<tr>
<td>0.96&lt;S/K≤0.98</td>
<td>10,761</td>
<td>17,202</td>
<td>17,000</td>
<td>7,167</td>
<td>52,130</td>
</tr>
<tr>
<td>S/K&gt;0.98</td>
<td>13,052</td>
<td>16,137</td>
<td>15,628</td>
<td>6,485</td>
<td>51,302</td>
</tr>
<tr>
<td>All</td>
<td>28,640</td>
<td>59,366</td>
<td>81,220</td>
<td>38,872</td>
<td>208,098</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Average price of call option contracts</th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤0.92</td>
<td>13.6200</td>
<td>15.5478</td>
<td>23.0998</td>
<td>47.0797</td>
<td>24.8368</td>
</tr>
<tr>
<td>0.92&lt;S/K≤0.94</td>
<td>11.7434</td>
<td>16.1440</td>
<td>26.2574</td>
<td>56.2993</td>
<td>27.6110</td>
</tr>
<tr>
<td>0.94&lt;S/K≤0.96</td>
<td>9.9935</td>
<td>18.0151</td>
<td>34.2459</td>
<td>69.4400</td>
<td>32.9236</td>
</tr>
<tr>
<td>0.96&lt;S/K≤0.98</td>
<td>11.5532</td>
<td>24.4015</td>
<td>44.6126</td>
<td>82.1867</td>
<td>40.6885</td>
</tr>
<tr>
<td>S/K&gt;0.98</td>
<td>18.5235</td>
<td>35.5330</td>
<td>57.9296</td>
<td>95.6642</td>
<td>51.9126</td>
</tr>
<tr>
<td>All</td>
<td>13.0867</td>
<td>21.9283</td>
<td>37.2290</td>
<td>70.1340</td>
<td>35.5945</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Average implied volatility of call option contracts</th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤0.92</td>
<td>0.4071</td>
<td>0.2299</td>
<td>0.1894</td>
<td>0.1791</td>
<td>0.2514</td>
</tr>
<tr>
<td>0.92&lt;S/K≤0.94</td>
<td>0.3163</td>
<td>0.2034</td>
<td>0.1760</td>
<td>0.1831</td>
<td>0.2197</td>
</tr>
<tr>
<td>0.94&lt;S/K≤0.96</td>
<td>0.2213</td>
<td>0.1792</td>
<td>0.1770</td>
<td>0.1881</td>
<td>0.1914</td>
</tr>
<tr>
<td>0.96&lt;S/K≤0.98</td>
<td>0.1784</td>
<td>0.1741</td>
<td>0.1833</td>
<td>0.1958</td>
<td>0.1829</td>
</tr>
<tr>
<td>S/K&gt;0.98</td>
<td>0.1715</td>
<td>0.1829</td>
<td>0.1900</td>
<td>0.2028</td>
<td>0.1868</td>
</tr>
<tr>
<td>All</td>
<td>0.2589</td>
<td>0.1939</td>
<td>0.1831</td>
<td>0.1898</td>
<td>0.2064</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: Average delta of call option contracts</th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤0.92</td>
<td>0.2316</td>
<td>0.2302</td>
<td>0.2724</td>
<td>0.3726</td>
<td>0.2767</td>
</tr>
<tr>
<td>0.92&lt;S/K≤0.94</td>
<td>0.2329</td>
<td>0.2549</td>
<td>0.3121</td>
<td>0.4268</td>
<td>0.3067</td>
</tr>
<tr>
<td>0.94&lt;S/K≤0.96</td>
<td>0.2381</td>
<td>0.2984</td>
<td>0.3832</td>
<td>0.4827</td>
<td>0.3506</td>
</tr>
<tr>
<td>0.96&lt;S/K≤0.98</td>
<td>0.2996</td>
<td>0.3843</td>
<td>0.4608</td>
<td>0.5319</td>
<td>0.4191</td>
</tr>
<tr>
<td>S/K&gt;0.98</td>
<td>0.4422</td>
<td>0.4976</td>
<td>0.5377</td>
<td>0.5771</td>
<td>0.5136</td>
</tr>
<tr>
<td>All</td>
<td>0.2889</td>
<td>0.3331</td>
<td>0.3932</td>
<td>0.4782</td>
<td>0.3733</td>
</tr>
</tbody>
</table>

Note to Table: This table reports the summary statistics of out-of-the-money S&P 500 call option contracts in our sample, from January 1, 1996 to December 31, 2011. The implied volatilities and the deltas are from the OptionMetrics volatility surface data set. S denotes the price of the S&P 500 index, K is the option strike price, and DTM is the number of calendar days to maturity.
Table 2: S&P 500 Index Put Option Data Characteristics by Moneyness and Maturity

Panel A: Number of put option contracts

<table>
<thead>
<tr>
<th></th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤1.02</td>
<td>10,776</td>
<td>13,499</td>
<td>13,463</td>
<td>5,904</td>
<td>43,642</td>
</tr>
<tr>
<td>1.02&lt;S/K≤1.04</td>
<td>7,163</td>
<td>10,951</td>
<td>12,018</td>
<td>5,008</td>
<td>35,140</td>
</tr>
<tr>
<td>1.04&lt;S/K≤1.06</td>
<td>3,699</td>
<td>8,083</td>
<td>10,399</td>
<td>5,317</td>
<td>27,498</td>
</tr>
<tr>
<td>1.06&lt;S/K≤1.08</td>
<td>1,248</td>
<td>5,334</td>
<td>8,105</td>
<td>3,908</td>
<td>18,595</td>
</tr>
<tr>
<td>S/K&gt;1.08</td>
<td>385</td>
<td>3,173</td>
<td>5,591</td>
<td>3,588</td>
<td>12,737</td>
</tr>
<tr>
<td>All</td>
<td>23,271</td>
<td>41,040</td>
<td>49,576</td>
<td>23,725</td>
<td>137,612</td>
</tr>
</tbody>
</table>

Panel B: Average price of put option contracts

<table>
<thead>
<tr>
<th></th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤1.02</td>
<td>18.7121</td>
<td>30.3521</td>
<td>44.9423</td>
<td>63.5550</td>
<td>39.3904</td>
</tr>
<tr>
<td>1.02&lt;S/K≤1.04</td>
<td>13.9689</td>
<td>25.4113</td>
<td>40.1731</td>
<td>59.5418</td>
<td>34.7738</td>
</tr>
<tr>
<td>1.04&lt;S/K≤1.06</td>
<td>12.7334</td>
<td>21.7862</td>
<td>34.1231</td>
<td>55.3294</td>
<td>30.9930</td>
</tr>
<tr>
<td>1.06&lt;S/K≤1.08</td>
<td>14.0224</td>
<td>20.8254</td>
<td>30.5229</td>
<td>44.3883</td>
<td>27.4397</td>
</tr>
<tr>
<td>S/K&gt;1.08</td>
<td>16.1005</td>
<td>20.9994</td>
<td>30.9259</td>
<td>43.7921</td>
<td>27.9545</td>
</tr>
<tr>
<td>All</td>
<td>15.1075</td>
<td>23.8749</td>
<td>36.1375</td>
<td>53.3213</td>
<td>32.1103</td>
</tr>
</tbody>
</table>

Panel C: Average implied volatility of put option contracts

<table>
<thead>
<tr>
<th></th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤1.02</td>
<td>0.1929</td>
<td>0.1933</td>
<td>0.1992</td>
<td>0.2121</td>
<td>0.1994</td>
</tr>
<tr>
<td>1.02&lt;S/K≤1.04</td>
<td>0.2194</td>
<td>0.2134</td>
<td>0.2158</td>
<td>0.2127</td>
<td>0.2153</td>
</tr>
<tr>
<td>1.04&lt;S/K≤1.06</td>
<td>0.2646</td>
<td>0.2314</td>
<td>0.2233</td>
<td>0.2313</td>
<td>0.2376</td>
</tr>
<tr>
<td>1.06&lt;S/K≤1.08</td>
<td>0.3342</td>
<td>0.2599</td>
<td>0.2367</td>
<td>0.2200</td>
<td>0.2627</td>
</tr>
<tr>
<td>S/K&gt;1.08</td>
<td>0.4255</td>
<td>0.2904</td>
<td>0.2583</td>
<td>0.2343</td>
<td>0.3021</td>
</tr>
<tr>
<td>All</td>
<td>0.2873</td>
<td>0.2377</td>
<td>0.2266</td>
<td>0.2221</td>
<td>0.2434</td>
</tr>
</tbody>
</table>

Panel D: Average delta of put option contracts

<table>
<thead>
<tr>
<th></th>
<th>DTM≤30</th>
<th>30&lt;DTM≤91</th>
<th>91&lt;DTM≤182</th>
<th>DTM&gt;182</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/K≤1.02</td>
<td>-0.3931</td>
<td>-0.3988</td>
<td>-0.3931</td>
<td>-0.3631</td>
<td>-0.3870</td>
</tr>
<tr>
<td>1.02&lt;S/K≤1.04</td>
<td>-0.2860</td>
<td>-0.3221</td>
<td>-0.3403</td>
<td>-0.3334</td>
<td>-0.3204</td>
</tr>
<tr>
<td>1.04&lt;S/K≤1.06</td>
<td>-0.2348</td>
<td>-0.2699</td>
<td>-0.2932</td>
<td>-0.3060</td>
<td>-0.2760</td>
</tr>
<tr>
<td>1.06&lt;S/K≤1.08</td>
<td>-0.2194</td>
<td>-0.2395</td>
<td>-0.2579</td>
<td>-0.2612</td>
<td>-0.2445</td>
</tr>
<tr>
<td>S/K&gt;1.08</td>
<td>-0.2175</td>
<td>-0.2209</td>
<td>-0.2431</td>
<td>-0.2547</td>
<td>-0.2341</td>
</tr>
<tr>
<td>All</td>
<td>-0.2702</td>
<td>-0.2902</td>
<td>-0.3055</td>
<td>-0.3037</td>
<td>-0.2924</td>
</tr>
</tbody>
</table>

Note to Table: This table reports the summary statistics of out-of-the-money S&P 500 put option contracts in our sample, from January 1, 1996 to December 31, 2011. The implied volatilities and delta are from the OptionMetrics volatility surface data set. $S$ denotes the price of the S&P 500 index, $K$ is the option strike price, and DTM is the number of calendar days to maturity.
Table 3: Data Sample Summary

<table>
<thead>
<tr>
<th>Company</th>
<th>Ticker</th>
<th>Call</th>
<th>Put</th>
<th>All Options</th>
<th>Avg DTM</th>
<th>Avg IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500 Index</td>
<td>SPX</td>
<td>208,098</td>
<td>137,612</td>
<td>345,710</td>
<td>141</td>
<td>22.49%</td>
</tr>
<tr>
<td>Alcoa</td>
<td>AA</td>
<td>134,112</td>
<td>106,732</td>
<td>240,844</td>
<td>130</td>
<td>35.16%</td>
</tr>
<tr>
<td>American Express</td>
<td>AXP</td>
<td>143,880</td>
<td>109,422</td>
<td>253,302</td>
<td>132</td>
<td>31.62%</td>
</tr>
<tr>
<td>Boeing</td>
<td>BA</td>
<td>149,949</td>
<td>116,967</td>
<td>266,916</td>
<td>131</td>
<td>30.52%</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>CAT</td>
<td>145,951</td>
<td>113,189</td>
<td>259,140</td>
<td>130</td>
<td>32.04%</td>
</tr>
<tr>
<td>Cisco</td>
<td>CSCO</td>
<td>127,223</td>
<td>100,605</td>
<td>227,828</td>
<td>128</td>
<td>36.92%</td>
</tr>
<tr>
<td>Chevron</td>
<td>CVX</td>
<td>178,737</td>
<td>132,901</td>
<td>311,638</td>
<td>135</td>
<td>24.56%</td>
</tr>
<tr>
<td>Dupont</td>
<td>DD</td>
<td>162,592</td>
<td>122,417</td>
<td>285,009</td>
<td>135</td>
<td>27.43%</td>
</tr>
<tr>
<td>Disney</td>
<td>DIS</td>
<td>145,656</td>
<td>114,062</td>
<td>259,718</td>
<td>138</td>
<td>29.84%</td>
</tr>
<tr>
<td>General Electric</td>
<td>GE</td>
<td>151,825</td>
<td>112,771</td>
<td>264,596</td>
<td>141</td>
<td>27.74%</td>
</tr>
<tr>
<td>Home Depot</td>
<td>HD</td>
<td>145,260</td>
<td>113,691</td>
<td>258,951</td>
<td>134</td>
<td>30.92%</td>
</tr>
<tr>
<td>Hewlett-Packard</td>
<td>HPQ</td>
<td>127,524</td>
<td>101,302</td>
<td>228,826</td>
<td>131</td>
<td>35.36%</td>
</tr>
<tr>
<td>IBM</td>
<td>IBM</td>
<td>164,543</td>
<td>125,043</td>
<td>289,586</td>
<td>135</td>
<td>27.09%</td>
</tr>
<tr>
<td>Intel</td>
<td>INTC</td>
<td>123,444</td>
<td>98,783</td>
<td>222,227</td>
<td>135</td>
<td>30.09%</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>JNJ</td>
<td>189,496</td>
<td>137,546</td>
<td>327,042</td>
<td>140</td>
<td>21.73%</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>JPM</td>
<td>149,895</td>
<td>110,342</td>
<td>260,237</td>
<td>132</td>
<td>31.60%</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>KO</td>
<td>178,611</td>
<td>131,747</td>
<td>310,358</td>
<td>141</td>
<td>23.03%</td>
</tr>
<tr>
<td>McDonald’s</td>
<td>MCD</td>
<td>163,745</td>
<td>126,156</td>
<td>290,102</td>
<td>138</td>
<td>26.05%</td>
</tr>
<tr>
<td>3M</td>
<td>MMM</td>
<td>176,339</td>
<td>131,127</td>
<td>307,466</td>
<td>135</td>
<td>24.82%</td>
</tr>
<tr>
<td>Merck</td>
<td>MRK</td>
<td>160,622</td>
<td>120,662</td>
<td>281,284</td>
<td>134</td>
<td>27.68%</td>
</tr>
<tr>
<td>Microsoft</td>
<td>MSFT</td>
<td>138,523</td>
<td>106,266</td>
<td>244,789</td>
<td>140</td>
<td>30.69%</td>
</tr>
<tr>
<td>Pfizer</td>
<td>PFE</td>
<td>145,288</td>
<td>112,480</td>
<td>258,118</td>
<td>141</td>
<td>28.63%</td>
</tr>
<tr>
<td>Procter &amp; Gamble</td>
<td>PG</td>
<td>186,969</td>
<td>137,111</td>
<td>324,080</td>
<td>139</td>
<td>22.12%</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>T</td>
<td>174,932</td>
<td>123,359</td>
<td>298,291</td>
<td>135</td>
<td>25.85%</td>
</tr>
<tr>
<td>United Technologies</td>
<td>UTX</td>
<td>166,534</td>
<td>126,111</td>
<td>292,645</td>
<td>134</td>
<td>26.64%</td>
</tr>
<tr>
<td>Verizon</td>
<td>VZ</td>
<td>167,457</td>
<td>117,498</td>
<td>284,955</td>
<td>138</td>
<td>26.02%</td>
</tr>
<tr>
<td>Walmart</td>
<td>WMT</td>
<td>165,015</td>
<td>127,833</td>
<td>292,848</td>
<td>138</td>
<td>25.74%</td>
</tr>
<tr>
<td>Exxon Mobil</td>
<td>XOM</td>
<td>177,667</td>
<td>133,517</td>
<td>311,184</td>
<td>137</td>
<td>24.07%</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>157,111</td>
<td>118,889</td>
<td>275,999</td>
<td>135</td>
<td>28.52%</td>
</tr>
<tr>
<td>Minimum</td>
<td></td>
<td>123,444</td>
<td>98,783</td>
<td>222,227</td>
<td>128</td>
<td>21.83%</td>
</tr>
<tr>
<td>Maximum</td>
<td></td>
<td>189,496</td>
<td>137,546</td>
<td>327,042</td>
<td>141</td>
<td>36.92%</td>
</tr>
</tbody>
</table>

Note to Table: This table reports the number of available call and put options for index and for each firm in our sample. Our sample contains options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM) and the average Black-Scholes implied volatility (Avg IV) of available contracts.
Table 4: Data Sample Summary - Call Options

<table>
<thead>
<tr>
<th>Ticker</th>
<th>Avg Price</th>
<th>Min Price</th>
<th>Max Price</th>
<th>Avg IV</th>
<th>Min. IV</th>
<th>Max IV</th>
<th>Avg Delta</th>
<th>Avg Vega</th>
<th>Avg DTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPX</td>
<td>35.59</td>
<td>1.876</td>
<td>195.53</td>
<td>20.64%</td>
<td>7.03%</td>
<td>74.98%</td>
<td>0.373</td>
<td>251.02</td>
<td>143</td>
</tr>
<tr>
<td>AA</td>
<td>2.256</td>
<td>0.110</td>
<td>14.121</td>
<td>34.20%</td>
<td>16.93%</td>
<td>153.65%</td>
<td>0.442</td>
<td>8.385</td>
<td>130</td>
</tr>
<tr>
<td>AXP</td>
<td>3.218</td>
<td>0.375</td>
<td>27.372</td>
<td>30.28%</td>
<td>12.72%</td>
<td>148.17%</td>
<td>0.436</td>
<td>13.612</td>
<td>133</td>
</tr>
<tr>
<td>BA</td>
<td>3.022</td>
<td>0.375</td>
<td>14.928</td>
<td>29.57%</td>
<td>16.06%</td>
<td>89.57%</td>
<td>0.429</td>
<td>13.062</td>
<td>131</td>
</tr>
<tr>
<td>CAT</td>
<td>3.351</td>
<td>0.376</td>
<td>15.375</td>
<td>30.98%</td>
<td>16.01%</td>
<td>103.28%</td>
<td>0.432</td>
<td>13.882</td>
<td>131</td>
</tr>
<tr>
<td>CSCO</td>
<td>2.364</td>
<td>0.093</td>
<td>32.268</td>
<td>35.87%</td>
<td>15.93%</td>
<td>107.08%</td>
<td>0.441</td>
<td>7.251</td>
<td>129</td>
</tr>
<tr>
<td>CVX</td>
<td>3.196</td>
<td>0.375</td>
<td>15.509</td>
<td>23.45%</td>
<td>12.79%</td>
<td>94.43%</td>
<td>0.416</td>
<td>16.718</td>
<td>137</td>
</tr>
<tr>
<td>DD</td>
<td>2.319</td>
<td>0.375</td>
<td>13.407</td>
<td>26.25%</td>
<td>12.90%</td>
<td>92.26%</td>
<td>0.427</td>
<td>10.961</td>
<td>136</td>
</tr>
<tr>
<td>DIS</td>
<td>1.899</td>
<td>0.375</td>
<td>17.498</td>
<td>28.56%</td>
<td>6.95%</td>
<td>95.86%</td>
<td>0.441</td>
<td>8.422</td>
<td>139</td>
</tr>
<tr>
<td>GE</td>
<td>2.385</td>
<td>0.375</td>
<td>27.865</td>
<td>26.38%</td>
<td>6.90%</td>
<td>148.93%</td>
<td>0.438</td>
<td>10.855</td>
<td>143</td>
</tr>
<tr>
<td>HD</td>
<td>2.215</td>
<td>0.375</td>
<td>15.933</td>
<td>29.72%</td>
<td>14.84%</td>
<td>100.91%</td>
<td>0.435</td>
<td>9.111</td>
<td>136</td>
</tr>
<tr>
<td>HPQ</td>
<td>2.869</td>
<td>0.375</td>
<td>46.162</td>
<td>34.47%</td>
<td>15.32%</td>
<td>97.89%</td>
<td>0.445</td>
<td>9.303</td>
<td>132</td>
</tr>
<tr>
<td>IBM</td>
<td>4.976</td>
<td>0.361</td>
<td>36.790</td>
<td>25.83%</td>
<td>11.93%</td>
<td>86.82%</td>
<td>0.416</td>
<td>23.901</td>
<td>136</td>
</tr>
<tr>
<td>INTC</td>
<td>2.946</td>
<td>0.375</td>
<td>28.764</td>
<td>35.20%</td>
<td>17.34%</td>
<td>90.86%</td>
<td>0.455</td>
<td>9.389</td>
<td>136</td>
</tr>
<tr>
<td>JNJ</td>
<td>2.391</td>
<td>0.375</td>
<td>14.911</td>
<td>20.44%</td>
<td>9.66%</td>
<td>70.84%</td>
<td>0.409</td>
<td>14.260</td>
<td>142</td>
</tr>
<tr>
<td>JPM</td>
<td>2.759</td>
<td>0.131</td>
<td>19.016</td>
<td>30.02%</td>
<td>11.19%</td>
<td>160.94%</td>
<td>0.431</td>
<td>11.158</td>
<td>133</td>
</tr>
<tr>
<td>KO</td>
<td>2.080</td>
<td>0.375</td>
<td>10.651</td>
<td>21.73%</td>
<td>8.27%</td>
<td>69.30%</td>
<td>0.416</td>
<td>11.767</td>
<td>143</td>
</tr>
<tr>
<td>MCD</td>
<td>2.008</td>
<td>0.375</td>
<td>13.560</td>
<td>24.80%</td>
<td>11.58%</td>
<td>78.87%</td>
<td>0.429</td>
<td>10.308</td>
<td>139</td>
</tr>
<tr>
<td>MMM</td>
<td>3.608</td>
<td>0.375</td>
<td>17.730</td>
<td>23.66%</td>
<td>12.51%</td>
<td>79.62%</td>
<td>0.413</td>
<td>18.890</td>
<td>136</td>
</tr>
<tr>
<td>MRK</td>
<td>2.797</td>
<td>0.375</td>
<td>23.758</td>
<td>26.56%</td>
<td>14.29%</td>
<td>85.20%</td>
<td>0.432</td>
<td>12.354</td>
<td>136</td>
</tr>
<tr>
<td>MSFT</td>
<td>3.143</td>
<td>0.375</td>
<td>29.554</td>
<td>29.44%</td>
<td>12.22%</td>
<td>87.86%</td>
<td>0.450</td>
<td>11.448</td>
<td>141</td>
</tr>
<tr>
<td>PFE</td>
<td>2.175</td>
<td>0.375</td>
<td>22.262</td>
<td>27.57%</td>
<td>14.20%</td>
<td>100.98%</td>
<td>0.441</td>
<td>8.982</td>
<td>143</td>
</tr>
<tr>
<td>PG</td>
<td>2.770</td>
<td>0.375</td>
<td>19.779</td>
<td>20.77%</td>
<td>9.28%</td>
<td>64.34%</td>
<td>0.409</td>
<td>16.262</td>
<td>142</td>
</tr>
<tr>
<td>T</td>
<td>1.611</td>
<td>0.075</td>
<td>9.373</td>
<td>24.41%</td>
<td>10.04%</td>
<td>82.25%</td>
<td>0.432</td>
<td>7.657</td>
<td>137</td>
</tr>
<tr>
<td>UTX</td>
<td>3.247</td>
<td>0.375</td>
<td>22.284</td>
<td>25.34%</td>
<td>13.16%</td>
<td>82.34%</td>
<td>0.417</td>
<td>16.273</td>
<td>135</td>
</tr>
<tr>
<td>VZ</td>
<td>2.078</td>
<td>0.375</td>
<td>12.448</td>
<td>24.58%</td>
<td>9.22%</td>
<td>86.98%</td>
<td>0.444</td>
<td>9.779</td>
<td>141</td>
</tr>
<tr>
<td>WMT</td>
<td>2.199</td>
<td>0.375</td>
<td>17.836</td>
<td>24.52%</td>
<td>11.16%</td>
<td>67.26%</td>
<td>0.418</td>
<td>11.103</td>
<td>140</td>
</tr>
<tr>
<td>XOM</td>
<td>2.688</td>
<td>0.375</td>
<td>15.079</td>
<td>22.92%</td>
<td>12.58%</td>
<td>84.79%</td>
<td>0.414</td>
<td>14.474</td>
<td>139</td>
</tr>
<tr>
<td>Avg.</td>
<td>2.688</td>
<td>0.334</td>
<td>20.527</td>
<td>27.32%</td>
<td>12.42%</td>
<td>96.71%</td>
<td>0.430</td>
<td>12.169</td>
<td>137</td>
</tr>
</tbody>
</table>

Note to Table: This table reports the number of available call option contracts for the index and for each firm in our sample. Our sample contains call options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM), the average Black-Scholes implied volatility (Avg IV), the average Black-Scholes delta (Avg Delta), and the average Black-Scholes vega (Avg Vega) of available contracts.
Table 5: Data Sample Summary - Put Options

<table>
<thead>
<tr>
<th>Ticker</th>
<th>Avg Price</th>
<th>Min Price</th>
<th>Max Price</th>
<th>Avg IV</th>
<th>Min IV</th>
<th>Max IV</th>
<th>Avg Delta</th>
<th>Avg Vega</th>
<th>Avg DTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPX</td>
<td>32.11</td>
<td>2.640</td>
<td>195.53</td>
<td>24.34%</td>
<td>8.90%</td>
<td>82.74%</td>
<td>-0.292</td>
<td>227.67</td>
<td>136</td>
</tr>
<tr>
<td>AA</td>
<td>1.908</td>
<td>0.110</td>
<td>14.121</td>
<td>36.13%</td>
<td>17.39%</td>
<td>159.25%</td>
<td>-0.342</td>
<td>7.840</td>
<td>129</td>
</tr>
<tr>
<td>AXP</td>
<td>2.821</td>
<td>0.375</td>
<td>27.372</td>
<td>32.95%</td>
<td>12.20%</td>
<td>149.37%</td>
<td>-0.340</td>
<td>11.851</td>
<td>130</td>
</tr>
<tr>
<td>BA</td>
<td>2.604</td>
<td>0.375</td>
<td>14.928</td>
<td>31.47%</td>
<td>17.43%</td>
<td>93.33%</td>
<td>-0.339</td>
<td>12.114</td>
<td>130</td>
</tr>
<tr>
<td>CAT</td>
<td>2.981</td>
<td>0.376</td>
<td>15.375</td>
<td>33.11%</td>
<td>17.86%</td>
<td>104.41%</td>
<td>-0.340</td>
<td>12.959</td>
<td>130</td>
</tr>
<tr>
<td>CSCO</td>
<td>2.120</td>
<td>0.093</td>
<td>32.268</td>
<td>37.97%</td>
<td>16.34%</td>
<td>112.08%</td>
<td>-0.351</td>
<td>6.862</td>
<td>128</td>
</tr>
<tr>
<td>CVX</td>
<td>2.754</td>
<td>0.375</td>
<td>15.509</td>
<td>25.67%</td>
<td>11.68%</td>
<td>98.59%</td>
<td>-0.327</td>
<td>15.499</td>
<td>134</td>
</tr>
<tr>
<td>DD</td>
<td>1.978</td>
<td>0.375</td>
<td>13.407</td>
<td>28.61%</td>
<td>13.70%</td>
<td>94.19%</td>
<td>-0.333</td>
<td>10.133</td>
<td>133</td>
</tr>
<tr>
<td>DIS</td>
<td>1.618</td>
<td>0.375</td>
<td>17.498</td>
<td>31.11%</td>
<td>14.31%</td>
<td>99.48%</td>
<td>-0.343</td>
<td>7.738</td>
<td>137</td>
</tr>
<tr>
<td>GE</td>
<td>2.018</td>
<td>0.375</td>
<td>27.865</td>
<td>29.09%</td>
<td>7.10%</td>
<td>149.59%</td>
<td>-0.337</td>
<td>10.048</td>
<td>140</td>
</tr>
<tr>
<td>HD</td>
<td>1.946</td>
<td>0.375</td>
<td>15.933</td>
<td>32.12%</td>
<td>14.03%</td>
<td>103.50%</td>
<td>-0.343</td>
<td>8.508</td>
<td>133</td>
</tr>
<tr>
<td>HPQ</td>
<td>2.368</td>
<td>0.375</td>
<td>46.162</td>
<td>36.25%</td>
<td>16.45%</td>
<td>94.06%</td>
<td>-0.350</td>
<td>8.721</td>
<td>129</td>
</tr>
<tr>
<td>IBM</td>
<td>4.535</td>
<td>0.361</td>
<td>36.790</td>
<td>28.35%</td>
<td>12.38%</td>
<td>90.96%</td>
<td>-0.336</td>
<td>22.422</td>
<td>134</td>
</tr>
<tr>
<td>INTC</td>
<td>2.596</td>
<td>0.375</td>
<td>28.764</td>
<td>36.97%</td>
<td>16.35%</td>
<td>92.03%</td>
<td>-0.353</td>
<td>9.103</td>
<td>134</td>
</tr>
<tr>
<td>JNJ</td>
<td>2.081</td>
<td>0.375</td>
<td>14.911</td>
<td>23.22%</td>
<td>9.61%</td>
<td>77.42%</td>
<td>-0.327</td>
<td>13.112</td>
<td>137</td>
</tr>
<tr>
<td>JPM</td>
<td>2.471</td>
<td>0.131</td>
<td>19.016</td>
<td>33.19%</td>
<td>11.99%</td>
<td>169.06%</td>
<td>-0.337</td>
<td>10.568</td>
<td>131</td>
</tr>
<tr>
<td>KO</td>
<td>1.827</td>
<td>0.375</td>
<td>10.651</td>
<td>24.34%</td>
<td>9.52%</td>
<td>67.51%</td>
<td>-0.330</td>
<td>10.878</td>
<td>139</td>
</tr>
<tr>
<td>MCD</td>
<td>1.727</td>
<td>0.375</td>
<td>13.560</td>
<td>27.30%</td>
<td>12.47%</td>
<td>74.29%</td>
<td>-0.336</td>
<td>9.455</td>
<td>136</td>
</tr>
<tr>
<td>MMM</td>
<td>3.175</td>
<td>0.375</td>
<td>17.730</td>
<td>25.99%</td>
<td>13.82%</td>
<td>86.39%</td>
<td>-0.329</td>
<td>17.609</td>
<td>134</td>
</tr>
<tr>
<td>MRK</td>
<td>2.316</td>
<td>0.375</td>
<td>23.758</td>
<td>28.80%</td>
<td>9.07%</td>
<td>88.64%</td>
<td>-0.334</td>
<td>11.504</td>
<td>132</td>
</tr>
<tr>
<td>MSFT</td>
<td>2.821</td>
<td>0.375</td>
<td>29.554</td>
<td>31.94%</td>
<td>11.20%</td>
<td>94.44%</td>
<td>-0.349</td>
<td>11.241</td>
<td>139</td>
</tr>
<tr>
<td>PFE</td>
<td>1.864</td>
<td>0.375</td>
<td>22.262</td>
<td>29.68%</td>
<td>13.95%</td>
<td>75.78%</td>
<td>-0.343</td>
<td>8.501</td>
<td>140</td>
</tr>
<tr>
<td>PG</td>
<td>2.435</td>
<td>0.375</td>
<td>19.779</td>
<td>23.47%</td>
<td>9.58%</td>
<td>74.12%</td>
<td>-0.327</td>
<td>15.103</td>
<td>137</td>
</tr>
<tr>
<td>T</td>
<td>1.400</td>
<td>0.075</td>
<td>9.373</td>
<td>27.30%</td>
<td>10.25%</td>
<td>86.45%</td>
<td>-0.334</td>
<td>7.206</td>
<td>134</td>
</tr>
<tr>
<td>UTX</td>
<td>2.904</td>
<td>0.375</td>
<td>22.284</td>
<td>27.94%</td>
<td>13.62%</td>
<td>87.87%</td>
<td>-0.333</td>
<td>15.167</td>
<td>133</td>
</tr>
<tr>
<td>VZ</td>
<td>1.728</td>
<td>0.375</td>
<td>12.448</td>
<td>27.45%</td>
<td>10.94%</td>
<td>89.81%</td>
<td>-0.330</td>
<td>9.118</td>
<td>135</td>
</tr>
<tr>
<td>WMN</td>
<td>1.979</td>
<td>0.375</td>
<td>17.836</td>
<td>26.97%</td>
<td>11.44%</td>
<td>72.69%</td>
<td>-0.353</td>
<td>10.324</td>
<td>136</td>
</tr>
<tr>
<td>XOM</td>
<td>2.309</td>
<td>0.375</td>
<td>15.079</td>
<td>25.22%</td>
<td>12.79%</td>
<td>97.18%</td>
<td>-0.329</td>
<td>13.299</td>
<td>136</td>
</tr>
</tbody>
</table>

| Avg.   | 2.344     | 0.334     | 20.527    | 29.73% | 12.87% | 99.35% | -0.337    | 11.366  | 134    |

Note to Table: This table reports the number of available put option contracts for the index and for each firm in our sample. Our sample contains put options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM), the average Black-Scholes implied volatility (Avg IV), the average Black-Scholes delta (Avg Delta), and the average Black-Scholes vega (Avg Vega) of available contracts.
Table 6: Market Parameter Estimates

Panel A: Parameter Estimates (Physical) - Joint Estimation

<table>
<thead>
<tr>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4271</td>
<td>3.5874</td>
<td>0.0026</td>
<td>0.0171</td>
<td>0.0855</td>
<td>0.3496</td>
<td>-0.6918</td>
<td>-0.2173</td>
<td>-1.0798</td>
<td>-1.0355</td>
</tr>
</tbody>
</table>

Panel B: Parameter Estimates (Risk Neutral) - Options-based Estimation

<table>
<thead>
<tr>
<th>$\tilde{\kappa}_1$</th>
<th>$\tilde{\kappa}_2$</th>
<th>$\tilde{\theta}_1$</th>
<th>$\tilde{\theta}_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2267</td>
<td>2.9137</td>
<td>0.0590</td>
<td>0.0100</td>
<td>0.0958</td>
<td>0.5678</td>
<td>-0.9135</td>
<td>-0.4934</td>
</tr>
</tbody>
</table>

Note to Table: This table reports the structural parameter estimates of the S&P 500 Index for the two-factor stochastic volatility model. The reported results in Panel A are from the joint estimation using the daily S&P 500 index returns and options data. Structural parameters in Panel B are estimated using only options data. In both panels, we use OTM call and put options with moneyness up to 10% over the period 1996-2011. As in Proposition (2), $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1 \theta_1}{k_1 + \lambda_1}$, $\tilde{\theta}_2 = \frac{k_2 \theta_2}{k_2 + \lambda_2}$. Therefore, risk-neutral parameters from the joint estimation are $\tilde{\kappa}_1 = 0.3473$, $\tilde{\kappa}_2 = 2.5520$, $\tilde{\theta}_1 = 0.0106$, $\tilde{\theta}_2 = 0.0240$.  

64
### Table 7: Individual Equity Parameter Estimates

<table>
<thead>
<tr>
<th>Company</th>
<th>Ticker</th>
<th>$\bar{\kappa}$</th>
<th>$\bar{\theta}$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alcoa</td>
<td>AA</td>
<td>0.7253</td>
<td>0.0202</td>
<td>0.1612</td>
<td>-0.87</td>
<td>0.3850</td>
<td>1.3159</td>
</tr>
<tr>
<td>American Express</td>
<td>AXP</td>
<td>0.7663</td>
<td>0.0128</td>
<td>0.1009</td>
<td>-0.91</td>
<td>0.3430</td>
<td>1.3203</td>
</tr>
<tr>
<td>Boeing</td>
<td>BA</td>
<td>0.7692</td>
<td>0.0235</td>
<td>0.1757</td>
<td>-0.97</td>
<td>0.4108</td>
<td>1.3046</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>CAT</td>
<td>0.6354</td>
<td>0.0291</td>
<td>0.1984</td>
<td>-0.84</td>
<td>0.3608</td>
<td>1.3215</td>
</tr>
<tr>
<td>Cisco</td>
<td>CSCO</td>
<td>0.6804</td>
<td>0.0653</td>
<td>0.3599</td>
<td>-0.81</td>
<td>0.4420</td>
<td>1.2508</td>
</tr>
<tr>
<td>Chevron</td>
<td>CVX</td>
<td>0.9390</td>
<td>0.0097</td>
<td>0.0913</td>
<td>-0.88</td>
<td>0.5816</td>
<td>1.1538</td>
</tr>
<tr>
<td>Dupont</td>
<td>DD</td>
<td>0.8702</td>
<td>0.0137</td>
<td>0.1310</td>
<td>-0.92</td>
<td>0.4949</td>
<td>1.2888</td>
</tr>
<tr>
<td>Disney</td>
<td>DIS</td>
<td>0.6995</td>
<td>0.0247</td>
<td>0.1841</td>
<td>-0.89</td>
<td>0.4962</td>
<td>1.2854</td>
</tr>
<tr>
<td>General Electric</td>
<td>GE</td>
<td>0.5694</td>
<td>0.0093</td>
<td>0.0670</td>
<td>-0.85</td>
<td>0.4968</td>
<td>1.3111</td>
</tr>
<tr>
<td>Home Depot</td>
<td>HD</td>
<td>0.6912</td>
<td>0.0340</td>
<td>0.2379</td>
<td>-0.83</td>
<td>0.4278</td>
<td>1.3097</td>
</tr>
<tr>
<td>Hewlett-Packard</td>
<td>HPQ</td>
<td>0.6159</td>
<td>0.0756</td>
<td>0.3967</td>
<td>-0.64</td>
<td>0.4432</td>
<td>1.2458</td>
</tr>
<tr>
<td>IBM</td>
<td>IBM</td>
<td>0.7717</td>
<td>0.0186</td>
<td>0.1676</td>
<td>-0.78</td>
<td>0.6798</td>
<td>1.2853</td>
</tr>
<tr>
<td>Intel</td>
<td>INTC</td>
<td>0.8160</td>
<td>0.0295</td>
<td>0.2123</td>
<td>-0.84</td>
<td>0.4322</td>
<td>1.2652</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>JNJ</td>
<td>0.6492</td>
<td>0.0238</td>
<td>0.2015</td>
<td>-0.95</td>
<td>0.5574</td>
<td>1.0197</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>JPM</td>
<td>0.8606</td>
<td>0.0193</td>
<td>0.1836</td>
<td>-0.99</td>
<td>0.4483</td>
<td>1.3466</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>KO</td>
<td>0.3920</td>
<td>0.0291</td>
<td>0.1895</td>
<td>-0.87</td>
<td>0.6077</td>
<td>1.0897</td>
</tr>
<tr>
<td>McDonald’s</td>
<td>MCD</td>
<td>0.9305</td>
<td>0.0262</td>
<td>0.2109</td>
<td>-0.97</td>
<td>0.4754</td>
<td>1.1359</td>
</tr>
<tr>
<td>3M</td>
<td>MMM</td>
<td>1.7078</td>
<td>0.0107</td>
<td>0.1569</td>
<td>-0.86</td>
<td>0.5886</td>
<td>1.1752</td>
</tr>
<tr>
<td>Merck</td>
<td>MRK</td>
<td>1.2259</td>
<td>0.0105</td>
<td>0.1073</td>
<td>-0.89</td>
<td>0.5018</td>
<td>1.2276</td>
</tr>
<tr>
<td>Microsoft</td>
<td>MSFT</td>
<td>0.7777</td>
<td>0.0108</td>
<td>0.0710</td>
<td>-0.81</td>
<td>0.4513</td>
<td>1.2739</td>
</tr>
<tr>
<td>Pfizer</td>
<td>PFE</td>
<td>0.8957</td>
<td>0.0210</td>
<td>0.1724</td>
<td>-0.88</td>
<td>0.5067</td>
<td>1.2166</td>
</tr>
<tr>
<td>Procter &amp; Gamble</td>
<td>PG</td>
<td>0.5107</td>
<td>0.0470</td>
<td>0.3056</td>
<td>-0.85</td>
<td>0.5782</td>
<td>1.0125</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>T</td>
<td>0.6972</td>
<td>0.0098</td>
<td>0.0830</td>
<td>-0.93</td>
<td>0.5116</td>
<td>1.2126</td>
</tr>
<tr>
<td>United Technologies</td>
<td>UTX</td>
<td>0.9778</td>
<td>0.0271</td>
<td>0.2606</td>
<td>-0.83</td>
<td>0.5221</td>
<td>1.2668</td>
</tr>
<tr>
<td>Verizon</td>
<td>VZ</td>
<td>0.8423</td>
<td>0.0102</td>
<td>0.0970</td>
<td>0.51</td>
<td>0.4719</td>
<td>1.1838</td>
</tr>
<tr>
<td>Walmart</td>
<td>WMT</td>
<td>0.6533</td>
<td>0.0314</td>
<td>0.2136</td>
<td>-0.86</td>
<td>0.4695</td>
<td>1.1724</td>
</tr>
<tr>
<td>Exxon Mobil</td>
<td>XOM</td>
<td>1.0785</td>
<td>0.0148</td>
<td>0.1849</td>
<td>-0.94</td>
<td>0.5925</td>
<td>1.1764</td>
</tr>
</tbody>
</table>

**Average**: 0.8055 0.0244 0.1823 -0.820 0.4899 1.2284

**Min**: 0.3920 0.0093 0.0670 -0.990 0.3430 1.0125

**Max**: 1.7078 0.0756 0.3967 0.512 0.6798 1.3466

Note to Table: This table reports the risk-neutral structural parameter estimates for individual equities conditional on the structural parameters of the S&P 500 index and the vectors of filtered spot market variance components. This table also reports the persistent beta $\beta_1^i$ and the transient beta $\beta_2^i$ for individual equity $i$. The market parameters and spot variance components are estimated using OTM call and put options over the period 1996-2011 with moneyness up to 10%. For individual equities, we use OTM call and put options with moneyness up to 10% over the period 1996-2011, where we drop the first five months.
<table>
<thead>
<tr>
<th>Company</th>
<th>Ticker</th>
<th>Mean</th>
<th>Std dev</th>
<th>Max</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alcoa</td>
<td>AA</td>
<td>0.1259</td>
<td>0.1387</td>
<td>0.6879</td>
<td>0.0900</td>
</tr>
<tr>
<td>American Express</td>
<td>AXP</td>
<td>0.1068</td>
<td>0.1489</td>
<td>0.7138</td>
<td>0.0692</td>
</tr>
<tr>
<td>Boeing</td>
<td>BA</td>
<td>0.0633</td>
<td>0.0442</td>
<td>0.2484</td>
<td>0.0521</td>
</tr>
<tr>
<td>Caterpillar</td>
<td>CAT</td>
<td>0.0783</td>
<td>0.0628</td>
<td>0.4395</td>
<td>0.0587</td>
</tr>
<tr>
<td>Cisco</td>
<td>CSCO</td>
<td>0.1497</td>
<td>0.1328</td>
<td>0.8274</td>
<td>0.0987</td>
</tr>
<tr>
<td>Chevron</td>
<td>CVX</td>
<td>0.0293</td>
<td>0.0267</td>
<td>0.2126</td>
<td>0.0260</td>
</tr>
<tr>
<td>Dupont</td>
<td>DD</td>
<td>0.0460</td>
<td>0.0476</td>
<td>0.2526</td>
<td>0.0292</td>
</tr>
<tr>
<td>Disney</td>
<td>DIS</td>
<td>0.0636</td>
<td>0.0515</td>
<td>0.2661</td>
<td>0.0460</td>
</tr>
<tr>
<td>General Electric</td>
<td>GE</td>
<td>0.0618</td>
<td>0.0938</td>
<td>0.6134</td>
<td>0.0413</td>
</tr>
<tr>
<td>Home Depot</td>
<td>HD</td>
<td>0.0741</td>
<td>0.0600</td>
<td>0.3230</td>
<td>0.0510</td>
</tr>
<tr>
<td>Hewlett-Packard</td>
<td>HPQ</td>
<td>0.1250</td>
<td>0.1231</td>
<td>0.4893</td>
<td>0.0903</td>
</tr>
<tr>
<td>IBM</td>
<td>IBM</td>
<td>0.0439</td>
<td>0.0482</td>
<td>0.2620</td>
<td>0.0260</td>
</tr>
<tr>
<td>Intel</td>
<td>INTC</td>
<td>0.1206</td>
<td>0.0882</td>
<td>0.6408</td>
<td>0.0927</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>JNJ</td>
<td>0.0225</td>
<td>0.0257</td>
<td>0.2340</td>
<td>0.0116</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>JPM</td>
<td>0.1070</td>
<td>0.1325</td>
<td>0.9138</td>
<td>0.0786</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>KO</td>
<td>0.0268</td>
<td>0.0308</td>
<td>0.1729</td>
<td>0.0133</td>
</tr>
<tr>
<td>McDonald’s</td>
<td>MCD</td>
<td>0.0389</td>
<td>0.0345</td>
<td>0.1638</td>
<td>0.0277</td>
</tr>
<tr>
<td>3M</td>
<td>MMM</td>
<td>0.0297</td>
<td>0.0304</td>
<td>0.1645</td>
<td>0.0180</td>
</tr>
<tr>
<td>Merck</td>
<td>MRK</td>
<td>0.0438</td>
<td>0.0367</td>
<td>0.2189</td>
<td>0.0358</td>
</tr>
<tr>
<td>Microsoft</td>
<td>MSFT</td>
<td>0.0749</td>
<td>0.0614</td>
<td>0.4605</td>
<td>0.0647</td>
</tr>
<tr>
<td>Pfizer</td>
<td>PFE</td>
<td>0.0490</td>
<td>0.0425</td>
<td>0.2021</td>
<td>0.0356</td>
</tr>
<tr>
<td>Procter &amp; Gamble</td>
<td>PG</td>
<td>0.0256</td>
<td>0.0326</td>
<td>0.2411</td>
<td>0.0103</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>T</td>
<td>0.0522</td>
<td>0.0532</td>
<td>0.5365</td>
<td>0.0359</td>
</tr>
<tr>
<td>United Technologies</td>
<td>UTX</td>
<td>0.0399</td>
<td>0.0374</td>
<td>0.2126</td>
<td>0.0258</td>
</tr>
<tr>
<td>Verizon</td>
<td>VZ</td>
<td>0.0428</td>
<td>0.0438</td>
<td>0.3520</td>
<td>0.0280</td>
</tr>
<tr>
<td>Walmart</td>
<td>WMT</td>
<td>0.0436</td>
<td>0.0550</td>
<td>0.2870</td>
<td>0.0193</td>
</tr>
<tr>
<td>Exxon Mobil</td>
<td>XOM</td>
<td>0.0234</td>
<td>0.0210</td>
<td>0.1556</td>
<td>0.0204</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>0.0633</td>
<td>0.0631</td>
<td>0.3812</td>
<td>0.0443</td>
</tr>
<tr>
<td>Minimum</td>
<td></td>
<td>0.0225</td>
<td>0.0210</td>
<td>0.1556</td>
<td>0.0103</td>
</tr>
<tr>
<td>Maximum</td>
<td></td>
<td>0.1497</td>
<td>0.1489</td>
<td>0.9138</td>
<td>0.0987</td>
</tr>
</tbody>
</table>

Note to Table: This table reports the mean, median, standard deviation, and maximum of spot idiosyncratic variance for every firm \( i \) conditional on the structural parameters of the S&P 500 index and filtered spot market variance components. The reported results are based on OTM call and put index option and individual equity option contracts with moneyness up to 10% over the period 1996-2011.
### Table 9: Goodness of Fit

<table>
<thead>
<tr>
<th>Panel A: Goodness of Fit - Call Option Contracts</th>
<th>Option Based Estimation</th>
<th>Joint Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTM $\leq$ 30</td>
<td></td>
<td>2.7171</td>
</tr>
<tr>
<td>30 $&lt;$ DTM $\leq$ 91</td>
<td>28,640</td>
<td>1.2956</td>
</tr>
<tr>
<td>91 $&lt;$ DTM $\leq$ 182</td>
<td>59,366</td>
<td>0.8695</td>
</tr>
<tr>
<td>DTM $&gt;$ 182</td>
<td>81,220</td>
<td>0.6913</td>
</tr>
<tr>
<td>All</td>
<td>38,872</td>
<td>0.8943</td>
</tr>
<tr>
<td></td>
<td>208,098</td>
<td>0.8846</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9132</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.4244</td>
</tr>
</tbody>
</table>

### Panel B: Goodness of Fit - Put Option Contracts

| DTM $\leq$ 30                                   |                         | 2.8857           |
| 30 $<$ DTM $\leq$ 91                            | 23,271                  | 1.6193           |
| 91 $<$ DTM $\leq$ 182                           | 41,040                  | 1.0712           |
| DTM $>$ 182                                     | 49,576                  | 0.8342           |
| All                                             | 23,725                  | 1.0440           |
|                                                 | 137,612                 | 1.1064           |
|                                                 |                         | 1.1167           |
|                                                 |                         | 4.5879           |

### Panel C: Goodness of Fit - All Option Contracts

| DTM $\leq$ 30                                   |                         | 2.7946           |
| 30 $<$ DTM $\leq$ 91                            | 51,911                  | 1.4497           |
| 91 $<$ DTM $\leq$ 182                           | 100,406                 | 0.9571           |
| DTM $>$ 182                                     | 130,796                 | 0.7486           |
| All                                             | 62,597                  | 0.9538           |
|                                                 | 345,710                 | 0.9790           |
|                                                 |                         | 0.9992           |
|                                                 |                         | 4.4428           |

Note to Table: This table reports goodness-of-fit statistics for individual equity options. In-sample statistics are computed using options over the entire sample, 1996-2011. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ratio of IVRMSE over the average Black-Scholes implied volatility. We also report out-of-sample Vega RMSE over the period 2004-2011, given the in-sample parameter estimates, market spot variance components, and spot idiosyncratic variance over the period 1996-2003.
Table 10: Subsample Parameter Estimates

<table>
<thead>
<tr>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2138</td>
<td>3.2780</td>
<td>0.0033</td>
<td>0.0195</td>
<td>0.0855</td>
<td>0.3220</td>
<td>-0.6514</td>
<td>-0.2985</td>
<td>-1.1008</td>
<td>-0.9755</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.1274</td>
<td>4.2337</td>
<td>0.0069</td>
<td>0.0289</td>
<td>0.0793</td>
<td>0.4675</td>
<td>-0.5102</td>
<td>-0.3086</td>
<td>-1.0684</td>
<td>-1.0351</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel B: Joint Estimation (2003 - 2011)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1794</td>
<td>2.6176</td>
<td>0.0437</td>
<td>0.0104</td>
<td>0.0912</td>
<td>0.3732</td>
<td>-0.8891</td>
<td>-0.4434</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1117</td>
<td>3.4731</td>
<td>0.0623</td>
<td>0.0247</td>
<td>0.0837</td>
<td>0.6692</td>
<td>-0.7550</td>
<td>-0.6497</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel D: Options-based Estimation (2003-2011)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note to Table: This table reports the structural parameter estimates of the S&P 500 Index for the two-factor stochastic volatility model over two subsample period. The first subsample is from January 1996 to December 2003 and the second one is from January 2004 to December 2011. The point estimates in Panel A and Panel B are from the joint estimation using the daily S&P 500 index returns and options data. Entries in Panel C and Panel D are estimated using only options data. In both panels, we use OTM call and put options with moneyness up to 10% over the period 1996-2011.
Table 11: Subsample Goodness of Fit (1996-2003)

<table>
<thead>
<tr>
<th>Option Based Estimation</th>
<th>Joint Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Obs.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>DTM≤30</td>
<td>14,267</td>
</tr>
<tr>
<td>30&lt;DTM≤91</td>
<td>30,414</td>
</tr>
<tr>
<td>91&lt;DTM≤182</td>
<td>39,160</td>
</tr>
<tr>
<td>DTM&gt;182</td>
<td>18,237</td>
</tr>
<tr>
<td>All</td>
<td>102,078</td>
</tr>
<tr>
<td>Panel B: Subsample Goodness of Fit (1996-2003) - Put Option Contracts</td>
<td></td>
</tr>
<tr>
<td>DTM≤30</td>
<td>11,775</td>
</tr>
<tr>
<td>30&lt;DTM≤91</td>
<td>20,282</td>
</tr>
<tr>
<td>91&lt;DTM≤182</td>
<td>24,137</td>
</tr>
<tr>
<td>DTM&gt;182</td>
<td>11,528</td>
</tr>
<tr>
<td>All</td>
<td>67,722</td>
</tr>
<tr>
<td>Panel C: Subsample Goodness of Fit (1996-2003) - All Option Contracts</td>
<td></td>
</tr>
<tr>
<td>DTM≤30</td>
<td>26,042</td>
</tr>
<tr>
<td>30&lt;DTM≤91</td>
<td>50,696</td>
</tr>
<tr>
<td>91&lt;DTM≤182</td>
<td>63,297</td>
</tr>
<tr>
<td>DTM&gt;182</td>
<td>29,765</td>
</tr>
<tr>
<td>All</td>
<td>169,800</td>
</tr>
</tbody>
</table>

Note to Table: This table reports in-sample goodness-of-fit statistics for our two-factor stochastic volatility model over the entire sample, 1996 through 2011 for various maturities. We also report in-sample fit for calls and puts separately. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ration of IVRMSE over the average implied volatility. To provide a basis for caparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.
Table 12: Subsample Goodness of Fit (2004-2011)

<table>
<thead>
<tr>
<th>Panel A: Subsample Goodness of Fit (2004-2011) - Call Option Contracts</th>
<th>Option Based Estimation</th>
<th>Joint Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Obs.</td>
<td>Vega RMSE</td>
</tr>
<tr>
<td>DTM $\leq$ 30</td>
<td>14,373</td>
<td>1.3526</td>
</tr>
<tr>
<td>30 $&lt;$ DTM $\leq$ 91</td>
<td>28,952</td>
<td>0.8998</td>
</tr>
<tr>
<td>91 $&lt;$ DTM $\leq$ 182</td>
<td>42,060</td>
<td>0.6640</td>
</tr>
<tr>
<td>DTM $&gt;$ 182</td>
<td>20,635</td>
<td>0.9985</td>
</tr>
<tr>
<td>All</td>
<td>106,020</td>
<td>0.9155</td>
</tr>
</tbody>
</table>

Note to Table: This table reports goodness-of-fit statistics for our two-factor stochastic volatility model over the subsample from January 2004 through December 2011 for various maturities. We also report in-sample fit for calls and puts separately. All numbers are in percentage points. We compute Vega-weighted root mean squared error (Vega RMSE) along with implied volatility root mean squared error (IVRMSE). We also report the ration of IVRMSE over the average implied volatility. To provide a basis for comparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.
Table 13: Out of Sample Goodness of Fit (2004-2011)

<table>
<thead>
<tr>
<th>Panel A: Out of Sample Goodness of Fit (2004-2011) - Call Option Contracts</th>
<th>Option Based Estimation</th>
<th>Joint Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Obs.</td>
<td>Vega RMSE</td>
</tr>
<tr>
<td>DTM ≤ 30</td>
<td>14,373</td>
<td>1.4764</td>
</tr>
<tr>
<td>30 &lt; DTM ≤ 91</td>
<td>28,952</td>
<td>0.9372</td>
</tr>
<tr>
<td>91 &lt; DTM ≤ 182</td>
<td>42,060</td>
<td>0.6902</td>
</tr>
<tr>
<td>DTM &gt; 182</td>
<td>20,635</td>
<td>1.0797</td>
</tr>
<tr>
<td>All</td>
<td>106,020</td>
<td>0.9753</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Out of Sample Goodness of Fit (2004-2011) - Put Option Contracts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>DTM ≤ 30</td>
</tr>
<tr>
<td>30 &lt; DTM ≤ 91</td>
</tr>
<tr>
<td>91 &lt; DTM ≤ 182</td>
</tr>
<tr>
<td>DTM &gt; 182</td>
</tr>
<tr>
<td>All</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Out of Sample Goodness of Fit (2004-2011) - All Option Contracts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>DTM ≤ 30</td>
</tr>
<tr>
<td>30 &lt; DTM ≤ 91</td>
</tr>
<tr>
<td>91 &lt; DTM ≤ 182</td>
</tr>
<tr>
<td>DTM &gt; 182</td>
</tr>
<tr>
<td>All</td>
</tr>
</tbody>
</table>

Note to Table: This table reports out-of-sample goodness-of-fit statistics for our two-factor stochastic volatility model over the period from January 2004 through December 2011 for various maturities. We also report out-of-sample fit for calls and puts separately. All numbers are in percentage points. Out-of-sample daily spot persistent and transient variance components are filtered with Particle Filter method given the in-sample structural parameter estimates over the period January 1996 through December 2003. The Vega RMSE along with the IVRMSE are computed given in-sample structural parameters and filtered variance components. We also report the ratio of IVRMSE over the average implied volatility. To provide a basis for comparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.
Table 14: Goodness of Fit - Individual Equities

<table>
<thead>
<tr>
<th>Ticker</th>
<th>Vega RMSE</th>
<th>IV RMSE</th>
<th>IVRMSE/Avg. IV</th>
<th>Vega RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
<td>1.84</td>
<td>1.87</td>
<td>5.32</td>
<td>2.24</td>
</tr>
<tr>
<td>AXP</td>
<td>1.82</td>
<td>1.79</td>
<td>5.66</td>
<td>2.14</td>
</tr>
<tr>
<td>BA</td>
<td>1.41</td>
<td>1.35</td>
<td>4.42</td>
<td>1.97</td>
</tr>
<tr>
<td>CAT</td>
<td>1.50</td>
<td>1.47</td>
<td>4.59</td>
<td>1.68</td>
</tr>
<tr>
<td>CSCO</td>
<td>2.14</td>
<td>2.12</td>
<td>5.74</td>
<td>2.23</td>
</tr>
<tr>
<td>CVX</td>
<td>2.02</td>
<td>1.95</td>
<td>7.94</td>
<td>2.24</td>
</tr>
<tr>
<td>DD</td>
<td>1.42</td>
<td>1.41</td>
<td>5.14</td>
<td>1.53</td>
</tr>
<tr>
<td>DIS</td>
<td>1.75</td>
<td>1.69</td>
<td>5.66</td>
<td>1.97</td>
</tr>
<tr>
<td>GE</td>
<td>1.84</td>
<td>1.86</td>
<td>6.71</td>
<td>1.93</td>
</tr>
<tr>
<td>HD</td>
<td>1.58</td>
<td>1.54</td>
<td>4.98</td>
<td>1.72</td>
</tr>
<tr>
<td>HPQ</td>
<td>1.53</td>
<td>1.53</td>
<td>4.33</td>
<td>1.87</td>
</tr>
<tr>
<td>IBM</td>
<td>1.46</td>
<td>1.42</td>
<td>5.24</td>
<td>1.61</td>
</tr>
<tr>
<td>INTC</td>
<td>1.56</td>
<td>1.58</td>
<td>4.38</td>
<td>1.68</td>
</tr>
<tr>
<td>JNJ</td>
<td>1.42</td>
<td>1.40</td>
<td>6.41</td>
<td>1.65</td>
</tr>
<tr>
<td>JPM</td>
<td>1.85</td>
<td>1.82</td>
<td>5.76</td>
<td>2.08</td>
</tr>
<tr>
<td>KO</td>
<td>1.54</td>
<td>1.46</td>
<td>6.34</td>
<td>1.62</td>
</tr>
<tr>
<td>MCD</td>
<td>1.34</td>
<td>1.33</td>
<td>5.11</td>
<td>1.59</td>
</tr>
<tr>
<td>MMM</td>
<td>1.41</td>
<td>1.39</td>
<td>5.60</td>
<td>1.74</td>
</tr>
<tr>
<td>MRK</td>
<td>1.36</td>
<td>1.41</td>
<td>5.09</td>
<td>1.46</td>
</tr>
<tr>
<td>MSFT</td>
<td>1.67</td>
<td>1.64</td>
<td>5.34</td>
<td>1.75</td>
</tr>
<tr>
<td>PFE</td>
<td>1.49</td>
<td>1.46</td>
<td>5.10</td>
<td>1.73</td>
</tr>
<tr>
<td>PG</td>
<td>1.39</td>
<td>1.37</td>
<td>6.19</td>
<td>1.39</td>
</tr>
<tr>
<td>T</td>
<td>1.98</td>
<td>1.96</td>
<td>7.58</td>
<td>2.21</td>
</tr>
<tr>
<td>UTX</td>
<td>1.48</td>
<td>1.44</td>
<td>5.41</td>
<td>1.54</td>
</tr>
<tr>
<td>VZ</td>
<td>1.56</td>
<td>1.55</td>
<td>5.96</td>
<td>1.59</td>
</tr>
<tr>
<td>WMT</td>
<td>1.57</td>
<td>1.55</td>
<td>6.02</td>
<td>1.76</td>
</tr>
<tr>
<td>XOM</td>
<td>1.66</td>
<td>1.63</td>
<td>6.77</td>
<td>1.82</td>
</tr>
<tr>
<td>Average</td>
<td>1.61</td>
<td>1.59</td>
<td>5.66</td>
<td>1.81</td>
</tr>
</tbody>
</table>

Note to Table: This table reports goodness-of-fit statistics for individual equity options. In-sample results are over the entire sample, 1996 through 2011. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ratio of IVRMSE over the average implied volatility. We also report out-of-sample Vega RMSE over the period of 2004 to 2011, given the in-sample parameter estimates, market variance components, and equity idiosyncratic variance over the period of 1996 to 2003.
Figure 1: Market Delta of Equity Call Options

Panel A

Panel B

Panel C

Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the level of market index for different sets of betas. Panel A shows this sensitivity following the calibration in a one-factor structure model of Christoffersen et al. (2017) while Panels B and C are the sensitivity in our two-factor structure model. Panel B shows market delta when persistent beta is constant and Panel C is market delta when transient beta is constant.
Figure 2: Persistent Market Vega of Equity Call Options

Panel A

Panel B

Panel C

Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the persistent variance component for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows the persistent market vega when transient beta is constant and Panel C is the persistent market vega when persistent beta is constant. Note also that for all the graphs the total unconditional equity variance is fixed, \( \tilde{\sigma}^2 = (\beta_1)^2 \tilde{\theta}_1 + (\beta_2)^2 \tilde{\theta}_2 + \theta_i = 0.11 \).
Figure 3: Transient Market Vega of Equity Call Options

Panel A

Panel B

Panel C

Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the transient variance component for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows the transient market vega when persistent beta is constant and Panel C is the transient market vega when transient beta is constant. Note also that for all the graphs the total unconditional equity variance is fixed, $\tilde{\sigma}^2 = (\beta_1)^2\tilde{\theta}_1 + (\beta_2)^2\tilde{\theta}_2 + \theta^2 = 0.11$. 

75
Figure 4: Persistent and Transient Betas and Implied Volatility Term Structure

Note to Figure: This figure plots the model-implied volatility for at-the-money equity call options with respect to the time-to-maturity for different sets of betas. Panel A shows the term-structure effect following the one-factor structure model and Panel B replicates the same IV structure with our two-factor structure model. Panels C shows IV term structure when persistent beta $\beta^1_i$ is constant and Panel D shows IV term structure when transient beta $\beta^2_i$ is constant. Note that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta^1_i)^2\tilde{\theta}_1 + (\beta^2_i)^2\tilde{\theta}_2 + \tilde{\theta}^i = 0.11$. We also fix the total unconditional risk-neutral market variances to $0.05$, with $\tilde{\theta}_1 = 0.006$ and $\tilde{\theta}_2 = 0.044$. Therefore, the unconditional idiosyncratic equity variance for every set of betas can be defined by $\theta^i = \tilde{v}^i - (\beta^1_i)^2\tilde{\theta}_1 - (\beta^2_i)^2\tilde{\theta}_2$. The spot market variance components are set equal to $v^1_{1,t} = 0.012$ and $v^2_{2,t} = 0.048$ and the total spot equity variance is $v^i_t = 0.05$. Consequently, we define the spot idiosyncratic variance for different sets of betas as $\xi^i_t = v^i_t - (\beta^1_i)^2v^1_{1,t} - (\beta^2_i)^2v^2_{2,t}$. We choose the remaining structural parameters of the market and equity dynamics as follows: $\{\tilde{\kappa}_1 = 0.18, \tilde{\kappa}_2 = 2.8, \sigma_1 = 3.6, \sigma_2 = 0.29, \rho_1 = -0.96, \rho_2 = -0.83\}$ and $\{\tilde{\kappa}^i = 0.8, \sigma^i = 0.2, \rho^i = 0\}$. We keep the risk-free rate at $4\%$ per year and the ratio of spot index price over spot equity price is equal to $S^i_t/S_t = 0.1$. Note that the Y axis is Implied Volatility.
Figure 5: Persistent and Transient Betas and Implied Volatility Across Moneyness

Panel A

Panel B

Panel C

Panel D

Note to Figure: This figure plots the model-implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. Panel A shows the IV moneyness slope following the one-factor structure model and Panel B replicates the same IV moneyness slope with our two-factor structure model. Panels C shows IV moneyness slope when persistent beta $\beta_1$ is constant and Panel D shows IV moneyness slope when transient beta $\beta_2$ is constant. Note that for all the graphs the total unconditional equity variance is fixed at $\tilde{\nu} = (\beta_1)^2\tilde{\theta}_1 + (\beta_2)^2\tilde{\theta}_2 + \theta^2 = 0.11$. Note also that the Y axis is Implied Volatility.
Figure 6: Persistent and Transient Variances Risk Premiums and Implied Volatility Smile

Panel A

Panel B

Panel C

Panel D

Note to Figure: This figure plots the difference between model-implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is the difference between IV when $\lambda_1 = \lambda_2 = -0.5$ and when $\lambda_1 = \lambda_2 = 0$. Panel A shows the effect of market variance risk premium on equity option skew (slope of IV curve) following the calibration in one-factor structure model while Panel B replicates the same effect in our two-factor structure model. Panels C shows IV difference when persistent beta $\beta_1$ is constant and Panel D shows IV difference when transient beta $\beta_2$ is constant. Note that for all the graphs the total unconditional equity variance is fixed, $\tilde{\nu} = (\beta_1^2 \tilde{\theta}_1 + (\beta_2^2 \tilde{\theta}_2 + \theta^i = 0.11$. 

\[ \hat{\nu}_i = \beta_1 \lambda_1 \tilde{\theta}_1 + \beta_2 \lambda_2 \tilde{\theta}_2 + \theta^i = 0.11. \]
Figure 7: The S&P 500 Index Spot Variance Components Paths

Panel A: Persistent Variance Component

Panel B: Transient Variance Component

Note to Figure: We plot time series of risk-neutral spot variances for the S&P 500 index in the two-factor stochastic volatility model. Panel A shows time series of persistent variance component and Panel B shows time series of transient variance component. The blue plots are based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation). The red plots are filtered spot variances using data from S&P 500 option market only.
Figure 8: The S&P 500 Index Total Spot Variance Path Versus VIX

Panel A: Joint Estimation Versus VIX

Panel B: Option-Based Estimation Versus VIX

Note to Figure: We plot time series of risk-neutral total spot variance for the S&P 500 index by combining persistent and transient variance components of the two-factor stochastic volatility model. The blue plots in Panel A is based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation). The blue plot in Panel B is based on data from S&P 500 option market only. Red plots in both panels are time series of the VIX option implied volatility index.