The Information Value of Distress

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January 30, 2020

Abstract

We propose a novel framework for investigating the dynamic feedback loop between firm behavior and the opinion of a competitive debt market, the latter being represented by a rating agency. Observing the survival of apparently distressed periods, the rating agency eliminates estimates of low asset value, which creates a feedback channel in a form of a lower cost of borrowing for the firm. Eventually, the expected default threshold persistently undercuts the true default threshold, leading to an underestimation of the firm’s default risk. In a specific example calibrated to market data, the credit spread decreases by more than 30%.

JEL classification: D83, G24, G33

Keywords: Asymmetric Information, Learning, Feedback Effect

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We propose a novel framework for investigating the dynamic feedback loop between firm behavior and the opinion of a competitive debt market, the latter being represented by a rating agency. The rating agency observes a distorted version of the firm’s current state and maximizes the accuracy of default prediction. The firm’s manager and owner of the private equity perfectly knows the true state and maximizes the firm’s equity value. By not defaulting during apparent distress (i.e., states of low estimated asset value), he signals a higher true asset value to the rating agency, often at the cost of injecting further cash into the distressed firm. Consequently, the rating agency adjusts the credit rating upwards, which in turn reduces the firm’s cost of capital. To complete the feedback loop, the change of the firms’ financing conditions also affects the firm’s default decision, and thus the rating agency’s estimate of the firm’s distance to default. The corresponding credit spreads are therefore both a direct result of the agency’s rating policy and an input into the firm’s endogenous default decision, making the two players’ strategies interdependent in a dynamic continuous-time game.

Our two main building blocks are, first, rating-dependent performance-sensitive debt as an implicit variant of introducing finite maturity, to capture the feedback effects between ratings and the firm’s cost of capital. We build on Manso, Strulovici, and Tchistyi (2010) and Manso (2013). We extend their setting by allowing for asymmetric information between firm and rating agency. The modeling of information asymmetry is the second building block and is similar to Grenadier, Malenko, and Malenko (2016). We share their concept of directional learning of private information. Importantly, and going beyond their setting, we introduce the feedback of the principal’s learning on the agent’s compensation via the firm’s cost of capital and thus its net cash flow stream over the whole lifetime, rather than only upon the end of the model, when the firm defaults.

In contrast to the literature on rating agencies’ distorted incentives leading to rating inflation, but in line with Manso (2013), we take as a starting point a rating agency aiming for objective and unbiased ratings. We show that as soon as asymmetric information is taken into account, even an unbiased rating agency will always overestimate the firm’s true asset value just upon the firm’s default. This behavior could be perceived as rating inflation, despite being an inherent consequence of asymmetric information.
We characterize the rating agency’s learning mechanism as follows. We model the firm’s relevant set of information in reduced form as a Markovian cash flow process, or, equivalently, the firm’s asset value.\(^1\) This information is available to the public, as well as to the rating agency, only with a measurement error around the true cash flow. We show that the firm does not default up to a measurement error dependent cutoff threshold. This allows the rating agency to learn as the firm survives distress: Once the observed cash flow and thus the public assessment of the firm’s asset value hits a new low, a firm subject to the largest feasible overestimation of asset value will default, but a less overestimated firm remains in business. The rating agency consequently adjusts its estimate of the firm’s distance to default via Bayesian updating. Note that the rating agency does not observe the private equity owner’s cash injection, which would also reveal the true state. Rather, it can only learn from observing survival combined with the current level of observed cash flow.

A key feature of the learning mechanism is that the rating agency rules out the firms with most overestimated asset values, which would have defaulted for the observed level of distress. We coin this mechanism Bayesian directional learning. In general, the rating agency will overestimate the firm’s true asset value from a specific minimum level of the observed cash flow onwards. The extent of overestimation will dynamically become more severe when new minima of the observed cash flow are reached. In particular, just upon the firm’s default, the firm’s asset value is maximally overestimated, as the worst case of the remaining probable states applies. An implication of our model is that learning happens particularly in the high-yield end of the rating spectrum, whereas there is little to learn in the investment-grade segment. In other words, the perceived rating inflation arises in particular during financial distress and persists in a subsequent recovery, despite the rating agency’s aim to rate as precisely as possible. Although it is common knowledge in our model that overestimation will eventually arise, an unbiased rating agency cannot mitigate it in any way, as it can identify a firm’s asset value as overestimated only ex-post at the time of default.

A direct empirical implication of our theory is that survival of an apparently distressed period reduces information asymmetry, as firms with low true asset values are ruled out. This means that

\(^{1}\) In structural credit risk models, it generally holds that the cash flow is given by the asset value times the payout ratio.
subsequent to an apparently distressed period, credit spreads are reduced for non-defaulted firms, and that upon default, the downward jump of bond prices on the default day shown empirically by Jankowitsch, Nagler, and Subrahmanyam (2014) is mitigated for firms that have survived an apparently distressed period before the default event. In a specific example calibrated with market data, learning from the firm’s signaling decreases the credit spread by more than 30%.\(^2\)

Moreover, we relax the assumption of the rating agency aiming for objective and unbiased ratings. Then, we can explain what happens when rating agencies become more conservative and downgrade more frequently, as a response to sharpened regulatory rules such as the Dodd-Frank Act in the aftermath of the 2008-2009 financial crisis, see Dimitrov, Palia, and Tang (2015) and Bedendo, Cathcart, and El-Jahel (2018). When a rating agency changes its policy from pleasing the issuer towards an unbiased assessment, we show how this leads to an immediate upward adjustment of the firm’s expected default threshold. Naturally, this, in turn, leads to lower ratings and higher interest payments. The feedback effect on the firm’s policy leads also to an upwards adjustment of the firm’s true default threshold. In general, a shift towards a more conservative rating policy leads to more accurate default prediction, which can be interpreted as a positive effect of regulation on rating quality. However, this comes at the cost of increasing debt service.

Our model literally describes a firm with public debt and private equity. Glushkov et al. (2018) analyze private firms that go public through an initial public debt offering (IPDO) as an alternative to going public through equity (initial public offering [IPO]). Over the 1987–2016 period, they have a sample of 635 IPDO firms, which corresponds to around 21% of all IPO firms that issue equity in the same industry and year, with the total IPDO amount raised over the sample period being around 45% of the amount raised in IPOs. They find that only a quarter of these firms eventually conduct an IPO. Summarizing, firms with public debt and private equity constitute a relevant part of the overall firm population.

\(^2\)In the example of Section 4.1, the credit spread decreases from 0.0744 to 0.0503, which is a difference of 241 basis points or 32% in relative terms.
Related literature

Our paper contributes to several strands of the literature. First, we contribute to the literature on credit ratings and debt pricing. We can explain the dynamic evolution of ratings and credit spreads over time, similar to, for example, Jarrow, Lando, and Turnbull (1997). Thus, we relate to the asset pricing perspective on credit risk. We calibrate our model to actual credit spreads for the respective rating classes, and we can make predictions on how much learning matters in terms of credit spread reduction for the rated entity. This is why we rely on a rather involved mathematical framework in continuous time. Still, we are able to stay within the class of structural credit risk models with endogenous default, in the tradition of Leland (1994) and Goldstein, Ju, and Leland (2001). Note that also with perfect information, models introducing finite maturity into infinite-horizon structural models feature a feedback effect between the firm’s cash flow or asset value and its cost of capital. Finite maturity can be introduced explicitly as the rollover of short-term or finite-maturity debt at market prices; see Leland and Toft (1996), He and Xiong (2012a,b), and He and Milbradt (2016).

In our paper, we capture the feedback between ratings and the firm’s cost of capital by employing rating-dependent performance-sensitive debt as in Manso, Strulovici, and Tchistyi (2010) and Manso (2013). This can be seen as an implicit variant of introducing finite maturity. Manso (2013) models the rated entity’s cash flow process in continuous time as we do, but he does not consider information asymmetry between the rated entity and either the rating agency or the financial market. Consequently, he does not analyze learning. In contrast, our paper offers Bayesian directional learning and explains the dynamic emergence of rating inflation. Our point of overestimating a firm’s true cash flow just upon the firm’s default, which leads to a sudden downward adjustment in debt value upon default, is related to Duffie and Lando (2001), who also study a structural model of debt valuation that features imperfect information. What distinguishes our contribution from Duffie and Lando (2001) is that in their model, the debt contract (in particular, the interest payment) is fixed ex-ante, and the creditors’ (secondary bond market’s) learning of the asset process feeds back

3Similar to us, he shows that the firm employs a cutoff strategy, and in case that “the cash-flow process of the firm follows a geometric Brownian motion, equilibrium of the game is unique” (Manso (2013, p. 543)). Consistent with this aspect of his paper, we employ a geometric Brownian motion and derive a unique equilibrium. However, the case of multiple equilibria, which drives a large part of his paper’s results, requires a mean-reverting cash-flow process.
neither into the cost of debt nor the firm’s endogenous default decision. In contrast, our model is more suitable in describing a repeated interaction in which the firm’s rating affects its financing conditions, and thus also its considerations whether to inject new equity or to default already today.

Second, our paper contributes to the literature on the strategic exercise of real options in financial economics, which has previously been studied in the context of initial public offerings by Bustamante (2012), for corporate investments by Grenadier and Malenko (2011), Morellec and Schürhoff (2011), and Grenadier, Malenko, and Strebulaev (2014), and for dynamic agency problems for real options in Gryglewicz and Hartman-Glaser (2019). We model an optimal stopping problem with private information, following Kruse and Strack (2015). We consider the firm’s default decision as a dynamic real option exercise game in the spirit of Grenadier, Malenko, and Malenko (2016). While we employ a similar equilibrium concept as Grenadier, Malenko, and Malenko (2016), we extend the setup substantially by a feedback effect between the cost of capital and the decision to default, as well as introducing performance-sensitive debt. The resulting equilibrium is partially separating due to the Bayesian directional learning, which prevents the rating agency from fully inferring the measurement error. Our innovation over the literature is that we allow a firm’s exercise policy to affect its own cost of capital, which can have a dynamic effect on firm value before exercise, rather than only upon exercise.

Third, we contribute to the literature on the economics of credit rating agencies and the causes of rating inflation. Typically, this literature focuses on the rating agencies’ incentives to be honest or inflate ratings, rather than the learning of issuer quality; see, for example, Bolton, Freixas, and Shapiro (2012) and Hirth (2014). Opp, Opp, and Harris (2013), Fulghieri, Strobl, and Xia (2014), and Frenkel (2015), model the market’s learning of the rating agencies’ type with Bayesian updating. However, their frameworks are not suitable for explaining the dynamics of credit spreads. Similar to our paper, Goldstein and Huang (2018) consider the impact of credit ratings on firms in a feedback loop, in which the rating agency considers its impact on the firms’ financing conditions. They show that ratings remain informative, even when the rating agency is biased. In contrast, we show that
ratings end up inflated after a period of financial distress despite the agency’s incentive to rate unbiased.

Finally, our paper contributes to the literature on learning in general economic context. We model the firm through its Markovian cash flow process, which the rating agency can only observe with a persistent measurement error as in Fershtman and Pakes (2012). Due to the nature of our cash flow process, we specifically exclude statistical learning of the process parameters or the state of the world as in Pastor and Veronesi (2009) or David (2007), respectively. That is, observing the cash flow process on its own does not provide valuable information on the measurement error. Thus, we are able to restrict the learning mechanism to learning from strategic actions, namely the firm’s choice of default threshold.

As in Hörner and Lambert (2018), our rating does not have to be a function of a single Markov process, such as our cash flow process. Rather, it is augmented by a second dimension, which in our context is the historical minimum of the observed cash flow. Thus, the rating agency accounts for the survival of past times of distress in its estimated default threshold despite the Markovian cash flow that itself does not carry valuable information besides its current state. Bar-Isaac (2003) considers a learning game between a monopolistic seller with a persistent type and a set of buyers with limited information, that, similar to ours, yields a history dependence. However, Different from Bar-Isaac (2003), we feature Bayesian directional learning.

The remainder of the paper is organized as follows: In Section 1, we introduce the rating game between the rating agency and the firm. In Section 2, we introduce the best response strategies of both players. Subsequently, we derive and compute the rating game equilibrium in Section 3. In Section 4, we provide an extensive analysis of the economic implications of our model. We conclude in Section 5.
1 A model of learning on a competitive debt market

In this section, we first present the firm, its performance-sensitive debt structure, and its expected net present value dependent on its liquidation strategy. Then, we turn to the rating agency that aims for a high rating precision, when estimating the firm’s default threshold. In the last subsection we present the Markov Bayesian Nash Equilibrium in pure strategies as our solution concept.

1.1 Firm: performance sensitive debt and information structure

First, consider a levered firm managed by the sole owner of the firm’s equity. It generates a stream of non-negative cash flow at the rate $X = (X_t)_{t \geq 0}$ and pays interest on its outstanding debt at the rate $C = (C_t)_{t \geq 0}$. The cash flow rate $X$ satisfies

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \text{ for } t > 0, X_0 \in \mathbb{R}^+,$$

(1)

where $\mu$ and $\sigma$ represent the cash flow’s growth rate and volatility, respectively, with $\mu < r$, $r$ being the risk-free interest rate, and $W = (W_t)_{t \geq 0}$ being a Wiener process. The growth rate $\mu$ and volatility $\sigma$ are common knowledge, but $X$ is known only by the firm’s manager-owner. The rating agency observes $D = (D_t)_{t \geq 0}$ which corresponds to the firm’s cash flow $X$ distorted by a measurement error $\tilde{\theta}$. The observed cash flow $D$ is given as

$$D_t = \tilde{\theta} X_t, \text{ for } t \geq 0,$$

(2)

and it has the same dynamics as the cash flow $X$, that is, $dD_t = \mu D_t \, dt + \sigma D_t \, dW_t$, for $t > 0$. As a consequence, the imperfect observation $D$ contains no information on $\tilde{\theta}$. Note that, while $X$ and $D$ have the same dynamics, updating the prior information obtained from observing the firm’s strategic behavior improves the rating agency’s estimate of $X$ over time. Modeling the measurement

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We consider a multiplicative impact of the measurement error to ensure tractability of the remaining analysis. An additive impact of the agent’s private information component is considered, e.g., by Grenadier and Malenko (2011), in an otherwise similar framework.
error as persistent following Fershtman and Pakes (2012), the structure of $X$ and $D$ ensures that learning only happens on the basis of observing strategic behavior. Pure statistical learning in the form of observing the imperfectly observed cash flow process alone can by definition not reveal any information. In contrast, think of the well-known case of a mean reverting process with an unknown mean. In such a situation, a mere observation of the process realization over time improves the estimates of the unknown parameter. For further examples on those types of learning, see David (2007) and Pastor and Veronesi (2009).

The firm knows the realization of the persistent measurement error $\tilde{\theta}$. The rating agency does not know $\tilde{\theta}$. Rather, it overestimates the true cash flow for a measurement error $\theta > 1$, while a measurement error $\theta < 1$ leads the rating agency to underestimate the true cash flow. However, the law $\mathbb{P}_{\tilde{\theta}}$ of $\tilde{\theta}$ on $\Theta = [\underline{\theta}, \overline{\theta}]$, with $0 < \underline{\theta} < \overline{\theta} < \infty$, is common knowledge; see, for example, Grenadier, Malenko, and Malenko (2016) and Fershtman and Pakes (2012) for a related setup. We assume that the distribution $\mathbb{P}_{\tilde{\theta}}$ admits a density, which is bounded from above and away from zero, which is our prior $\phi$. 5

The firm issues performance-sensitive debt, see Manso, Strulovici, and Tchistyi (2010). This can be interpreted as the rollover of maturing debt, which similarly connects the firm’s cost of debt capital to the outsiders’ current perception of the firm’s distance to default. Formally, the interest payment rate $C$ depends on the rating of the firm $R = (R_t)_{t \geq 0}$. In particular, $C$ is a non-increasing function $C : [1, \infty] \to \mathbb{R}_0^+$, which we assume bounded away from zero and bounded from above. Hence, it satisfies $0 < C = C(\infty) \leq \overline{C} = C(1) = C < \infty$. The perpetual debt contract does not repay the principal. Apart from the rating-dependence of $C$, it is a consol bond as in Black and Cox (1976) and Leland (1994). For the subsequent equilibrium analysis, the sensitivity of $C$ must be smooth:6

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5 $\tilde{\theta}$ is independent of the Wiener process $W$. The filtrations $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ capture the information generated by $X$ and $D$, respectively. Formally, the firm’s information set at $t$ is given by $\sigma(\tilde{\theta}) \vee \mathcal{F}_t$, for $t \geq 0$. Since $\tilde{\theta}$ is known to the firm at $t = 0$, we can condition on $\tilde{\theta} = \theta$ and work with $\mathcal{F}$.

6 Assumption 1 imposes Lipschitz continuity on the log-log scale for interest changes, that is, for some $L_C$, with $0 < L_C < 1$, it holds that $|\log C(z) - \log C(z')| \leq L_C |\log(z) - \log z'|$, for all $z, z' \geq 1$. In Appendix I, we confirm the validity of Assumption 1 with market data.
**Assumption 1.** Assume that the interest payment rate $C$ satisfies for some $0 < L_C < 1$ that

$$C(z) \leq C(z') \leq (z/z')^{L_C}C(z), \text{ for } 1 \leq z' \leq z. \tag{3}$$

The rating agency estimates the firm's critical cash flow level $\hat{D}^\star = (\hat{D}^\star_t)_{t \geq 0}$ where default occurs from observing $D$. It issues as its strategy the rating as distance to default $R_t = D_t/\hat{D}^\star_t$, which we discuss in more detail below.

The firm owner is risk-neutral and maximizes the net present value of the cash flows net the interest payments on outstanding debt based on the cash flow process $X$ and the realized measurement error $\theta$. The default timing forms the firm’s strategy $\tau(\theta)$. Implicitly, the default timing decides if and how much cash is injected by the financially-unconstrained firm owner. Because the firm is aware of the measurement error at the start and $\breve{\theta}$ and the Wiener process $W$ are independent, we can write $\tau = (\tau(\theta))_{\theta \in \Theta}$, where $\tau(\theta)$ is a stopping time, for $\theta \in \Theta$.

Formally, remembering that $X = D/\breve{\theta}$ and $R = D/\hat{D}^\star$, the firm’s payoff is the net present value

$$U_F^{(\theta)}(\tau, \hat{D}^\star) = \mathbb{E} \left[ \int_0^{\tau(\theta)} e^{-r t} \left( D_t/\theta - C(D_t/\hat{D}^\star_t) \right) \, dt \right], \text{ for } \theta \in \Theta. \tag{4}$$

Note the feedback effect: the rating affects the firms payoff via the interest payment $C$, which in turn influences the default decision. If the interest payments exceed the true cash flow at a given time, the firm does not have to default, but the owners can inject further cash.\(^7\) The firm chooses the default time $\tau$ with the objective to maximize $U_F^{(\theta)}(\tau, \hat{D}^\star)$, namely, the present value of cash flows after interest payments to creditors. See Equation (13) below for the formal specification of the firm’s optimization problem. Thus, the decision to delay default includes the decision of how much cash to inject into the firm in a time of distress.

\(^7\) Assuming financially constrained or even cashless owners would not necessarily shut down our learning mechanism. Outsiders could still infer from non-default that the true cash flow is higher than the worst case, if it is still sufficient to cover the coupon payments. Though, if the degree of the owners’ financial constraints is also subject to asymmetric information, this will add another layer of complexity to the model.
1.2 Rating Agency: Dynamic learning and Bayesian belief updating

The rating agency’s success is determined by its accuracy relative to a precise and unbiased rating. On a competitive debt market, creditors aim to price credit precisely to be successful. The rating agency in our model faces the same information asymmetry towards the firm as the investors on the competitive debt market. Thus, the function of the rating agency in our model simplifies the exposition of the interaction between outsiders’ perception of the firm’s distance to default and cost of debt financing.

The rating agency assigns the rating $R_t$ based on the estimated default level $\hat{D}_t^\star$ as the distance to default

$$R_t = \frac{D_t}{\hat{D}_t^\star}, t \geq 0. \quad (5)$$

A high observed cash flow $D$ relative to the predicted default level $\hat{D}^\star$ increases the rating. The rating $R$ takes values in $[1, \infty]$, where $R_t = 1$ implies imminent expected default. $R_t = \infty$ corresponds to a default-free firm from the rating agency’s perspective. A change in the imperfectly observed cash flow adjusts the rating because the distance to the estimated default threshold $\hat{D}^\star$ changes. Note that a distance-to-default type of rating resembles rather internal rating approaches, whereas the major rating agencies communicate discrete rating scales.\(^8\) In the language of Manso, Strulovici, and Tchistyi (2010), our rating scheme resembles more the “Asset-Based PSD” with the important distinction that our outsiders face asymmetric information regarding the firm’s asset value. See their paper for a more formal discussion of the transition between “Asset-Based PSD” and the case of a discrete rating scale, which they call “Ratings-based PSD”.

The risk-neutral rating agency maximizes its accuracy over estimated default threshold $\hat{D}^\star$. Conditional on its own information captured by the belief $\pi$ and the firm’s measurement error-dependent liquidation strategy $\tau(\theta)$, the rating agency expects the costs from a rating with an

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\(^8\)A discretized rating scale would imply a step-function for the interest payment levels. For technical tractability, we require some sensitivity constraints for the interest response to rating changes, which is formalized in Assumption 1.
estimated default threshold $\hat{D}^\star$ to be

\[
U_{\text{RA}}^\pi(\tau, \hat{D}^\star) = -\mathbb{E}\left[\int_0^\tau e^{-\rho t} k^\pi_t \, dt\right].
\] (6)

Similar to Manso (2013), the rating agency is concerned with its reputation that depends on its rating precision at any given time. Formally, the cost rate

\[
k^\pi_t = \int_\Theta \frac{1}{2} (\hat{D}^\star_t - f(\theta))^2 \phi^\pi_t(\theta) \, d\theta, \text{ for } t \geq 0,
\] (7)

defines the expected costs as the squared distance between the estimated default threshold, conditional on the belief, and the true default threshold $f(\theta)$, dependent on the measurement error.\textsuperscript{10} The rating agency’s discount parameter $\rho$ measures the time preference over future lack of accuracy.

Central to the rating agency is the learning of the firm’s distance to default from the observed cash flow and the implied updating of its belief $\pi$. The rating agency learns from the firm’s endogenous survival in low cash flow states. Apart from the observed current cash flow level $D$, the lowest observed cash flow $E = (E_t)_{t \geq 0}$, with $E_t = \inf_{0 \leq s \leq t} D_s$, $t \geq 0$ provides valuable information. For the rating agency, $E$ is the information generating process, and its estimate of the firm’s default threshold is a function of this variable, that is, $\hat{D}^\star = g(E)$. The extended state space therefore consists of both the current observed cash flow $D_t$ and the running minimum $E_t$. The rating agency only employs the information contained in the observed cash flow $D$ but cannot access the firm’s private information of the real cash flow $X$, or, equivalently $\tilde{\theta}$ (with $\tilde{\theta} = D/X$).\textsuperscript{11}

\textsuperscript{9}Note that the true default threshold $f(\theta)$ will be formally introduced in Section 1.3 below.

\textsuperscript{10}Later, we will generalize the cost rate by allowing for an asymmetric effect of over- and underestimation of the true default threshold. This generalized structure captures different attitudes of the rating agency toward its stakeholders: If it is more concerned with serving regulatory authorities, the rating agency would be biased toward avoiding overestimation and estimate the default threshold conservatively. In the extreme case, it would always assume the worst possible measurement error. On the other hand, a rating agency aiming for maximizing short-term revenues in the issuer-pays model would publish the best-case rating to please the issuers. A detailed discussion of this generalization can be found in Section 4.3.

\textsuperscript{11}Formally, the default level $\hat{D}^\star$ is strictly positive and $G$-adapted.
Observing the firm’s decision either to default or to signal a higher true cash flow by not defaulting, the rating agency gradually learns the measurement error of the firm’s cash flow over time. Hence, the rating agency forms its belief \( \pi = (\pi_t)_{t \geq 0} \) regarding the measurement error \( \tilde{\theta} \), which we interpret as type in the given signaling game. We restrict our analysis to beliefs that are absolutely continuous with respect to the prior \( \mathbb{P}_{\tilde{\theta}} \), and thus also \( \pi \) has a density, say \( \phi^\pi \), with

\[
\phi^\pi_t(\theta) = L^\pi_t(\theta) \phi(\theta), \quad \text{for } \theta \in \Theta, t \geq 0,
\]

where \( L^\pi_t = (L^\pi_t(\theta))_{\theta \in \Theta} \) describes the evolution of the probabilities for each measurement error by using the information available in the market.\(^{12}\) While we formulate the beliefs here in general, we will later use perfect Bayesian Markov equilibrium as the equilibrium concept. Thus, the rating agency will update its belief about the measurement error according to Bayes’ rule whenever possible. Because Bayesian updating is infeasible for observed firm actions inconsistent with any possible measurement error, we need to specify off-path beliefs.\(^{13}\) Following Grenadier, Malenko, and Malenko (2016), we make the standard assumption that in such a case the beliefs remain unchanged:

**Assumption 2.** If at any \( t \), the rating agency’s belief \( \pi_t \) and the firm’s action are such that this action could not occur for any possible measurement error in equilibrium, then the belief is unchanged.

Figure 1 displays an example of the rating agency’s Bayesian updating of its belief: Setting off from a prior (solid line), by observing the firm’s default or survival, the rating agency learns and reassesses the probabilities for each measurement error, which in turn sharpens its belief. As the cash flow evolves, the beliefs become more and more precise (dotted and dashed lines).

\(^{12}\)\( L^\pi_t \) is a family of non-negative \( \mathcal{G}_t \)-measurable random variables with \( \int_{\Theta} L^\pi_t(\theta) \phi(\theta) \, d\theta = 1, t \geq 0. \)

\(^{13}\)For example, we will show in later sections that for each measurement error, there will be a default threshold for the firm. If the rating agency observes that the firm does not default, even though firms with all measurement errors should have defaulted, this strategy should not occur in equilibrium, so we need to specify beliefs in this case.
Figure 1: Bayesian updating of beliefs. This figure displays an example of how the rating agency updates the prior (solid line) to the updated belief at times $t_1$ and $t_2$ (dotted and dashed line, respectively). It plots the density for each measurement error $\tilde{\theta}$, which has a support of $\Theta = [\underline{\theta}, \overline{\theta}] = [0.50, 1.50]$. As time evolves, the rating agency rules out overestimated measurement errors.

1.3 Markov Perfect Bayesian Equilibrium and Admissible Strategies

We focus on equilibria in pure strategies. The equilibrium concept is perfect Bayesian equilibrium in Markov strategies in the extended state space. It requires that the rating agency’s strategies are sequentially optimal, beliefs are updated according to Bayes’ rule whenever possible, and the equilibrium strategies are Markov. The formal definition of the perfect Bayesian equilibrium in Markov strategies is presented in Appendix II and is subsequently referred to simply as equilibrium.

The Markov property of the extended state processes, $(D, E)$ with running minimum $E = (E_t)_{t \geq 0}$ of $D$, that is, $E_t = \inf_{0 \leq s \leq t} D_s$, $t \geq 0$, as observed by the rating agency, suggests that it is sufficient to consider Markov strategies. The set of admissible Markov strategies $\mathcal{A}_f$ for the firm is given by default levels $f(\theta)$ of firm cash flow, as observed by the rating agency, for $\theta \in \Theta$. The first time the cash flow as observed by the rating agency $D$ falls below $f(\theta)$, the firm defaults; that is, $\tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\}$, for $\theta \in \Theta$.\(^{14}\)

\(^{14}\)The set of admissible strategies is formalized by $\mathcal{A}_f = \{f : \Theta \to \mathbb{R}_0^+, f \text{ is measurable}\}$. Strictly speaking, the set of Markov strategies is much larger and consists of stopping times that are given by first entry times in a measurable set $B(\theta) \subseteq \mathbb{R}^2$, i.e., $\tau(\theta) = \inf\{t \geq 0 : (D_t, E_t) \in B(\theta)\}$, $\theta \in \Theta$. However, the subsequent Proposition 2 shows that this restriction is innocent. Also note that we are taking the perspective of the rating agency to avoid problems when discussing the matters from the perspective of both parties, the firm and the rating agency. The critical default level in the firm’s cash flow is then $f(\theta)/\theta$, i.e. $\tau(\theta) = \inf\{t \geq 0 : X_t \leq f(\theta)/\theta\}$, since $D = \theta X$.\(^{13}\)
The admissible strategies for the rating agency $\mathcal{A}_g$ are functions of the information-generating process, namely the minimum observed cash flow $E$, that is, $\hat{D}^* = g(E)$, and $\mathcal{A}_g = \{g : \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0, g \text{ is measurable} \}$. Further, $g$ should be reasonable from a financial economics perspective, that is, in $(D_t, E_t)$, the predicted default at $g(E_t)$ is attainable, hence $g(E_t) \leq E_t$.

Firm survival in bad times, that is, for a decreasing running minimum $E_t$, potentially signals a higher true cash flow. Then the estimated default threshold $\hat{D}^* = g(E_t)$ should be adjusted downward or remain constant; that is, $g$ is non-decreasing. However, we demand that a rating $R = D_t/g(E_t)$ should not improve when the cash flow hits a new all-time low. Formally, a new all-time low occurs at time $t$ for $D_t = E_t$. Then the rating is $R_t = D_t/g(E_t) = E_t/g(E_t)$, and has to be non-increasing in $E_t$ to avoid a better rating in case of a new all-time low. Thus a reasonable strategy $g \in \mathcal{A}_g$ satisfies\footnote{If $g(e)$ is non-decreasing and $g(e)/e$ is non-increasing in $e$, then $g$ is also continuous.}

$$g(e) \leq e, \quad g(e) \text{ non-decreasing in } e \quad \text{and} \quad g(e)/e \text{ non-increasing in } e \quad (9)$$

The Markov property requires that the firm’s and the rating agency’s strategies are only functions of the payoff-relevant information at any time $t$. It contains the measurement error $\tilde{\theta}$ and the current value of the suitably extended state process $(D_t, E_t)$ for the firm, and the beliefs $\phi_{\pi t}$ about $\tilde{\theta}$ and the current value of the state process $(D_t, E_t)$ for the rating agency. Note that our state space processes for firm and rating agency contain both the observed cash flow process and its running minimum. Therefore, our players have a memory beyond the current value of the cash flow process, although the Markov property holds for the extended state space.

## 2 Best Responses and Learning

Using both the firm and rating agency’s strategies from Section 1, this section presents the best responses of both players and describes the rating agency’s learning of the firm’s true cash flow. In particular, the rating agency’s best estimate of the firm’s optimal liquidation decision is a
continuously updated default barrier. For the observed cash flow trajectory, the rating agency infers from the running minimum up to which degree of overestimation the firm would have defaulted and rules out the most overestimated types in its consistent belief. Observing survival at new historical lows of observed cash flow makes the rating agency update its beliefs about the firm’s true cash flow and thus leads to lower financing costs for the same given observed cash flow in the future. The firm specifies its optimal type-dependent liquidation decision as a best response to a rating strategy. The firm’s best response accounts for the feedback effect: lower future financing costs because of the rating agency’s learning imply that the firm delays default for lower present cash flows.

2.1 Best Response and Learning of the Rating Agency

This section characterizes the best response of the rating agency, that is, the estimated default level \( \hat{D}^* = g(E) \), and the rating agency’s consistent belief \( \pi \), given the firm’s liquidation strategy \( \tau \). The firm’s strategy is a type-dependent cutoff threshold for all \( \theta \in \Theta \).\(^{16}\)

The minimum observed cash flow \( E \) drives the rating agency’s consistent belief. At time \( t \), types \( \theta \) with \( f(\theta) \geq E_t \) can be discarded, since default obviously has not occurred yet. The consistent belief is therefore \( \pi_t = \mathbb{P}_{\theta} | f(\theta) < E_t \), with density given in (11) below. The rating agency optimizes its expected payoff given in Equation (6) over admissible estimated default thresholds of the firm, that is,

\[
\sup_{\hat{D}^* = g(E), g \in \mathcal{A}_g} U_{RA}^\pi(\tau, \hat{D}^*) = - \inf_{g \in \mathcal{A}_g} \mathbb{E} \left[ \int_0^\tau e^{-\rho t} \int_\Theta \frac{1}{2} (g(E_t) - f(\theta))^2 \phi_\pi(\theta) d\theta dt \right]. 
\tag{10}
\]

For each \( t \), the optimal rating agency strategy \( g(E_t) \) minimizes the mean squared error for estimating the type-dependent default threshold \( f(\theta) \) using the current belief \( \pi \). The latter is given by the respective conditional expectation; see (12). This allows us to formulate both a consistent belief and the rating agency’s best response in the following proposition.

**Proposition 1** (Rating Agency’s Best Response). Let a firm strategy \( \tau \) be given by a function \( f \in \mathcal{A}_f \), with \( \tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\} \), \( \theta \in \Theta \). Then the rating agency’s consistent belief is

\(^{16}\)Formally, it is given by \( f \in \mathcal{A}_f \) with \( \tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\} \), for all \( \theta \in \Theta \).
given by
\[ \phi_t^\pi(\theta) = \frac{1_{f(\tilde{\theta}) < E_t}}{\int_\Theta 1_{f(\theta') < E_t} \phi(\theta) \, d\theta'} \phi(\theta), \quad \text{for } \theta \in \Theta \text{ and } 0 \leq t < \tau. \] (11)

The rating agency's corresponding best response to \( f \) is given by \( \hat{D}^* = g(E; f) \), with

\[ g(e; f) = \begin{cases} 
\mathbb{E} \left[ f(\tilde{\theta}) \mid f(\tilde{\theta}) < e \right], & \text{for } e > \inf_{\theta \in \Theta} f(\theta), \\
e, & \text{else.} 
\end{cases} \] (12)

and \( g(e; f) \leq e, \) for \( e \geq 0, \) as well as non-decreasing. Moreover, if \( f \) is strictly increasing, then \( g(\cdot; f) \) is continuous and strictly increasing on \( f(\Theta) \).

Sketch of Proof of Proposition 1. For a given \( f \), the form of the consistent belief \( \pi \) in (11) follows directly from Bayes’ rule. Using the consistent belief \( \pi \), the rating agency maximizes the respective utility in (10), where the key term is the quadratic loss \( \int_{\Theta} \frac{1}{\pi} (g(E_t) - f(\theta))^2 \phi_t^\pi(\theta) \, d\theta \). We look for \( g(E_t) \), which minimizes the squared distance to the random variable \( f(\tilde{\theta}) \mid f(\tilde{\theta}) < E_t \). Therefore, the optimal \( g(E_t; f) \) is the expected value of \( f(\tilde{\theta}) \mid f(\tilde{\theta}) < E_t \), which is in essence (12). The detailed proof is given in Appendix II.

Proposition 1 specifies that the rating agency’s best response to a type-dependent firm liquidation strategy is a rating based on an estimated liquidation barrier based on its belief consistent with observing which types should have defaulted for the observed cash flow process. A belief of the rating agency is consistent if the likelihood for the type is in accordance with the firm’s default behavior for the observed cash flow process. Hence, if the rating strategy induces a specific default barrier and the imperfectly observed cash flow reaches this level, a firm of a sufficiently overestimated type defaults while a firm of an underestimated type may not. This observed default behavior provides useful information to the rating agency, and allows it to update its rating strategy consistent with its belief about the types.

Comparing two states with identical current observed cash flow but with different historical cash flow paths, the state with a lower historical minimum cash flow exhibits a better rating. However,
this does not imply that a decreasing cash flow improves the current rating. Rather, the rating is worsening for a decreasing cash flow, see the specification of the set of admissible rating strategies \( \mathcal{A}_g \) as well as the constraint in (9) and the discussion there. However, the survival of low observed cash flow levels in the past improves future ratings spurred by the rating agency’s learning. Note that learning occurs when the observed cash flow falls to a level where a given type, or, measurement error, may default, which are typically very low levels with a short distance to default.

2.2 Best Response of the Firm

In the following, we characterize the best response of the firm, that is, its default time \( \tau \), for an admissible rating strategy \( g \in \mathcal{A}_g \) that satisfies (9). Since the firm knows its own cash flow and the type \( \theta \), we can specify the firm’s type-dependent best response \( \tau(\theta; g) \).

For a given rating agency strategy \( g \in \mathcal{A}_g \) satisfying (9) and type \( \theta \), the best response \( \tau(\theta; g) \) is the stopping strategy that maximizes its expected payoff given in Equation (4). Denote by \( v(\cdot, \cdot; \theta, g) \) the value function, which is given by

\[
v(d, e; \theta, g) = \sup_{\tau \in \mathcal{T}_d(e)} \mathbb{E}_{(d, e)} \left[ \int_0^\tau e^{-rt} \left( D_t/\theta - C(D_t/g(E_t)) \right) \, dt \right], \text{ for } (d, e) \in \mathcal{C},
\]

where \( \mathcal{C}\{(d, e) \in \mathbb{R}^2 : 0 \leq e \leq d\} \) is the convex cone for feasible values of the imperfectly observed cash flow \( D \) and its running minimum \( E \); \( \mathcal{T}_d(e) \) is the set of all stopping times with respect to the information generated by \( (D, E) \) with starting value \( (d, e) \). Formally, equation (13) poses an optimal stopping problem. The firm’s manager-owner maximizes the expected firm value by choosing the optimal default timing. The solution of this problem is rather involved and includes a non-standard boundary condition.\(^\text{17}\)

The best response of the firm to a given rating agency rating strategy \( g \) is the collection of optimal stopping times \( (\tau(\theta; g))_{\theta \in \Theta} \). Under the condition \( g \in \mathcal{A}_g \) and satisfying (9), we show that the optimal stopping rule \( \tau(\theta; g) \) is a cutoff rule in the observed cash flow \( D \) and does not depend

\(^{17}\)See (II.5) in Appendix II.
on its running minimum $E$. Specifically, the firm liquidates at the first hitting time of a threshold $f(\theta;g)$, for all $\theta \in \Theta$, as is shown in the following proposition. The cutoff rule balances the firm’s trade-off between continuing in unfavorable conditions now in order to reap lower interest payments once the rating agency updates the imperfectly observed cash flow. If the imperfectly observed cash flow exceeds the threshold, continuing is attractive. Once it hits the cutoff, the firm liquidates.

**Proposition 2** (Firm’s Best Response). For $g \in \mathcal{A}_g$ satisfying (9), $\theta \in \Theta$, and $d > 0$, the optimal stopping time of (13) is given by

$$
\tau_{(d,d)}(\theta;g) = \inf\{t \geq 0 : D(t) \leq f(\theta;g)\},
$$

where $f(\theta;g)$ is some positive real constant, that is, $\tau_{(d,d)}(\theta;g)$ is the first hitting time of the imperfectly observed cash flow $D$ with respect to the barrier $f(\theta;g)$.

**Sketch of Proof of Proposition 2.** We obtain a cutoff rule for the firm’s best response by analyzing the early exercise region using the conditions in (9). Firstly, $g$ being non-decreasing implies that the firm defaults when the observed cash flow falls below a critical level. Secondly, $g(e)/e$ being non-increasing in $e$ ensures that this critical level is unique as a better rating in case of a new all-time low is prohibited. The detailed proof is given in Appendix II.

This result implies that the best response of the firm $\tau(\theta;g)$ for a rating agency’s strategy $g$ is characterized by a specific default barrier $f(\theta;g)$, for each $\theta \in \Theta$.\(^{18}\) For any cash flow, the firm is aware of its over- or underestimation by the rating agency based on the imperfectly observed cash flow. If the observed cash flow decreases beyond a type-dependent level, the firm defaults.

How does the firm’s best response to a given rating strategy $\hat{D}^* = g(E)$ look like? So far, Proposition 2 characterizes the firm’s optimal strategy in terms of a default barrier $f(\theta;g)$ for one specific $\theta$. But the firm’s best response is the mapping $\theta \mapsto f(\theta;g)$ for all $\theta \in \Theta$. We provide more structure on how the barrier changes in the type, see Lemma 2 in Appendix II. These results are crucial for the subsequent equilibrium analysis. From an economic perspective, a high measurement

\(^{18}\)Solving the associated free boundary value problem determines the point-wise solution.
error $\theta$ implies a lower cutoff threshold. Hence, an overestimated firm survives longer because it underpays its interest.

### 3 Rating Equilibrium

The results of the previous section are fundamental for the subsequent equilibrium analysis. Both the rating agency and the firm use Markov strategies\(^\text{19}\) characterized by real-valued functions in Propositions 1 and 2. We obtain a perfect Bayesian equilibrium in Markov strategies as introduced by Maskin and Tirole (1988) in spirit of Grenadier, Malenko, and Malenko (2016) by the Schauder fixed-point theorem. However, the application of this fixed-point theorem requires well-behaved interest payments in form of the growth constraint in Assumption 1 to arrive at our rating equilibrium.

In the following, we first provide the existence of an equilibrium candidate in Proposition 3. Then, Proposition 4 characterizes the solution and verifies the existence and uniqueness, provided that specific technical conditions hold. The latter non-trivially extends the result of Manso (2013), giving the existence and uniqueness of an equilibrium for the cash flow that follows a geometric Brownian motion to the case of information asymmetry. The underlying reason for the uniqueness is the non-stationarity assumption for the cash flow process. Multiple equilibria would be likely for a mean-reverting specification, see Manso (2013). Furthermore, notice that $f(\theta)$ is strictly increasing according to Lemma 2 in Appendix II; that is, a best response has a minimal slope of $l_f > 0$ for all given strategies $g$ of the rating agency. This eliminates classical semi-pooling equilibria, in which some observed types play the same strategy and default at the same time. While we do not have a semi-pooling equilibrium, the rating agency’s learning implies that the information is only revealed fully at the time the firm actually defaults, prior to which the rating agency can only rule out some types but cannot fully infer the firm’s type. The rating agency learns the exact type at default, but the game ends simultaneously, leaving the rating agency without the possibility to react.

\(^{19}\)For the rating agency, recall the extension of the state space by the running minimum of the cash flow.
To establish our equilibrium, we apply the Schauder fixed-point theorem to the best responses. To do this, we must ensure that the best response \( g(\cdot; f) \in \mathcal{A}_g \) satisfies all conditions in (9). Whereas \( g(e; f) \) is non-decreasing in \( e \) and \( g(e) \leq e \), the important constraint that \( g(e; f)/e \) is non-increasing in \( e \) cannot be shown to hold in general. Instead of imposing the constraint by requiring \( g(e; f)/e \) to be non-increasing in \( e \) in (10), we rather modify \( g(\cdot; f) \) such that (9) holds, by a suitable transformation \( \mathcal{R} \). \(^{20}\) We apply the Schauder fixed-point theorem to the mapping \( T: (f, g) \mapsto (f(\cdot; g), \mathcal{R}(g(\cdot; f))) \), where \( f(\cdot; g) \) is the firm’s best response given in Proposition 2, \( g(\cdot; f) \) is the rating agency’s best response given in Proposition 1.

**Proposition 3.** Suppose Assumption 1 holds. Then \( T: (f, g) \mapsto (f(\cdot; g), \mathcal{R}(g(\cdot; f))) \) has at least one fixed point in \( \mathcal{A}_f \times \mathcal{A}_g \). Let \((f^*, g^*)\) be such a fixed point, if \( \mathcal{R} \circ g(\cdot; f^*) = g(\cdot; f^*) \), then \((f^*, g^*)\) is an equilibrium.

**Sketch of Proof of Proposition 3.** For the Schauder fixed-point theorem, we identify a sufficiently rich subset \( \mathcal{K} \subseteq \mathcal{A}_f \times \mathcal{A}_g \) and prove that \( \mathcal{K} \) is a nonempty convex compact subset of a Banach space, here, the space of continuous functions on a compact set endowed with the sup-norm. Using structural properties of \((f, g) \in \mathcal{K}\), we prove that \( f(\cdot; g) \) and \( \mathcal{R}(g(\cdot; f)) \) are both continuous functionals. Then Schauder gives us the existence of at least one fixed point \((f^*, g^*)\). If, in addition, \( \mathcal{R} \circ g(\cdot; f^*) = g(\cdot; f^*) \), then \((f^*, g^*) = (f(\cdot; g^*), g(\cdot; f^*))\), establishing an equilibrium. The detailed proof is given in Appendix II.

Proposition 3 provides us with a candidate for an equilibrium in this general setup with function-valued strategies, which then has to be verified to be an equilibrium, that is, \( \mathcal{R}(g(\cdot; f^*)) = g(\cdot; f^*) \).

\(^{20}\)Note that solving the constrained problem is much more evolving as path dependencies arise, leading to a departure from the Markovian setup. For the sake of tractability, we limit the exposition to the case in which implicitly it is assumed that the constraint holds. The transform \( \mathcal{R} \) maps \( \{ g \in \mathcal{A}_g : g \text{ non-decreasing}, g(e) \leq e \} \) to \( \{ g \in \mathcal{A}_g : g \text{ satisfies (9)} \} \) and is the identity on the latter set, that is, \( \mathcal{R}(g) = g \), for \( g \in \{ g \in \mathcal{A}_g : g \text{ satisfies (9)} \} \), see Lemma 1 in Appendix II. It is defined by:

\[
\mathcal{R}(g)(e) = \begin{cases} 
    e \inf \{ g(z)/z : 0 < z \leq e \}, & \text{for } e > 0, \\
    0, & \text{for } e = 0.
\end{cases}
\]

(15)

Although the constraint that \( g(e; f)/e \) is non-increasing in \( e \) cannot be shown to hold in general, our numerical implementation and all cases that we could think of as practically relevant satisfy this constraint. That is, the transformation is formally needed but typically is the identity.

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Practically, for the subsequent analysis of special cases of the rating game’s equilibrium, the transform $R$ regulating the rating agency’s strategy $g$ always takes the form of the identity function, so that no transformation is necessary. In turn, without employing the transformation, we immediately end up with an equilibrium for the rating game without further ado for all relevant cases.

This result is a pure existence result, and no particular guidance is given on how to actually compute such an equilibrium candidate. Proposition 4 characterizes an equilibrium candidate as the solution to a two-dimensional ordinary differential equation (ODE) under some technical assumptions. Further, it states an inequality condition, and given that this condition holds, the candidate is indeed the unique equilibrium. This result is the basis for computing an equilibrium strategy in a fast and efficient way, which we rely on heavily in the following analysis.

In order to facilitate the subsequent analysis, we require that $g^*$ of the fixed point $(f^*, g^*)$ in Proposition 3 posses some desired properties. In particular, we require that the value function $v(\cdot, \cdot; \theta, g^*)$ has to be suitably differentiable.\footnote{We call the collection of solutions $(v(\cdot, \cdot; \theta, g))_{\theta \in \Theta}$ of the optimal stopping problem (13) sufficiently differentiable, if: (i) $v(\cdot, \cdot; g)$ is continuously differentiable in $\theta$, (ii) $v(\cdot, \cdot; g)$ allows for interchanging the order of differentiation with respect to $d$ and $\theta$ on the interior of $\bigcup_{\theta \in \Theta} C^{(0, g)} \times \{ \theta \}$, and (iii) the collection of boundary functions $(b(\cdot, \cdot; g))_{\theta \in \Theta}$ is continuously differentiable with respect to $e$ and $\theta$. Note that the boundary value problem associated with the optimal stopping problem is given by (II.1-II.5) in Appendix II.} For the latter notion, we use the boundary $b$ separating the default region from the non-default region. The boundary $b$ is formally defined for any $g \in \mathcal{A}_g$ satisfying (9) by

$$b(e, \theta; g) = \inf\{d \geq e : v(d, e; \theta, g) > 0\}, \text{ for } e \in [0, f(\theta; g)]. \quad (16)$$

The following proposition characterizes an equilibrium candidate as the solution to an implicit two-dimensional ODE and gives a condition for the candidate being indeed an equilibrium. This ODE is the centerpiece to the numerical computation of the equilibrium. Its proof is given in Appendix III.

**Proposition 4.** Given the setting of Proposition 3, denote by $(f^*, g^*)$ a fixed point of $T$. Suppose $f^*$, $g^*$, and $\phi$ are continuously differentiable, and the collection of solutions $(v(\cdot, \cdot; \theta, g^*))_{\theta \in \Theta}$ of the
boundary value problem (II.1-II.5) is sufficiently differentiable. Then \((f, \hat{g}) = (f^*, g^* \circ f^*)\) satisfies

\[
\begin{pmatrix}
f'(\theta) \\
\hat{g}'(\theta)
\end{pmatrix} = \begin{pmatrix}
(1 + \eta) \sigma^2 \\
2(r - \mu)
\end{pmatrix} \frac{f(\theta)^2}{\theta^2} - \frac{f(\theta)}{\theta} \frac{1}{\Phi(\theta)} \frac{f'(\theta)}{f^*(\theta; \hat{g} \circ f^{-1})} - \frac{f'(\theta)}{f^*(\theta; \hat{g} \circ f^{-1})} \phi(\theta) (f(\theta) - \hat{g}(\theta)),
\end{pmatrix}
\]

on \((\theta, \overline{\theta})\) with initial condition \((f(\theta), \hat{g}(\theta)) = \theta(f^*_1, f^*_1)\), where \((f^*_1, g^*_1)\) denotes the unique equilibrium of the perfect information case, that is, \(\Theta_1 = \{1\}\), \(\Phi(\theta) = \int_\theta^\theta \phi(t) \, dt\), and \(\eta = \frac{1}{\sigma^2} (\mu - \frac{1}{2} \sigma^2 + \sqrt{(\mu - \frac{1}{2} \sigma^2)^2 + 2 r \sigma^2}) > 0\). If

\[\hat{g}' \leq \frac{f^* \hat{g}'}{f}, \text{ on } (\theta, \overline{\theta}),\]

then the fixed point \((f^*, g^*)\) is an equilibrium.

Equation (17) is an implicit ODE.\(^{22}\) If this implicit ODE admits a unique solution \((f, \hat{g})\), satisfying the differentiability assumptions in Proposition 4, then the fixed point is given by \((f^*, g^*) = (f, \hat{g} \circ f^{-1})\) on \(\Theta \times f(\Theta)\). Now, if condition (18) holds, then the fixed point is an equilibrium that is moreover unique within the set of strategies, satisfying the differentiability assumptions.\(^{23}\)

### 4 Implications of Rating Equilibrium

In this section, we summarize a number of equilibrium implications of our model and derive empirically testable implications. Following an illustration of the rating agency’s Bayesian directional learning, we turn to the perceived rating inflation, which in our model evolves dynamically. Then,

\(^{22}\)This holds since \(\frac{\partial h}{\partial \theta}(f(\theta), \theta; \hat{g} \circ f^{-1})\) does not depend on the entire function \(\hat{g} \circ f^{-1}\). Instead, at \(\theta\) it depends on \(f(\theta), \hat{g}(\theta), f'(\theta)\) and \(\hat{g}'(\theta)\), or, more precisely, \(\frac{\partial h}{\partial \theta}(f(\theta), \theta; \hat{g} \circ f^{-1}) = h(f(\theta), \hat{g}(\theta), \hat{g}'(\theta)/f'(\theta), \theta)\) for some function \(h\); see (III.7) in Proposition 6 in Appendix III.

\(^{23}\)In contrast to explicit ODEs, the existence and uniqueness of solutions of implicit ODEs, also known as differential algebraic equations (DAEs), is more delicate and hence beyond the scope of this paper. Here, we feel it is more appropriate to point at the related specialist literature, such as Kunkel and Mehrmann (2006).
we deviate from the assumption of an unbiased rating agency and examine the effect of the rating agency taking a biased view on the firm. Finally, we discuss empirically testable hypotheses.

The firm has debt outstanding with a face value scaled to unity. \( C \) then denotes the interest rate in percentage terms payable by the firm to the debt holders depending on the rating \( R \). The firm generates cash flow per unit debt at the rate \( X \) with expected growth rate of \( \mu = 0 \) and a volatility of \( \sigma = 0.30 \). The rating agency faces a firm whose cash flow it observes imperfectly. The corresponding type \( \tilde{\theta} \) is distributed according to a truncated normal distribution with parameters \( \mu_\theta, \sigma_\theta \), and truncation to \( \Theta = [\theta, \bar{\theta}] \). Accordingly, the density of \( \tilde{\theta} \) is given by

\[
\phi(\theta) = \begin{cases} 
0, & \text{for } \theta < \theta, \\
\frac{1}{\sqrt{2\pi} \sigma_\theta c_\theta} \exp \left( -\frac{1}{2} \frac{(\theta - \mu_\theta)^2}{\sigma_\theta^2} \right), & \text{for } \theta \leq \theta \leq \bar{\theta}, \text{ and} \\
0, & \text{for } \theta > \bar{\theta},
\end{cases}
\]

(19)

where \( c_\theta = N((\bar{\theta} - \mu_\theta)/\sigma_\theta) - N((\theta - \mu_\theta)/\sigma_\theta) \) and \( N \) is the standard normal cumulative distribution function. The base case parameters are \( \mu_\theta = 1.0, \sigma_\theta = 0.25 \) and \( \Theta = [\theta, \bar{\theta}] = [0.50, 1.50] \). For the original distribution (before updating of beliefs), this means that the rating agency is unbiased, as \( E[\tilde{\theta}] = 1 \). We combine this setup with a risk-free rate \( r \) set to 0.0211.\(^{24}\)

### 4.1 Learning Mechanism

This section turns to the evolution of the rating agency’s assessment of a firm over time. Figure 2 illustrates the best responses of the rating agency and firm, respectively, as well as the rating agency’s continuous learning. It shows a sample path of a strongly underestimated cash flow (\( \theta = 0.5 \)). The rating agency observes an initial cash flow of 0.1366, for which the firm pays interest of 0.0955. The true cash flow is 0.1366/0.5 = 0.2732. In Panel a), the solid black line represents the observed cash flow, and the dashed black line its running minimum. The solid gray line presents the default threshold estimated by the rating agency, and the dashed gray line is the true default threshold of

\(^{24}\) Appendix I provides details about the calibration.
Figure 2: **Best responses (strongly underestimated cash flow).** This graph displays the best response of the rating agency as an estimated default threshold, dependent on the observed cash flow. The firm has a strongly underestimated cash flow with $\theta = 0.5$. In Panel a), the solid black line represents the observed cash flow, and the dashed black line its running minimum. The solid gray line is the default threshold estimated by the rating agency, and the dashed gray line is the true default threshold of the firm, which is unobservable to the rating agency. Panel b) zooms into Panel a). In Panel c), the black line represents the firm’s privately known true cash flow, and the solid gray line is the firm’s interest payment. The dashed gray line indicates the default threshold.
the firm, which is unobservable to the rating agency. Panel 2b zooms into Panel 2a for the period of financial distress (years 2 - 5).

As the observed cash flow decreases, the rating worsens. However, as the running minimum falls, the rating agency continuously adjusts the best response default threshold, as it rules out the most overestimated types, who would have defaulted for the observed path. As the observed cash flow increases again, the rating improves and the corresponding interest payments decrease. Panel 2c shows that at year 10, the identical cash flow level to time zero requires interest payments decreased by 241 basis points. The interest payment (solid gray line) at year 10 lies below the initial interest payments at time zero. Between the two intersections (i.e., between about \( t = 2 \) and \( t = 8 \) years), the firm has to raise new equity to cover the interest payments. This means that the firm’s best response to the rating agency’s strategy is not to default over the 10-year horizon illustrated in the graph, visualized by the cash flow process staying above the true default threshold in either panel. The firm’s strategy results in a positive dividend income stream for the firm’s owner in the start and the end of the horizon. However, it means that between the two mentioned intersections, the firm’s owner prevents the firm from defaulting by injecting new cash. Finally, when the observed cash flow again reaches the level of 0.1366 as at the start of the observation period, the firm pays interest of 0.0714 and a credit spread of 0.0503. This is a reduction of 241 basis points compared to the initial interest of 0.0955 and credit spread of 0.0744. The credit spread reduces by 32% in relative terms. This substantial change is the direct result of the rating agency’s updated beliefs, after observing a temporarily low cash flow and ruling out a wide range of cash flow overestimations, see also Figure 1.

4.2 Rating Inflation

In this section, we focus on the impact of learning on perceived rating inflation, which dynamically emerges despite the rating agency’s aim to rate as precise and unbiased as possible. In our model, rating inflation arises directly from Bayesian directional learning. Observing the survival of apparently distressed periods allows the rating agency to subsequently update the prior. Rating
inflation in our setting causes above-average assessments of a firm approaching its true default threshold. The rating inflation becomes more severe in periods of financial distress, but persists during the potential recovery afterwards.

Figure 3 illustrates the best responses of the rating agency and firm, respectively, for a sample path of an only mildly underestimated cash flow ($\theta = 0.9$). Again, in Panel 3a, the solid black line represents the observed cash flow, and the dashed black line its running minimum. The solid gray line presents the default threshold estimated by the rating agency, and the dashed gray line the unobservable true default threshold of the firm. Panel 3b zooms into Panel 3a for the distressed period. The true cash flow is now $0.1366/0.9 = 0.1518$. Note though that the rating agency observes again the cash flow of 0.1366, just as in the underestimated cash flow case. Therefore it again assigns an identical rating for a much lower true cash flow in this scenario.

The firm’s best response to the rating agency’s strategy is not to default for the first 4.5 years illustrated in the graph, visualized by the cash flow process staying above the true default threshold in either panel. As Panel 3c shows, the firm’s strategy results in a positive dividend income stream for the firm’s owner only during the first 1.5 years. After the intersection of the true cash flow (black line) and the interest payment (gray line) and until the default point in the fifth year, the firm’s owner prevents the firm from defaulting by injecting new cash. The survival of apparently distressed periods signals a higher true cash flow, and the rating agency consequently lowers the firm’s estimated default threshold.

Figure 3 illustrates the rating inflation. As the minimum observed firm cash flow deteriorates below a certain point, the rating agency unconsciously inflates the firm’s ratings until the end of the rating game. As a consequence, not only will the firm have to pay lower coupons if it stays in a regime of low current cash flow. In case the cash flow recovers, the firm enjoys the well-deserved relief in debt payments, given the higher cash flow, but also an additional relief due to the inflation. The rating agency’s Bayesian directional learning drives the extent of the rating inflation. The
Figure 3: Best responses (mildly underestimated cash flow). This graph displays the best response of the rating agency as an estimated default threshold, dependent on the imperfectly observed cash flow. The firm has a mildly underestimated cash flow with $\theta = 0.9$. In Panel a), the solid black line represents the imperfectly observed cash flow, and the dashed black line its running minimum. The solid gray line is the default threshold estimated by the rating agency, and the dashed gray line is the true default threshold of the firm, which is unobservable to the rating agency. Panel b) zooms into Panel a). In Panel c), the black line represents the firm’s privately observed true cash flow, and the solid gray line is the firm’s interest payment. The dashed gray line indicates the default threshold.
inflation appears once the estimated default threshold falls below the true default threshold, which happens in Panel 3b from around year 3.4 onwards.\textsuperscript{25}

The inflation persists if the firm recovers, because the Bayesian directional learning only lowers the estimated default threshold. Once eliminated measurement errors remain excluded. Because the rating agency learns the true $\theta$ and its implied true default threshold only at default, it cannot mitigate the rating inflation in any way. While the estimated default threshold falling short of the true default threshold implies rating inflation, the extent of inflation $f(\theta) - D^*_t$ still varies dynamically. More precisely, it increases whenever the observed cash flow process reaches a new minimum and thus, an additional part of the $\theta$ distribution can be ruled out.

We emphasize that rating inflation occurs even though the rating agency has an unbiased objective function. In our setting, rating inflation is a consequence of the rating agency's learning mechanism, which requires asymmetric information between firm and rating agency. In contrast, the rating agency literature and public discussion typically focus on rating inflation as a result of distorted incentives or conflicts of interest. The theoretical contributions in that field often assume that rating agencies observe the rated entity’s quality perfectly, see, for example, Fulghieri, Strobl, and Xia (2014). Alternatively, they receive an imprecise signal about a binary type, see, for example, Bolton, Freixas, and Shapiro (2012). Such frameworks preclude rating inflation due to resolution of asymmetric information over time. In the following, we analyze the effect of a biased rating agency objective function, which can be seen as a combination of the two approaches.

### 4.3 Rating Attitude

How does a shift of the rating attitude affect learning, rating inflation, and firm survival? So far, the cost rate $k^\pi$ in equation (7) defines symmetric costs for an over- or underestimation of the true default threshold. In reality, the rating agency could have incentives corresponding to an asymmetric

\textsuperscript{25}Formally, for a fixed type $\theta$ with default threshold $f(\theta)$, rating inflation appears once the predicted default threshold $D^*_t = g(E_t)$ is smaller than or equal to the true threshold, i.e., $g(E_t) \leq f(\theta)$. The random time $T_f(\theta)$ defined as $T_f(\theta) = \inf\{t \geq 0 : g(E_t) \leq f(\theta)\}$ captures this event. From the time $T_f(\theta)$ on, the predicted default threshold is lower than or equal to the true one; that is, $D^*_t \leq f(\theta)$, for $T_f(\theta) \leq t < \tau(\theta)$, and $T_f(\theta) < \tau(\theta)$ almost surely. Accordingly, the rating of the firm for type $\theta$ is inflated from $T_f(\theta)$ on.
cost function. On the one hand, the rating agency could lean towards regulatory authorities, who are mostly concerned about avoiding an underestimation of the true default threshold ($\hat{D}_t^* < f(\theta)$) particularly in economic downturns. On the other hand, the rating agency could lean towards issuers who are more concerned about avoiding an overestimation of the true default threshold ($\hat{D}_t^* > f(\theta)$). The latter is particularly relevant if the rating agency applies the issuer-pays model and if issuers can shop for favorable ratings. Then, the issuers’ choice whether to obtain a rating has a direct effect on the rating agency’s revenues. The investors’ preferences regarding over- or underestimation are ambiguous.\footnote{If debt is fairly priced and the investors can obtain adequate compensation, then they should be indifferent regarding the rating agency’s assessment. In our framework, we assume that investors and the rating agency have the same information level. Thus, the debt investors can simply ask for the appropriate cost of debt, regardless of which scheme the rating agency applies. If debt investors have regulatory disadvantages of holding low-rated debt, they might even prefer an underestimation of the true default threshold and as such collude with the issuers; see Opp, Opp, and Harris (2013). However, in our way of modeling, the cost of debt is determined directly from the rating. When the debt contract is already fixed, then debt investors will prefer an overestimation of the true default threshold, as it yields higher cash flows to debt.}

To allow for such alternative rating attitudes, we generalize the cost rate $k^\pi$ to $k^\pi(\alpha)$ defined by

$$k^\pi_t(\alpha) = \int_\Theta Q(\hat{D}_t^*, f(\theta); \alpha) \phi^\pi_t(\theta) \, d\theta, \text{ for } t \geq 0,$$

(20)

with $Q \colon \mathbb{R} \times \mathbb{R} \times [0, 1]$ given by $Q(d, f; \alpha) = (1 - \alpha) 1_{d \leq f} (d - f)^2 + \alpha 1_{d > f} (d - f)^2$. For $\alpha = 0.5$, Equation (20) reduces to Equation (7), that is, the case of a symmetric objective function.

In the following, we focus on the two boundary cases $\alpha = 0$ and $\alpha = 1$. For $\alpha = 0$, the objective function exposes the rating agency to costs if and only if it underestimates the true default threshold ($\hat{D}_t^* \leq f(\theta)$). The rating agency then plays safe by assuming the worst case. This leads to the highest feasible interest payments for the firm. Hence, the rating agency imposes the earliest possible default. Therefore, it learns as fast as possible, as it can adjust its prior the most, whenever it observes non-default for new historical minima of observed cash flow. Further, the firm’s credit risk at default is assessed accurately and firms default at the perfect information default threshold regardless of the type. The improvement in default prediction comes at the cost that prior to default, all firms make higher interest payments than under perfect information, and the interest differential

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is increasing for decreasing type. For $\alpha = 1$, the objective function exposes the rating agency to costs if and only if it overestimates the true default threshold ($\hat{D}_t^* > f(\theta)$). The rating agency then assumes the best case and refuses to learn. Consequently, the firm fully benefits from the information asymmetry. The rating is always better than under perfect information. Accordingly, firms pay less interest than under perfect information, the interest differential is increasing for increasing type, and firms delay default regardless of the observed type. See Proposition 8 and Proposition 9 in Appendix IV for the related results.

As an important implication of our model, we can explain what happens when rating agencies become more conservative and downgrade more frequently, as a response to sharpened regulatory rules such as the Dodd-Frank Act in the aftermath of the 2008-2009 financial crisis, see Dimitrov, Palia, and Tang (2015) and Bedendo, Cathcart, and El-Jahel (2018). In our framework, we capture the event as a sudden and unexpected shift of the rating attitude, which all market participants assume to remain in place forever after the shift.\footnote{The shift does not change the information about measurement errors. A measurement error that the rating agency ruled out previously remains excluded; one that is considered possible continues to be possible. Only after the firm adjusts its best response to the changed rating strategy, the rating agency can potentially rule out some measurement errors, as it observes further combinations of historic low observed cash flows and firm survival. Note that observing the same combination of historic low observed cash flow and firm survival as before the shift can transmit new information, as the firm’s incentives and its outlook for future needs of cash injection are changed. In particular, when the rating agency becomes more conservative, the best response of a sufficiently overestimated firm can be to default on the spot, without a change in the observed cash flow relative to before the shift.}

We illustrate the event numerically as a shift from a best-case rating with $\alpha = 1$ to an unbiased one with $\alpha = 0.5$ after two and a half years.

First, we focus again on the case of a firm with a strongly underestimated cash flow $\theta = \underline{\theta} = 0.5$, leading to the firm surviving for the whole observation period of ten years, as discussed in Figure 2. For this case, Figure 4 shows the shift of the rating attitude. Before the shift, the rating agency assumes the lowest feasible default threshold, which for $\theta = \underline{\theta} = 0.5$ equals the true default threshold. From $t = 2.5$ onwards, the figure is identical with Figure 2.\footnote{Note this is not generally true, but it holds here because no learning has taken place in either case. See also Footnote 27.} We observe an immediate shift in the estimated default threshold: instead of the solid gray line, the rating agency estimates the default threshold as presented by the blue line in Panel 4a. The shift of the estimated default
Figure 4: Rating Attitude Shift (strongly underestimated cash flow). This figure displays the impact of a shift of the rating attitude from a best-case rating to an unbiased one at $t = 2.5$. The firm has a strongly underestimated cash flow with $\theta = 0.5$ as in Figure 2. Panel a) shows the observed cash flow (black solid line) and its running minimum (black dashed line). The solid gray and blue lines present the estimated default threshold before and after the shift. The dashed blue line indicates the true default threshold. Panel b) zooms into Panel a). Panel c) shows the true cash flow (black line), and the costs of capital before the shift (gray line) and afterwards (blue line). The dashed blue line indicates the default threshold.
threshold implies a sizeable downgrading: the rating jumps downwards with an associated upwards jump in interest payments in Panel 4c. While the firm’s current cash flow further deteriorates, also the best-case rating agency reports a closer distance to default, and thus the gap between the interest payments implied by the unbiased and the best-case rating policy is reduced. Moreover, the unbiased rating agency learns and adjusts its estimated default threshold downwards. As the firm’s cash flow reaches almost the true default threshold by \( t = 5 \), also the unbiased rating agency is able to exclude almost the full range of \( \theta > \theta_0 = 0.5 \). Therefore, interest payments after \( t = 5 \) are only slightly higher for the unbiased rating policy than for the best-case rating policy.

Figure 5 illustrates the case of the mildly underestimated firm with \( \theta = 0.9 \) known from Figure 3 with the same shift of the rating attitude. This firm responds to the shift after two and a half years by increasing its (true) default threshold (dotted blue line, Panels 5a and 5b). The downgrade causes unsustainable capital costs, which force the firm to bankrupt earlier, namely, after around 4.4 years, instead of a bit further towards 4.5 years in the best case. Panel 5b illustrates that the best-case rating leads to an underestimated default threshold throughout, which is intuitive and resembles much of the discussion on distorted incentives in the issuer-pays model preceding the financial crisis. After the policy change towards the unbiased rating, the situation is different: the rating agency starts off predicting a default threshold above the true one. Learning from the firm’s survival and eliminating infeasibly high measurement errors, the unbiased rating agency predicts the true default threshold around \( t = 3.4 \). For further deterioration of the cash flow, we observe the dynamic emergence of rating inflation described in the previous section.

For the generalized objective function proposed in the current section, rating inflation is only eliminated for the most conservative rating agency (\( \alpha = 0 \)). Such an agency’s policy makes the firm default at the perfect-information default threshold regardless of the observed type. Otherwise, all types and all objective functions except for the one with \( \alpha = 0 \) induce the dynamic emergence of rating inflation to varying degrees.
Figure 5: Rating Attitude Shift (mildly underestimated cash flow). This figure displays the impact of a shift of the rating attitude from a best-case rating to an unbiased one at \( t = 2.5 \). The firm has a mildly underestimated cash flow with \( \theta = 0.9 \) as in Figure 3. Panel a) shows the observed cash flow (black solid line) and its running minimum (black dashed line). The solid gray and blue lines present the estimated default threshold before and after the shift. The dashed gray and blue line indicate the true default threshold before and after the shift. Panel b) zooms into Panel a). Panel c) shows the true cash flow (black line), and the costs of capital before the shift (gray line) and afterwards (blue line). The dashed gray blue line indicate the default threshold as in Panel a).
4.4 Empirically Testable Implications

The strategic interaction between the firm and the debt market naturally leads to hypotheses that are based on the firm’s signaling. Both default probabilities and expected recovery rates and thus credit spreads are affected by asymmetric information and its dynamic mitigation through the feedback effect. The implication of Duffie and Lando (2001) is that increasing information asymmetry leads to an increase in the credit spread. This is also found empirically by Yu (2005) and Lu, Chen, and Liao (2010) among others. The new angle for an empirical test based on our theory is the dynamic evolution of asymmetric information. Specifically, a firm’s survival of an apparently distressed period should mitigate part of the asymmetric information, as bad firms are ruled out. Thus, the first empirical implication of our model is:

**Implication 1.** Survival of an apparently distressed period reduces subsequent credit spreads for non-defaulted firms.

In our context, and interpreting the state variable as the firm’s asset value rather than cash flow (see Footnote 1), an apparently distressed period can be defined based on a critical level of a Merton (1974) distance-to-default measure, see Bharath and Shumway (2008). In the specific example of Section 4.1 calibrated with market data, learning from the firm’s signaling decreases the credit spread from 0.0744 to 0.0503, which is a difference of 241 basis points or 32% in relative terms. Regarding the relation between severity of distress and the outsiders’ learning, we state a second implication:

**Implication 2.** The closer a firm comes to the true default threshold, the more learning about the true state of the firm takes place.

This implies also that according to our mechanism and with our nature of asymmetric information, not much can be learned about the firms’ true quality from observing healthy, or, in rating terminology, investment-grade firms. Rather, learning happens particularly in the high-yield spectrum. A third empirical implication of our model is:
**Implication 3.** *Survival of an apparently distressed period leads to less pronounced surprise effects upon default.*

This implication should be understood in the cross-section of firms that are subject to varying degrees of ex-ante information asymmetry. Testing the size of the surprise effect upon default will complement Jankowitsch, Nagler, and Subrahmanyam (2014), who document a significant downward jump of bond prices on the default day. We predict that the jump size is mitigated for firms that have survived an apparently distressed period before the default event, and thus are subject to a lower degree of information asymmetry. A fourth implication is:

**Implication 4.** *Higher information asymmetry has the potential to lead to higher rating inflation.*

Although our model focuses on corporate credit ratings, the fourth implication can be related to the theory that rating inflation is more severe for more complex assets, see Skreta and Veldkamp (2009). Finally, based on the analysis in Section 4.3, we pose the fifth implication:

**Implication 5.** *When rating agencies become more conservative, this leads to earlier defaults, faster learning and reduction of information asymmetry, i.e., a more accurate default prediction.*

When a shift towards a more conservative rating policy leads to more accurate default prediction, this can be interpreted as a positive effect of regulation on rating quality. Our final prediction is thus broadly in line with Bedendo, Cathcart, and El-Jahel (2018). In contrast, Dimitrov, Palia, and Tang (2015) argue that at least with respect to Dodd-Frank, sharpened regulation rather had a negative effect on rating quality.
5 Conclusion

In this paper, we analyze a continuous-time rating game between a rating agency as a representative of a competitive debt market and a rated firm. The main friction in the model is a measurement error, which implies imperfectly observed cash flows. The decisions of the rating agency and the firm are linked through a feedback effect: The rating influences the firm’s cost of capital, to which the firm’s default policy and the rating respond again. The firm’s non-default in periods of apparent distress has an information value: it signals that the observed cash flow is overestimated only to a certain extent. Hence, the rating agency’s optimal strategy is to issue a higher rating for the same current cash flow, if the historical minimum has been sufficiently low, yielding a strategy that is non-Markovian in the cash flow. The firm responds to the rating strategy by maximizing its firm value by defaulting at a type-dependent default threshold.

The central mechanism of the game is what we coin Bayesian directional learning. The rating agency rules out types over time by noting that the firm does not default for observed cash flow levels too low for more overestimated types to survive. This one-sided narrowing of the measurement error reduces information asymmetry. It also implies the dynamic emergence of rating inflation, in particular for apparently distressed firms. As the rating agency rules out more and more types, at some point the firm’s true cash flow is lower than the rating agency’s estimate, leaving the rating agency to unconsciously underestimate the default risk.

The paper provides a rich framework for studying feedback effects in dynamic structural models in potential extensions. One extension of our structural model framework could allow for a broader concept of asymmetric information by adding ambiguity on the type. Then, the rating agency is not only limited by its imperfect observation of the firm’s cash flow, but it is also uncertain about the exact distribution of the measurement error. In another direction, our model framework carries over to the valuation of real options, which allows for embedding feedback effects between a firm’s investment policy and its valuation by the capital market into such modeling approaches.
References


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Appendix I  Interest Rates and Rating Parametrization

We use Lemma 3.1.2 of Bielecki and Rutkowski (2004),

\[
PD(T) = \mathbb{P} \left( \inf_{t \in [0,T]} X_t \leq f^*_1 \right) = \mathbb{P} \left( \inf_{t \in [0,T]} X_t \leq X_0/R \right) = R^{-2 \frac{\mu - \sigma^2/2}{\sigma^2}} \left( -\ln(R) + \frac{\mu - \sigma^2/2}{\sigma \sqrt{T}} T \right) + N \left( \frac{-\ln(R) - \left( \mu - \sigma^2/2 \right) T}{\sigma \sqrt{T}} \right), \tag{I.1}
\]

where \( N \) is the standard normal cumulative distribution function, \( f^*_1 \) is the firm’s default threshold in the case in which the rating agency can observe the cash flow perfectly, that is, \( \Theta = \{1\} \), and \( R = X_0/f^*_1 \) is the distance-to-default type rating associated with the given default probability \( PD(T) \), for time horizon \( T > 0 \). When we solve this equation for each \( PD(10)_i \), with \( i \) denoting the rating class, we obtain the corresponding rating \( R_i \) in our scale. The results are given in Table I.1. The values of \( C \) on \([1, \infty)\) are obtained by linear interpolation and extrapolation on the log-scale.

Figure I.1 illustrates Assumption 1 in light of the above interest rate structure. Panel I.1a displays the average interest payment rate \( C \) as a function of default probability \( PD \). If the rating turns bad, that is, if the rating implies a high default probability, then the firm’s interest payments increase sharply. On the other hand, if the firm is far away from default, a further increase in the rating has only a small effect on the firm’s interest payments. Panel I.1b draws the slope on the log-log scale of the interest payment rate function \( C \) depending on \( R \) based on the values given in Table I.1. In particular, the slope in the log-log scale lies in \([-L_C, 0)\) with \( L_C = 0.8989 \) and thus satisfies Assumption 1.

For this specification, the interest payment rate determining function \( C \) is calculated under the assumption of perfect information, that is, \( D = X \), or \( \tilde{\theta} = 1 \) and \( \Theta_1 = \{1\} \), from available market data. For seven rating classes AAA, AA, A, BBB, BB, B, and C/CCC, the interest rate \( C_i \) is determined by the average effective yield for each rating class, that is, averaging over the time
Default Probabilities

Test of Assumption 1

Figure I.1: Interest rates for different ratings. Panel I.1a displays the average interest payment rate $C$ as a function of default probability $PD$. Linear interpolation and extrapolation, respectively, is applied on the log-log scale. Panel I.1b draws the slope on the log-log scale of the interest payment rate function $C$ depending on $R$ based on the values given in Table I.1.

Table I.1: Rating-Dependent Interest Rate

<table>
<thead>
<tr>
<th>Rating Class</th>
<th>10-Year Default Probability</th>
<th>Implied Rating $R$</th>
<th>Interest Rate $C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.0086</td>
<td>18.2180</td>
<td>0.0439</td>
</tr>
<tr>
<td>AA</td>
<td>0.0109</td>
<td>16.8281</td>
<td>0.0447</td>
</tr>
<tr>
<td>A</td>
<td>0.0195</td>
<td>13.7125</td>
<td>0.0494</td>
</tr>
<tr>
<td>BBB</td>
<td>0.0464</td>
<td>9.7811</td>
<td>0.0573</td>
</tr>
<tr>
<td>BB</td>
<td>0.1527</td>
<td>5.5756</td>
<td>0.0732</td>
</tr>
<tr>
<td>B</td>
<td>0.2746</td>
<td>3.9336</td>
<td>0.0916</td>
</tr>
<tr>
<td>C/CCC</td>
<td>0.5584</td>
<td>2.2507</td>
<td>0.1513</td>
</tr>
</tbody>
</table>

Notes: The 10-year default probabilities are obtained from Table 25 in Standard and Poor’s (2016) and refer to rated US companies in the period 1981 to 2015. The implied rating is determined using (I.1). The interest rates are the average of effective corporate yields provided by Federal Reserve Economic Data.

For each rating class $i$, the distance-to-default type rating $R_i$ is extracted from the 10-year probability of default $PD_i$, which is collected from Table 25 of Standard and Poor’s (2016) based on US data in the period 1981 through 2015. The respective Federal Reserve Economic Data identifiers for yield data are BAMLC0A1CAAEY, BAMLC0A2CAAEY, BAMLC0A3CAEY, BAMLC0A4CBBBEY, BAMLH0A1HYBBEY, BAMLH0A2HYBBEY, and BAMLH0A3HYCEY. Note that the data for calibration stems predominantly from non-performance-sensitive debt instruments with a higher default risk than comparable PSD instruments. However, the higher credit risk of non-performance-sensitive debt instruments is perhaps accompanied by a higher interest rate rewarding for the additional risk. Thus, the effect of using non-performance-sensitive debt instruments for calibration does not seem to be critical. This case is an equilibrium, see Proposition 4.

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risk-free rate $r$ is set to 0.0211, which is the average 3-month T-Bill rate (DTB3) over the time period 01 January 1997 through 31 December 2016 provided by Federal Reserve Economic Data.

Consider the perfect information case as benchmark, that is, $\Theta_1 = \{1\}$. In this case, the equilibrium $(f_1^*, g_1^*)$ is given by $f_1^* = g_1^* = 0.0490$. Comparing this number to Table I.1, we see that in the base case a company being rated B or better is having a cash flow $X$ of $f_1^* R_B = 0.0490 \times 3.9336 = 0.1928$ or greater. This cash flow exceeds the interest payment rate $C$, which amounts to 0.0916 for a B rated firm or is smaller for a firm with a better rating. However, for the C/CCC rated firm, the corresponding cash flow amounts to $f_1^* R_{C/CCC} = 0.0490 \times 2.2507 = 0.1103$, which is not sufficient to cover the interest of 0.1513. In this case, the equity holders keep the company alive by injecting additional funds. If the cash flow drops further and reaches the default threshold $f_1^* = 0.0490$, the firm defaults on its debt and creditors receive the assets.
Appendix II Proof of Main Results

Let $Id$ denote the identity on $\mathbb{R}_0^+$, $Id(x) = x$, for $x \in \mathbb{R}_0^+$.

**Definition 1** (Perfect Bayesian Equilibrium in Markov Strategies). Strategies $(\tau^*, \hat{D}^{**})$ and beliefs $\pi^*$ constitute a perfect Bayesian equilibrium in Markov strategies (PBEM) if:

1. For every $0 \leq t < \tau^*$, $\theta \in \Theta$, and strategy $\tau(\theta)$

\[
\mathbb{E} \left[ \int_t^{\tau^*(\theta)} e^{-rs} (D_s/\theta - C(D_s/\hat{D}^{**}_s)) \, ds \bigg| \mathcal{F}_t \right] \geq \mathbb{E} \left[ \int_t^{\tau^*(\theta)} e^{-rs} (D_s/\theta - C(D_s/\hat{D}^{**}_s)) \, ds \bigg| \mathcal{F}_t \right].
\]

2. For every $0 \leq t < \tau^*$ and strategy $\hat{D}^*$

\[
-\mathbb{E} \left[ \int_t^{\tau^*} e^{-r_s} \int_\Theta (\hat{D}^{**}_s - \mathbb{E} [D_{\tau^*(\theta)} | \mathcal{G}_s])^2 \phi_s^{\pi^*}(\theta) \, d\theta \, ds \bigg| \mathcal{F}_t \right],
\]

\[
\geq -\mathbb{E} \left[ \int_t^{\tau^*} e^{-r_s} \int_\Theta (\hat{D}^{**}_s - \mathbb{E} [D_{\tau^*(\theta)} | \mathcal{G}_s])^2 \phi_s^{\pi^*}(\theta) \, d\theta \, ds \bigg| \mathcal{F}_t \right].
\]

3. Bayes’ rule is used to update beliefs $\pi^*$ with density $(\phi_t^{\pi^*})_{t \geq 0} = (I_t^{\pi^*} \phi)_{t \geq 0}$ whenever possible:

For every $t \geq 0$, if there exists $\theta_0 \in \Theta$ such that $t < \tau^*(\theta_0)$, then

\[
\phi_t^{\pi^*}(\theta) = \frac{\phi_{t_{- \tau^*(\theta_0)}}^{\pi^*}(\theta) 1_{t < \tau^*(\theta)}}{\int_\Theta \phi_t^{\pi^*}(\theta') 1_{t < \tau^*(\theta')} \, d\theta'}, \quad \text{for } \theta \in \Theta,
\]

where $\phi_0^{\pi^*} = \phi$, i.e. $L_0^{\pi^*}(\theta) = 1$, for $\theta \in \Theta$.

4. The strategies are Markov, that is,

\[
\tau^*(\theta) = \inf \{ t \geq 0 : (D_t, E_t) \in \mathcal{E}(\theta) \}, \quad \text{for a Borel set } \mathcal{E}(\theta) \subseteq \mathbb{R}^+, \theta \in \Theta, \text{ and}
\]

\[
\hat{D}_t^{**} = g(D_t, E_t), \quad \text{for some function } g : \mathcal{E} \to \mathbb{R}_0^+ \text{ for } 0 \leq t < \tau^*.
\]
Proof of Proposition 1. The function \( f \in \mathcal{A}_f \) specifying the firm’s strategy is given, which is then \( \tau = (\tau(\theta))_{\theta \in \Theta} \) with \( \tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\} \), for \( \theta \in \Theta \). The structure of the rating agency’s belief \( \pi \) that is consistent with the firm strategy \( f \) as given in (11) follows from Bayes’ rule. Using the consistent belief \( \pi \), the rating agency maximizes the respective utility, that is,

\[
\sup_{g \in \mathcal{A}_g} U_{RA}^\pi(\tau, g(E)) = - \inf_{g \in \mathcal{A}_g} \mathbb{E} \left[ \int_0^\tau e^{-\rho t} \int_\Theta (g(E_t) - f(\theta))^2 \phi_t^\pi(\theta) \, d\theta \, dt \right].
\]

The expression above is minimized in case \( g(E_t) \) minimizes for each \( 0 \leq t < \tau \)

\[
\int_\Theta (g(E_t) - f(\theta))^2 \phi_t^\pi(\theta) \, d\theta = \frac{\int_\Theta (g(E_t) - f(\theta))^2 \mathbf{1}_{f(\theta) < E_t} \phi(\theta) \, d\theta}{\int_\Theta \mathbf{1}_{f(\theta) < E_t} \phi(\theta) \, d\theta}
\]

\[
= \mathbb{E} \left[ (g(e) - f(\tilde{\theta}))^2 \big| f(\tilde{\theta}) < e \right] \Big|_{e=E_t}.
\]

In fact, we look for \( g(E_t) \), which minimizes the squared distance to the random variable \( f(\tilde{\theta})\big|_{f(\tilde{\theta}) < E_t} \), which is a function \( f \) of the random variable describing the type \( \tilde{\theta} \) based on the consistent belief \( \pi_t \). Therefore, the optimal \( g(E_t; f) \) has to be the expected value of \( f(\tilde{\theta})\big|_{f(\tilde{\theta}) < E_t} \), which is in essence (12).

This result can also be obtained by solving the first-order condition and checking the second-order condition for a minimum. For the irrelevant case \( t \geq \tau \), or, equivalently, \( E_t < \inf_{\theta \in \Theta} f(\theta) \), we set the critical default level at \( g(e; f) = e \), for \( e \leq \inf_{\theta \in \Theta} f(\theta) \). Thus, default is predicted to happen immediately, as it should have been occurred before. Clearly, \( g(\cdot; f) \in \mathcal{A}_g \).

From (12) we see that \( g(\cdot; f) \) is bounded by \( Id \), that is, \( g(e; f) \leq e \), for \( e \geq 0 \), and that \( g(\cdot; f) \) is non-decreasing.
Assuming that $f$ is strictly increasing, $f^{-1}$ is well defined on $f(\Theta)$. Take $e > e' > \inf_{\theta \in \Theta} f(\theta)$ and then

$$g(e; f) - g(e'; f) = \frac{\int_{\theta < f^{-1}(e)} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e)} \phi(\theta) \, d\theta} - \frac{\int_{\theta < f^{-1}(e')} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}$$

$$= \frac{\int_{\theta < f^{-1}(e)} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e)} \phi(\theta) \, d\theta} + \frac{\int_{f^{-1}(e')} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}$$

$$- \frac{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta} = g(e'; f) \frac{\int_{\theta < f^{-1}(e)} \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e)} \phi(\theta) \, d\theta}$$

$$+ \frac{\int_{f^{-1}(e')} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta} - \frac{\int_{f^{-1}(e')} \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}.$$

The quantity above is non-negative and further strictly positive on the interior of $f(\Theta)$ and converges to zero for $e \searrow e'$. Accordingly, $g(\cdot; f)$ is continuous and strictly increasing as claimed.  

The subset of $\mathcal{A}_g$, where the constraints in (9) hold, is denoted by $\mathcal{A}_g^C = \{g \in \mathcal{A}_g : g \text{ satisfies (9)}\}$. 

**Lemma 1.** Suppose that $g \in \mathcal{A}_g$ is non-decreasing and is bounded by $Id$, that is, $g(e) \leq e$, for $e \geq 0$, then $\mathcal{R}(g) \in \mathcal{A}_g^C$. Moreover, $\mathcal{R}(g) = g$, for $g \in \mathcal{A}_g^C$. 

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Proof of Lemma 1. We see immediately \( \mathcal{R}(g) \leq g \leq \text{Id} \). To observe that \( \mathcal{R}(g) \) is non-decreasing take \( e' \geq e > 0 \) and

\[
\mathcal{R}(g)(e') = e' \inf\{t(z)/z : 0 < z \leq e'\} \\
= e' \left( \inf\{t(z)/z : 0 < z \leq e\} \wedge \inf\{g(z)/z : e < z \leq e'\} \right) \\
\geq \left( \frac{e'}{e} \mathcal{R}(g)(e) \right) \wedge \left( e' \inf\{g(e)/z : e < z \leq e'\} \right) \\
= \left( \frac{e'}{e} \mathcal{R}(g)(e) \right) \wedge g(e) \\
= \mathcal{R}(g)(e) + \left( \frac{e' - e}{e} \mathcal{R}(g)(e) \right) \wedge (g(e) - \mathcal{R}(g)(e)) \\
\geq \mathcal{R}(g)(e).
\]

By definition, \( \mathcal{R}(g)/\text{Id} \) is non-increasing. These two properties rule out negative jumps as well as positive jumps, and hence \( \mathcal{R}(g) \) is continuous on \( \mathbb{R}^+ \). To check the continuity at 0, we observe that \( 0 \leq \mathcal{R}(g)(e) \leq e \), and thus \( \lim_{e \searrow 0} \mathcal{R}(g)(e) = 0 = \mathcal{R}(g)(0) \), as defined in (15). To address the other claimed properties of \( \mathcal{R}(g) \), observe that by \( g \) being non-decreasing it holds that

\[
\mathcal{R}(g)(e) = e \inf\{g(z)/z : 0 < z \leq e\} \\
\leq e \inf\{g(e)/z : 0 < z \leq e\} = eg(e)/e = g(e).
\]

Optimal stopping problems are connected to free-boundary value problems; see Peskir and Shiryaev (2006). For the case that the running minimum of a diffusion process is included in the state variables, Heinricher and Stockbridge (1991) and Barron (1993) provide a characterization of the solution of optimal control problems in terms of the solution of an associated free boundary value problem. While we do have a partial differential equation system for the firm’s optimal stopping problem in Equation (13), this is a non-standard system of partial differential equations, as we have to incorporate not only the cash flow but also the running minimum of the imperfectly
observed cash flow that the rating agency uses. We characterize the value function $v$ through a viscosity solution to the following free boundary problem.

Define the differential operator $\mathcal{L}^{(\theta,g)}$ by

$$
\mathcal{L}^{(\theta,g)} h = \mu d \frac{\partial h}{\partial d} + \frac{1}{2} \sigma^2 d^2 \frac{\partial^2 h}{\partial d^2} + k^{(\theta,g)} - rh,
$$

(II.1)

where $k^{(\theta,g)}(d,e) = d/\theta - C(d/g(e))$, for $(d,e) \in \mathcal{C}$. Then

$$
\mathcal{L}^{(\theta,g)} v \leq 0, 
$$

(II.2)

$$
v \geq 0, \text{ and}$$

(II.3)

$$
v \cdot \mathcal{L}^{(\theta,g)} v = 0,
$$

(II.4)

with boundary conditions

$$
0 = \frac{\partial v}{\partial d} \text{ on } \partial C^{(\theta,g)}, \text{ and } 0 = \frac{\partial v}{\partial e} \text{ on } \mathcal{D},
$$

(II.5)

where $C^{(\theta,g)} = \{(d,e) \in \mathcal{C} : v(d,e;\theta,g) > 0\}$ is the continuation region and $\mathcal{D} = \{(d,d) \in \mathcal{C} : d > 0\}$ is the diagonal. The first boundary condition is due to a smooth fit at the edge of the continuation region $\partial C^{(\theta,g)}$ and the second condition is a normal reflection on the diagonal $\mathcal{D}$, addressing the dependence on the running minimum $E$.

**Proof of Proposition 2.** For $g \in \mathfrak{A}^C_g$ and $\theta \in \Theta$, the early exercise region associated with the optimal stopping time of (13) is denoted by $\mathcal{E}(\theta;g) = \{(d,e) \in \mathcal{C} : v(d,e;\theta,g) = 0\}$, see also (V.8). The optimal strategy can be written as first entry time of the state process $(D,E)$ with starting value $(d,d)$, $d \geq 0$, in the early exercise region $\mathcal{E}(\theta;g)$,

$$
\tau_{(d,d)}(\theta;g) = \inf\{t \geq 0 : (D(t),E(t)) \in \mathcal{E}(\theta;g)\}.
$$
Now, $g \in A^C_g$, and we can apply the results of Lemma 7. Recall the definition in (V.9)

$$\mathcal{D}(\theta; g) = \{d \in \mathbb{R}_0^+ : (d, d) \in \mathcal{E}(\theta; g)\}, \text{ and } D(\theta; g) = \sup \mathcal{D}(\theta; g).$$

According to Lemma 7 we have that $\mathcal{D}(\theta; g) = [0, D(\theta; g)]$. Set $f(\theta; g) = D(\theta; g)$. The stopping time $\tau_{(d, d)}(\theta; g)$ is dependent on the starting value $(d, d)$. Consider first the case $d \leq D(\theta; g) = f(\theta; g)$. Then $d \in \mathcal{D}(\theta; g)$ and by the definition of $\mathcal{D}(\theta; g)$ we conclude that $(d, d) \in \mathcal{E}(\theta; g)$. Accordingly, we start in the early exercise region and stop immediately at 0, that is, $\tau_{(d, d)}(\theta; g) = 0$. Since we also have $D_0 = d \leq D(\theta; g) = f(\theta; g)$, (14) holds true. Now, consider $d > D(\theta; g) = f(\theta; g)$. The starting value $(d, d)$ of $(D, E)$ is not in $\mathcal{E}(\theta; g)$. The process $(D, E)$ has continuous paths, and $E$ is the running minimum of $D$. Thus, if $(D, E)$ decreases in the second component, then it has to travel through the diagonal $\mathcal{D}$. Since $\{(d, d) : d \in \mathcal{D}(\theta; g)\} \subseteq \mathcal{E}(\theta; g)$, we have that

$$\tau_{(d, d)}(\theta; g) = \inf \{t \geq 0 : (D(t), E(t)) \in \mathcal{E}(\theta; g)\}$$

$$\leq \inf \{t \geq 0 : (D_t, E_t) \in \{(d, d) : d \in \mathcal{D}(\theta; g)\}\}$$

$$= \inf \{t \geq 0 : D_t \leq f(\theta; g)\}.$$

Suppose now that $\tau_{(d, d)}(\theta; g) < \inf \{t \geq 0 : D_t \leq f(\theta; g)\}$ with some non-negative probability. Thus, $(D, E)$ has to hit $\mathcal{E}(\theta; g)$ with some non-negative probability before it eventually hits $(f(\theta; g), f(\theta; g))$. Then there exists $(d', e') \in \mathcal{E}(\theta; g)$ with $e' > f(\theta; g)$. However, this would contradict (V.11). Therefore, $\tau_{(d, d)}(\theta; g) = \inf \{t \geq 0 : D_t \leq f(\theta; g)\}$ almost surely, and (14) holds also in this case.

For our specific default barrier, we can provide more structure on how the barrier changes in the type in Lemma 2.

**Lemma 2.** Given the setting of Proposition 2 and let $\theta, \theta' \in \Theta$ with $\theta' \leq \theta$, then

$$f(\theta'; g) \leq f(\theta; g), \text{ and } f \leq \frac{f(\theta; g)}{\theta} \leq \frac{f(\theta'; g)}{\theta'} \leq \bar{f},$$

(II.6)
uniformly in \( g \), where \( 0 < f \leq \bar{f} < \infty \). In particular, \( f(\cdot; g) \) is Lipschitz continuous.

\[
|f(\theta; g) - f(\theta'; g)| \leq L_f |\theta - \theta'|, \text{ for } \theta, \theta' \in \Theta, \tag{II.7}
\]

where \( L_f = \bar{f} > 0 \) is the uniform Lipschitz constant for all \( g \). Moreover, suppose that Assumption 1 holds, then for \( \theta, \theta' \in \Theta \) with \( \theta' \leq \theta \) it holds that

\[
f(\theta; g) - f(\theta'; g) \geq l_f (\theta - \theta'), \tag{II.8}
\]

where \( l_f = (1 - L_C) \bar{f}_f > 0 \) is the uniform constant for all \( g \).

Proof of Lemma 2. For \( \theta, \theta' \in \Theta \), with \( \theta' \leq \theta \), and \( g \in \mathcal{C}_g \) is non-decreasing and bounded by \( Id \), we have \( v(\cdot, \cdot; \theta', g) \geq v(\cdot, \cdot; \theta, g) \) by part 2 of Lemma 6 and \( v(\cdot, \cdot; \theta', g), v(\cdot, \cdot; \theta, g) \geq 0 \) by part 1 of Lemma 6. From this, the corresponding early exercise regions \( \mathcal{E}(\theta; g) = \{(d, e) \in \mathcal{C} : v(d, e; \theta, g) = 0\} \) and \( \mathcal{E}(\theta'; g) = \{(d, e) \in \mathcal{C} : v(d, e; \theta', g) = 0\} \) (see also (V.8) for definition) satisfy \( \mathcal{E}(\theta'; g) \subseteq \mathcal{E}(\theta; g) \). This holds in particular on the diagonal \( \mathcal{D} \), that is, \( \mathcal{E}(\theta'; g) \cap \mathcal{D} \subseteq \mathcal{E}(\theta; g) \cap \mathcal{D} \), and \( [0, D(\theta'; g)] = \mathcal{D}(\theta'; g) \subseteq \mathcal{D}(\theta; g) = [0, D(\theta; g)] \), thus \( D(\theta'; g) \leq D(\theta; g) \). Identifying \( f(\theta'; g) = D(\theta'; g) \) and \( f(\theta; g) = D(\theta; g) \), as is done in the proof of Proposition 2, gives \( f(\theta'; g) \leq f(\theta; g) \), establishing the first part of (II.6). For the second part, we rewrite (14) of Proposition 2 in the firm scale \((x, y)\). With \((d, e) = \theta(x, y)\), we obtain \( \tau_{(d,d)}(\theta; g) = \inf\{t \geq 0 : X_t \leq f(\theta; g)/\theta\} \), and \( f(\theta; g)/\theta \) is the default barrier in the firm scale, for \( \theta \in \Theta \). Regardless of the rating strategy \( g \), the interest payment rate function \( C \) is bounded from below by \( \bar{C} \) and from above by \( \bar{C} \). Denote by \( \underline{\theta} \) the default threshold in the firm scale for the constant interest \( \underline{C} \) (by setting \( g = 0 \)) and by \( \bar{\theta} \) the default threshold in the firm scale for the constant interest \( \bar{C} \) (by setting \( \bar{g} = \infty \), which is interpreted as limiting case, and noting that \( C \) is, as in Lemma 6, extended to \([0, \infty]\) by setting \( C|_{(0,1)} = C(1) \)). From \( \underline{g} \leq g \leq \bar{g} \) we obtain with part 3 of Lemma 6 that \( w(\cdot, \cdot; \underline{\theta}, \underline{g}) \geq w(\cdot, \cdot; \theta, g) \geq w(\cdot, \cdot; \theta, \bar{g}) \), and hence \( \mathcal{S}(\theta; g) \subseteq \mathcal{S}(\theta; \bar{g}) \subseteq \mathcal{S}(\theta; g') \), where \( \mathcal{S}(\theta; g') = \{(x, y) \in \mathcal{C} : w(x, y; \theta, g') = 0\} \), for \( g' \in \{\underline{g}, \bar{g}, \bar{g}\} \). The boundary cases \( \underline{g} \) and \( \bar{g} \) also admit a critical default level, which is given by
$f = \frac{\eta(r - \mu)}{(1 + \eta)r} C$ and $\tilde{f} = \frac{\eta(r - \mu)}{(1 + \eta)r} \tilde{C}$, respectively, where $\eta = \frac{b - \frac{1}{2} \sigma^2}{\sigma^2} + \sqrt{\left(\frac{b - \frac{1}{2} \sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0$, see, for example, Equation (C.5) in Manso (2013). Thus $f \leq f(\theta; g)/\theta \leq \tilde{f}$, for all $\theta \in \Theta$ and $g \in \mathcal{A}_g^C$ that are non-decreasing and bounded by $Id$. Part 2 of Lemma 6 gives $w(c; \theta') \leq w(c; \theta, g)$, yielding $f(\theta; g)/\theta \leq f(\theta'; g)/\theta'$, what completes the second part of (II.6). Finally, recall that $\theta' \leq \theta$ and use the established results given in (II.6) to see

$$0 \leq f(\theta; g) - f(\theta'; g) = \theta \frac{f(\theta; g)}{\theta} - f(\theta'; g) \leq \theta \frac{f(\theta'; g)}{\theta'} - f(\theta'; g) = (\theta - \theta') \frac{f(\theta'; g)}{\theta'} \leq (\theta - \theta') \tilde{f},$$

which proves (II.7). Now, assume that Assumption 1 holds. For $\theta, \theta' \in \Theta$, with $\theta' \leq \theta$, and $g \in \mathcal{A}_g^C$ is non-decreasing and bounded by $Id$, and $(d, e) \in \mathcal{C}$. From part 1 of Lemma 5 we see $g_{\theta'} \leq g_{\theta}$, and since $C$ is non-increasing, $C(x/g_{\theta'}(y)) \leq C((\theta'/\theta)x/g_{\theta}(y))$, for $(x, y) \in \mathcal{C}$. Assumption 1 gives

$$C(z) \leq C(z') \leq (z/z')^{Lc} C(z), \text{ for } 1 \leq z' \leq z.$$

With $z' = (\theta'/\theta)x/g_{\theta}(y)$ and $z = x/g_{\theta}(y)$, it holds that $z' \leq z$, and we obtain that $C((\theta'/\theta)x/g_{\theta}(y)) \leq (\theta/\theta')^{Lc} C(x/g_{\theta}(y))$, thus

$$C(x/g_{\theta'}(y)) \leq (\theta/\theta')^{Lc} C(x/g_{\theta}(y)), \text{ for } (x, y) \in \mathcal{C}. \quad \text{(II.9)}$$

Consider the optimal stopping problem in (13), but now scaled by $\hat{\theta} = (\theta/\theta')^{Lc} \geq 1$,

$$\bar{v}(d, e; \hat{\theta}, g_{\theta}) = \sup_{\tau \in \mathcal{F}_{d, e}} \mathbb{E}_{(d, e)} \left[ \int_0^\tau e^{-rt} (D_t - \hat{\theta} C(D_t/g_{\theta}(E_t))) \, dt \right],$$

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or, noting that \((E,D)\) has the same distribution as \((X,Y)\), provided the starting values are identical, we can express this alternatively as

\[
\tilde{v}(x,y; \hat{\theta}, g_\theta) = \sup_{\tau \in \mathcal{F}_{(x,y)}} \mathbb{E}_{(x,y)} \left[ \int_0^\tau e^{-rt} \left( X_t - \hat{\theta} C(X_t/g_\theta(Y_t)) \right) \, dt \right],
\]

and, as in (V.2),

\[
w(x,y; \theta', g) = \sup_{\tau \in \mathcal{F}_{(x,y)}} \mathbb{E}_{(x,y)} \left[ \int_0^\tau e^{-rt} \left( X_t - C(X_t/g_{\theta'}(Y_t)) \right) \, dt \right],
\]

By Proposition 2, the latter two optimal stopping problems admit a critical default level for describing the optimal default time. The value function \(\tilde{v}(\cdot,\cdot; \hat{\theta}, g_\theta)\) is in the rating agency-scale \((D,E)\) and the critical level is given by \(f(\hat{\theta}; g_\theta)\). The value function \(w(\cdot,\cdot; \theta', g)\) is in the firm-scale \((X,Y)\) and critical default level is given by \(f(\theta'; g)/\theta'\). Recalling that the interest payments are ordered uniformly (see (II.9)), we can order the critical default levels, as a higher interest payment leads to a higher critical default value, hence

\[f(\hat{\theta}; g_\theta) \geq f(\theta'; g)/\theta'.\]

Now, focus on \(f(\hat{\theta}; g_\theta)\) and recall that \(\hat{\theta} = (\theta/\theta')^{L_C} \geq 1\), and since \(f(\cdot; g)/Id\) is decreasing by (II.6),

\[f(\hat{\theta}; g_\theta) = \hat{\theta} \frac{f(\hat{\theta}; g_\theta)}{\hat{\theta}} \leq \hat{\theta} \frac{f(1; g_\theta)}{1} = \hat{\theta} f(1; g_\theta) = \hat{\theta} \frac{f(\theta; g)}{\theta},\]

where the last step follows from plugging \((1, g_\theta)\) in (13) and comparing this with (V.2), to see that \(f(1, g_\theta)\) is also the optimal default level in firm-scale for \((\theta, g)\), which is \(f(\theta; g)/\theta\). Using this, we
find a lower bound to

\[ f(\theta; g) - f(\theta'; g) \geq f(\theta; g) - \theta' f(\hat{\theta}; g) \geq f(\theta; g) - \theta' \frac{\hat{\theta}}{\theta} f(\theta; g) \]

\[ = f(\theta; g) \left(1 - (\theta'/\theta)^{1-L_C}\right) \geq f(\theta) \left(1 - (\theta'/\theta)^{1-L_C}\right). \]

Taking the limit gives us

\[ D_+ f(\theta'; g) = \liminf_{\theta \to \theta'} \frac{f(\theta; g) - f(\theta'; g)}{\theta - \theta'} \]

\[ \geq \liminf_{\theta \to \theta'} \frac{f(\theta) \left(1 - (\theta'/\theta)^{1-L_C}\right)}{\theta - \theta'} = (1 - L_C) f, \]

implying the claimed inequality. \qed

\textit{Proof of Proposition 3.} A fixed point is obtained by the Schauder fixed-point theorem: Let \( \mathcal{K} \) be a nonempty convex compact subset of a Banach space \( \mathcal{V} \); if \( \tilde{T} : \mathcal{K} \to \mathcal{K} \) is continuous, then \( \tilde{T} \) has a fixed point. Set \( \mathcal{V} = C(\Theta, \mathbb{R}) \times C(\Xi, \mathbb{R}) \) endowed with the sup-norm, that is, \( \|(f, g)\|_\infty = \max(\|f\|_\infty, \|g\|_\infty) \), for \( (f, g) \in \mathcal{V} \), where \( \|f\|_\infty = \sup_{\theta \in \Theta} |f(\theta)| \), \( \|g\|_\infty = \sup_{\xi \in \Xi} |g(\xi)| \), and the set \( \Xi = [\xi, \bar{\xi}] \) is defined by \( \xi = \theta f \) and \( \bar{\xi} = \theta f + \frac{\bar{\theta}^2 J^2}{(\theta f)} \). Since \( \Theta \) and \( \Xi \) are closed intervals, \( \mathcal{V} \) is a Banach space as required. Next, we define the set \( \mathcal{K} = \mathcal{K}_f \times \mathcal{K}_g \). The set \( \mathcal{K}_f \subseteq C(\Theta, \mathbb{R}^+) \) should contain the firm strategies in \( C(\Theta, \mathbb{R}^+) \subseteq \mathcal{A}_f \) that are relevant for fixed points of \( T \). Based on Lemma 2, define

\[ \mathcal{K}_f = \{ f \in C(\Theta, \mathbb{R}^+) : l_f (\theta - \theta') \leq f(\theta') - f(\theta) \leq L_f (\theta - \theta'), \quad (\text{II.10}) \]

\[ 0 \leq f(\theta) \leq \bar{\theta} \bar{f}, \text{ for } \theta, \theta' \in \Theta \text{ with } \theta' \leq \theta, \]

where \( 0 < l_f = (1 - L_C) \bar{f} \leq \bar{f} = L_f \), with \( 0 \leq L_C < 1 \). The set \( \mathcal{K}_g \subseteq C(\Xi, \mathbb{R}^+) \) should contain the rating agency strategies that are relevant for fixed points of \( T \). Proposition 1 and Lemma 4 suggest

\[ \mathcal{K}_g = \{ g \in C(\Xi, \mathbb{R}^+) : \xi \leq g \leq \bar{g} = \text{Id}, g \text{ non-decreasing, } g/\text{Id} \text{ non-increasing} \}. \quad (\text{II.11}) \]
It is sufficient to constrain the domain of the rating agency strategy $g$ from originally $\mathbb{R}_0^+$ to $\Xi$ for the following reason. By Proposition 1, we see that $g(\cdot;f)$, for $f \in \mathcal{H}_f \subseteq \mathcal{A}_f$, is continuous, non-decreasing, bounded by $Id$. Moreover, $g(\cdot;f)$ is determined by $f$ on $f(\Theta)$. For $e \geq \bar{e}_f = \sup_{\theta \in \Theta} f(\theta)$, it holds that $g(e;f) = g(\bar{e}_f;f)$, and $g(\cdot;f) = Id$ on $[0,e_f]$, with $e_f = \inf_{\theta \in \Theta} f(\theta)$. Furthermore, $f(\Theta) \subseteq [\theta_f, \bar{\theta}_f]$, for $f \in \mathcal{H}_f$. This property is preserved when considering $R \circ g(\cdot;f)$ and the set $[\theta_f, \bar{\theta}_f^2 f^2/(\theta_f^2)] = \Xi$ according to Lemma 1, that is, $R \circ g(e;f) = g(\bar{e}_f;f)$, for $e \geq \bar{\theta}_f^2 f^2/(\theta_f^2) = \bar{\theta}_f$, and $R \circ g(e;f) = e$, for $0 \leq e \leq \bar{\theta}_f = \bar{\xi}$, for all $f \in \mathcal{H}_f \subseteq \mathcal{A}_f$. For this reason we cut the non-relevant part of $T$ off, that is, we consider $\tilde{T} : \mathcal{H} \mapsto \mathcal{H}$, $(f,g) \mapsto (f(\cdot;g|_{\mathbb{R}_0^+}), R(g(\cdot;f))|_{\Xi})$, where $g|_{\mathbb{R}_0^+}$ is understood as the obvious continuation of $g \in \mathcal{H}_g$ from $\Xi$ to $\mathbb{R}_0^+$, with

$$g|_{\mathbb{R}_0^+}(e) = \begin{cases} e & \text{for } 0 \leq e < \bar{\xi}, \\ g(e) & \text{for } e \in \Xi, \\ g(\bar{\xi}) & \text{for } e > \bar{\xi}, \end{cases}$$

and thus $\tilde{T}$ is well-defined. To show that $\tilde{T}$ has a fixed point, we have to prove that $\mathcal{H} \subseteq \mathcal{V}$ is nonempty, convex, and compact, and $\tilde{T}$ is continuous.

The set $\mathcal{H} = \mathcal{H}_f \times \mathcal{H}_g$ is nonempty, convex and compact, if the factors $\mathcal{H}_f$ and $\mathcal{H}_g$ have these properties. By definition, $\mathcal{H}_f$ and $\mathcal{H}_g$ are both nonempty. The convexity and compactness follows from Lemma 8 and Lemma 9, respectively. It remains to show that $g \mapsto f(\cdot;g|_{\mathbb{R}_0^+})$ and $f \mapsto R(g(\cdot;f))|_{\Xi}$ are both continuous.

Consider the best response of the firm $f(\cdot;\cdot) : \mathcal{H}_g \rightarrow \mathcal{H}_f$, $g \mapsto f(\cdot;g|_{\mathbb{R}_0^+})$; see Proposition 2. Take $g, g' \in \mathcal{H}_g$ with $\|g - g'\|_\infty \leq \varepsilon$, for some $\varepsilon > 0$. Denote the continuations by $\bar{g} = g|_{\mathbb{R}_0^+}$ and $\bar{g}' = g'|_{\mathbb{R}_0^+}$, then $\|\bar{g} - \bar{g}'\|_\infty \leq \varepsilon$. Fix $\theta \in \Theta$, then

$$|\bar{g}_\theta(e) - \bar{g}'_\theta(e)| = \frac{1}{\theta} |\bar{g}(\theta e) - \bar{g}'(\theta e)| \leq \frac{\varepsilon}{\theta},$$

for all $e \geq 0$. 

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and hence $\|\hat{g}_\theta - \hat{g}'_\theta\|_\infty \leq \varepsilon / \theta$, for $\theta \in \Theta$. For now, we fix $\theta \in \Theta$ and estimate $|f(\theta; \hat{g}) - f(\theta; \hat{g}')|$. For doing so, we focus on $\hat{g}_\theta$ and $\hat{g}'_\theta$. For $0 \leq e < \xi / \theta$, we have $\hat{g}_\theta(e) = \hat{g}(\theta e) / \theta = \hat{g}'(\theta e) / \theta = \hat{g}'_\theta(e)$. For $e \geq \xi / \theta$, note that $\hat{g}_\theta(e) = \hat{g}(\theta e) / \theta \geq \xi / \theta$ and

$$\hat{g}'_\theta(e) \leq \hat{g}_\theta(e) + \varepsilon / \theta = \hat{g}_\theta(e)(1 + \varepsilon / (\theta \hat{g}_\theta(e))) \leq \hat{g}_\theta(e)(1 + \varepsilon / (\theta \hat{g}_\theta(e))) \leq \hat{g}_\theta(e)(1 + \varepsilon / \xi) .$$

Define $\lambda(\varepsilon) = 1 + \varepsilon / \xi \geq 1$, and by symmetry we have

$$\lambda(\varepsilon)^{-1} \hat{g}_\theta \leq \hat{g}'_\theta \leq \lambda(\varepsilon) \hat{g}_\theta .$$

The interest payment rate function $C$ is non-increasing, thus

$$C(\lambda(\varepsilon) x / \hat{g}_\theta(y)) \leq C(x / \hat{g}'_\theta(y)) \leq C(\lambda(\varepsilon)^{-1} x / \hat{g}_\theta) , \text{ for } (x, y) \in C ,$$

and Assumption 1 applied in the same fashion as in the proof of Lemma 2 gives us

$$\lambda(\varepsilon)^{-LC} C(x / \hat{g}_\theta(y)) \leq C(x / \hat{g}'_\theta(y)) \leq \lambda(\varepsilon)^{LC} C(x / \hat{g}_\theta(y)) .$$

Using the same arguments as in the proof of Lemma 2, we have

$$f(\lambda(\varepsilon)^{-LC}; \hat{g}_\theta) \leq f(\theta; \hat{g}') / \theta \leq f(\lambda(\varepsilon)^{LC}; \hat{g}_\theta) .$$

Recalling that $f(\cdot; h_\theta) / Id$ is non-increasing and $\lambda_\varepsilon(\varepsilon) \geq 1$, we proceed as in the proof of Lemma 2, to obtain

$$\lambda(\varepsilon)^{-LC} f(\theta; \hat{g}) \leq f(\theta; \hat{g}') \leq \lambda(\varepsilon)^{LC} f(\theta; \hat{g}) .$$
Now, we can estimate
\[
|f(\theta; \hat{g}) - f(\theta; \hat{g}')| \leq (\lambda_g(\varepsilon)L_C - 1) f(\theta; \hat{g}) \leq (1 + \varepsilon / \hat{\xi})L_C - 1) \bar{\theta} \bar{f} \leq \frac{L_C \bar{\theta} \bar{f}}{\hat{\xi}} \varepsilon ,
\]
where we used (II.6) of Lemma 2 in the second step and \(0 < L_C < 1\), which is given by Assumption 1, in the third step. Hence we obtain a uniform upper bound in \(\theta \in \Theta\), that is,
\[
\|f(\cdot; g)\|_{\mathbb{R}^n_0} - f(\cdot; g')\|_{\mathbb{R}^n_0} \|_\infty = \|f(\cdot; g) - f(\cdot; g')\|_\infty \leq \frac{L_C \bar{\theta} \bar{f}}{\hat{\xi}} \varepsilon .
\]

This implies that \(g \mapsto f(\cdot; g)\) is continuous on \(\mathcal{K}_g\).

Finally, consider the transformed best response of the rating agency, which is given by \(\mathcal{R}(g(\cdot))|_{\mathbb{R}} : \mathcal{K}_f \to \mathcal{K}_g, f \mapsto \mathcal{R}(g(\cdot; f))|_{\mathbb{R}}\), see Proposition 1. We focus on the best response \(f \mapsto g(\cdot; f)\) for now, the transformation \(\mathcal{R}\) is dealt with later, using Lemma 4. Take \(f, f' \in \mathcal{K}_f\) with \(\|f - f'\|_\infty \leq \varepsilon\), for some \(\varepsilon > 0\). Consider \(e \in [0, f(\theta) \land f'(\theta))\), then \(g(e; f) = g(e; f') = e\) and
\[
|g(e; f) - g(e; f')| = 0, \text{ for } e \in [0, f(g) \land f'(g)) .
\]

Now, \(e \in [f(\theta) \land f'(\theta), f(\theta) \lor f'(\theta))\). Without loss of generality, assume \(f(\theta) < f'(\theta)\), and thus \(f(\theta) \leq e < f'(\theta)\). From this we see by (12), that \(g(e; f') = e\) and \(f(\theta) \leq g(e; f) \leq e\), and since \(e < f'(\theta)\) and \(|f(\theta) - f'(\theta)| \leq \|f - f'\|_\infty = \varepsilon\), we have
\[
|g(e; f) - g(e; f')| \leq \varepsilon, \text{ for } e \in [f(\theta) \land f'(\theta), f(\theta) \lor f'(\theta)) .
\]

Consider \(e \in [f(\theta) \lor f'(\theta), e(\varepsilon, f, f')]\), where \(e(\varepsilon, f, f') = f(\theta) \lor f'(\theta) + L_f \varepsilon^{1/2}\). Then by the uniform Lipschitz continuity of \(\mathcal{K}_f\) with Lipschitz constant \(L_f\) (see Lemma 8), it follows that \(f(\theta) \lor f'(\theta) \leq g(e; f), g(e; f') \leq f(\theta) \lor f'(\theta) + L_f \varepsilon^{1/2}\), and thus
\[
|g(e; f) - g(e; f')| \leq \varepsilon + L_f \varepsilon^{1/2} , \text{ for } e[f(\theta) \lor f'(\theta), e(\varepsilon, f, f')] .
\]
Consider $e \geq e(\epsilon, f, f')$. Without loss of generality, assume $f^{-1}(e) \leq f'^{-1}(e)$ and note that by $f, f' \in \mathcal{X}_f$ both functions are continuous and strictly increasing with minimum slope $l_f$, and thus their respective inverse functions exist and are well-defined. Using the definition of $g(\cdot, f)$ in (12), we can write

$$
g(e; f') = \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'}
$$

$$
= \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'}
$$

$$
= \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'}
$$

$$
= g(e; f) - g(e; f) \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'}.
$$

Hence

$$
|g(e; f) - g(e; f')| \leq g(e; f) \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'} + \frac{\int_{\Theta} f'(e') \phi(e') \de e'}{\int_{\Theta} f'(e') \phi(e') \de e'}.
$$

All three expressions on the right-hand side need to become small for $\epsilon \searrow 0$. Observe that $0 \leq f'^{-1}(e) - f^{-1}(e) \leq l_f^{-1} \epsilon$, where the first inequality follows by assumption and the second by
the fact that \( f, f' \) are strictly increasing with a minimum slope of \( l_f > 0 \) and \( \| f - f' \|_\infty \leq \epsilon \). Also, \( f, f' \) are bounded by \( \overline{\theta} \overline{f} \) by (II.6), and so are \( g(\cdot; f) \) and \( g(\cdot; f') \). Since \( \phi \) is bounded away from zero and bounded from above by assumption, we see that

\[
\int_{\overline{\theta}}^{f^{-1}(e)} \phi(e') \, de' \geq \int_{\overline{\theta}}^{f^{-1}(e, f, f')} \phi(e') \, de' \geq \int_{\overline{\theta}}^{f^{-1}(f(\overline{e}) + L_f \epsilon)} \phi(e') \, de' = \phi(f^{-1}(f(\overline{e}) + L_f \epsilon) - \overline{\theta}) \geq \phi L_f \epsilon^{1/2} L_f = \phi \epsilon^{1/2}.
\]

Using this, we can bound the first expression on the right-hand side of (II.12),

\[
g(e; f) \int_{\overline{\theta}}^{f^{-1}(e)} \phi(e') \, de' \leq \overline{\theta} \overline{f} \int_{\overline{\theta}}^{f^{-1}(e)} \phi(e') \, de' = \overline{\theta} \overline{f} \phi \epsilon^{1/2}.
\]

For the second expression in (II.12) the same arguments apply, now aimed at \( f \) rather than \( g(\cdot; f) \), and

\[
\int_{\overline{\theta}}^{f^{-1}(e)} f'(e') \phi(e') \, de' \leq \overline{\theta} \overline{f} \int_{\overline{\theta}}^{f^{-1}(e)} \phi(e') \, de' = \overline{\theta} \overline{f} \phi \epsilon^{1/2}.
\]

The third expression in (II.12) can be estimated as follows:

\[
\frac{\int_{\overline{\theta}}^{f^{-1}(e)} |f'(e') - f(e')| \phi(e') \, de'}{\int_{\overline{\theta}}^{f^{-1}(e)} \phi(e') \, de'} \leq \frac{\| f - f' \|_\infty}{\phi \epsilon^{1/2}} \leq \frac{\epsilon^{1/2}}{\phi}.
\]

Adding up the three expressions, we obtain as bound

\[
|g(e; f) - g(e; f')| \leq \frac{2 \overline{\theta} \overline{f} \overline{\phi} + l_f}{\phi L_f} \epsilon^{1/2}, \text{ for } e \in [e(\epsilon, f, f'), \infty).
\]

From there the uniform bound can be expressed as follows:

\[
\| g(\cdot; f) - g(\cdot; f') \|_\infty \leq \epsilon + \frac{2 \overline{\theta} \overline{f} \overline{\phi} + l_f + \phi l_f L_f}{\phi l_f} \epsilon^{1/2}. \quad (II.13)
\]
This implies that $f \mapsto g(\cdot; f)$ is continuous. From Lemma 4 and noting that the restriction to $\Xi$ does no harm, the continuity of $\mathcal{R}(g(\cdot; f))|_{\Xi}$ follows.
Appendix III  ODE Characterization of Best Responses and Potential Equilibria

Proposition 4 gives an ODE characterization of an equilibrium in case a specified condition holds. In the following, this result is derived. First, the best responses of both, rating agency and firm, are characterized by solutions to ODEs systems. Based on these, the equilibrium characterization is established.

III.1 Best Response of Rating Agency ODE

**Proposition 5.** Given a strategy \( f \in \mathcal{A}_f \) that is continuous and strictly increasing, the best response \( g(\cdot; f) \), given in (12) of Proposition 1, and its transformation \( R(g(\cdot; f)) \), defined in (15), can be characterized as follows

\[
g(\cdot; f) = \hat{g}_f \circ f^{-1} \quad \text{and} \quad R(g(\cdot; f)) = \tilde{g}_f \circ f^{-1}, \quad \text{on} \quad f(\Theta),
\]

where \( \hat{g}_f \) and \( \tilde{g}_f \) have initial values

\[
\hat{g}_f(\hat{\theta}) = \tilde{\theta}_f(\hat{\theta}) = f(\theta)
\]

and derivative \( \hat{g}'_f \) and \( \tilde{g}'_f \) that satisfy

\[
\hat{g}'_f = \frac{\Phi}{\Phi}(f - \hat{g}_f) \quad \text{and} \quad \tilde{g}'_f = f' \tilde{g}_f \mathbf{1}_{\tilde{g}_f} < \hat{g}_f + \min \left( \tilde{g}'_f, f' \frac{\tilde{g}_f}{f} \right) \mathbf{1}_{\tilde{g}_f} = \hat{g}_f,
\]

Lebesgue almost everywhere on \((\theta, \tilde{\theta})\), where \( \Phi(g) = \int_{\theta}^{\tilde{\theta}} \phi(r) \, dr \), \( \theta \in \Theta \).
Corollary 1. Suppose that $f \in \mathcal{A}_f$ and $\phi$ are both continuously differentiable, then the function pair $(\hat{g}_f, \tilde{g}_f)$, given in Proposition 5, is the solution to the ODE

$$
(\hat{g}_f', \tilde{g}_f') = \left( \frac{\phi}{\Phi} (f - \hat{g}_f), \frac{f'}{f} \hat{g}_f < \hat{g}_f + \min \left( \hat{g}_f', \frac{f'}{f} \hat{g}_f \right) \right),
$$

(III.4)
on $(\bar{\theta}, \bar{\theta})$ with initial conditions

$$
(\hat{g}_f(\theta), \tilde{g}_f(\theta)) = (f(\theta), f(\theta)) \text{ and } (\hat{g}_f'(\theta), \tilde{g}_f'(\theta)) = \left( \frac{1}{2} f'(\theta), \frac{1}{2} f'(\theta) \right).
$$

(III.5)

Proof of Proposition 5. First, define $\hat{g}_f : \Theta \to \mathbb{R}^+$ by $\hat{g}_f(\theta) = g(f(\theta); f)$, for $\theta \in \Theta$, where $g(\cdot; f)$ is given in (12) of Proposition 1. Then $g(\cdot; f) = \hat{g}_f \circ f^{-1}$ as in (III.1) and furthermore,

$$
\hat{g}_f(\theta) = \frac{1}{\Phi(\theta)} \int_{\theta}^{\theta} f(t) \phi(t) \, dt, \text{ for } \theta \in (\theta, \bar{\theta}],
$$

and $\hat{g}_f(\theta) = g(f(\theta); f) = f(\theta)$, where the latter proves the first part of (III.2). Noting that $g(\cdot; f)$ is the quotient of two absolutely continuous and strictly positive functions on $(\theta, \bar{\theta}]$, we see that $g(\cdot; f)$ is Lebesgue almost everywhere differentiable with derivative $\hat{g}_f'$ that satisfies according to the quotient rule

$$
\hat{g}_f'(\theta) = \frac{1}{\Phi(\theta)^2} \left( f(\theta) \phi(\theta) \Phi(\theta) - \int_{\theta}^{\theta} f(t) \phi(t) \, dt \phi(\theta) \right) = \frac{\phi(\theta)}{\Phi(\theta)} \left( f(\theta) - \hat{g}_f(\theta) \right),
$$

for Lebesgue almost every $\theta \in (\theta, \bar{\theta})$, proving the first part of (III.3). Define $\tilde{g}_f$ by

$$
\tilde{g}_f(\theta) = \mathcal{R}(g(\cdot; f))(f(\theta)), \text{ for } \theta \in \Theta,
$$
The initial value is given by \( \tilde{\mathcal{R}}(g;f) = \tilde{g}_f \circ f^{-1} \) on \( f(\Theta) \), as claimed in (III.1). Note that \( g(z;f)/z = 1 \), for \( 0 \leq z \leq f(\mathcal{O}) \), \( g(z;f)/z \leq 1 \), for \( z > 0 \), and \( f \in \mathcal{A} \) is continuous and strictly increasing, to see that

\[
\tilde{g}_f(\theta) = \tilde{\mathcal{R}}(g;f)(f(\theta)) = e \inf_{0 < z \leq e} \frac{g(z;f)}{z} = f(\theta) \inf_{0 < z \leq f(\theta)} \frac{g(z;f)}{z} = f(\theta) \inf_{\theta \leq \theta' \leq \theta} \frac{\tilde{g}_f(\theta')}{f(\theta')} = f(\theta) \inf_{\theta \leq \theta' \leq \theta} f(\theta'), \quad \text{for } \theta \in \Theta.
\]

The initial value is given by \( \tilde{g}_f(\theta) = f(\theta) \tilde{g}_f(\theta)/f(\theta) = \hat{g}_f(\theta) = f(\theta) \), establishing the second part of (III.2). Furthermore, from Lemma 1 it follows that \( \tilde{g}_f \) is continuous and non-decreasing, bounded by \( f \), as well as, that \( \tilde{g}_f/f \) is nonincreasing. For \( \theta \geq \theta' \in \Theta \), write

\[
0 \leq \tilde{g}_f(\theta) - \tilde{g}_f(\theta') = f(\theta) \inf_{\theta \leq z \leq \theta} \frac{\hat{g}_f(z)}{f(z)} - f(\theta') \inf_{\theta \leq z \leq \theta'} \frac{\hat{g}_f(z)}{f(z)} \\
\leq (f(\theta) - f(\theta')) \inf_{\theta \leq z \leq \theta'} \frac{\hat{g}_f(z)}{f(z)} \leq (f(\theta) - f(\theta')) \leq L_f |\theta - \theta'|,
\]

by Lemma 2, for \( L_f = \underline{f} > 0 \). Thus, \( \tilde{g}_f \) is Lipschitz continuous and by this has a derivative \( \tilde{g}'_f \) Lebesgue almost everywhere on \( \Theta \). By the same rationale, \( f \) has a derivative \( f' \) Lebesgue almost everywhere on \( \Theta \). Since \( \tilde{g}_f/f \) is non-increasing and \( \underline{f} > 0 \), we have Lebesgue almost everywhere on \( \Theta \)

\[
0 \leq (\tilde{g}_f/f)' = \frac{\tilde{g}_f f' - \tilde{g}_f f'}{f^2} \quad \iff \quad \tilde{g}_f' \leq f' \frac{\tilde{g}_f}{f}.
\]

Denote by \( E_f \subseteq \Theta \) the set where \( \tilde{g}_f \) equals \( \hat{g}_f \), that is, \( E_f = \{ \theta \in \Theta : \tilde{g}_f(\theta) = \hat{g}_f(\theta) \} \). Since \( \tilde{g}_f \) and \( \hat{g}_f \) are both continuous and \( \Theta \) is bounded, \( E_f \) is compact. On \( \hat{E}_f = E_f \setminus \partial E_f \) we have Lebesgue almost everywhere that \( \tilde{g}_f' = \hat{g}_f' \). Now, the boundary \( \partial E_f \) has Lebesgue measure 0 and \( \tilde{g}_f' \leq f' \frac{\tilde{g}_f}{f} \) Lebesgue almost everywhere on \( \Theta \); hence we have Lebesgue almost everywhere on \( E_f \)

\[
\tilde{g}_f' = \min \left( \tilde{g}_f', f' \frac{\tilde{g}_f}{f} \right).
\]

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Next, consider \((\theta, \bar{\theta}) \setminus E_f\), which is open. Take \(\theta \in (\theta, \bar{\theta}) \setminus E_f\); then we find an \(\varepsilon > 0\) such that \(B_\varepsilon(\theta) = \{\theta' \in \mathbb{R} : |\theta - \theta'| < \varepsilon\} \in (\theta, \bar{\theta}) \setminus E_f\), that is, \(\tilde{g}_f < \hat{g}_f\) on \(B_\varepsilon(\theta)\). Then

\[
\tilde{g}_f(\theta) = f(\theta) \inf_{\theta < \theta' \leq \theta} \frac{\hat{g}_f(\theta')}{f(\theta')} = f(\theta) \frac{\hat{g}_f(\theta^*)}{f(\theta^*)},
\]

for some \(\theta^* < \theta\), since \(\tilde{g}_f(\theta)/f(\theta) < \hat{g}_f(\theta)/f(\theta)\) and \(\theta \mapsto \inf_{\theta < \theta' \leq \theta} \tilde{g}_f(\theta)/f(\theta)\) is continuous, where the latter follows from the continuity of \(\tilde{g}_f\) and \(f\). Moreover, the equality above extends by the latter mentioned continuity to \([\theta, \theta + \varepsilon^*]\), for some \(\varepsilon^* > 0\). And thus for \(\tilde{\varepsilon} = \varepsilon \wedge \varepsilon^*\), we have

\[
\tilde{g}_f(\theta') = f(\theta') \frac{\hat{g}_f(\theta^*)}{f(\theta^*)}, \text{ for } \theta' \in B_{\tilde{\varepsilon}}(\theta).
\]

Now, \(f\) and \(\tilde{h}_f\) are absolutely continuous and their derivatives satisfy Lebesgue almost surely on \(B_{\tilde{\varepsilon}}(\theta)\),

\[
\tilde{g}'_f = f' \frac{\hat{g}_f(\theta^*)}{f(\theta^*)} = f' \frac{\tilde{g}_f}{f}.
\]

Putting the pieces together gives the second part of (III.3). \(\square\)

**Proof of Corollary 1.** Since \(\phi\) and \(\Phi\) are continuous, the results in (III.3) hold for all \(\theta \in (\theta, \bar{\theta})\), which then also specifies an ODE. The initial values for the derivative of \(\hat{g}_f\) is obtained using L’Hospital rule

\[
\hat{g}'_f(\theta) = \lim_{\theta \searrow \theta} \hat{g}'_f(\theta) = \lim_{\theta \searrow \theta} \frac{\phi(\theta)}{\Phi(\theta)} (f(\theta) - \hat{g}_f(\theta)) = \phi(\theta) \lim_{\theta \searrow \theta} \frac{f(\theta) - \hat{g}_f(\theta)}{\Phi(\theta)} = \phi(\theta) \lim_{\theta \searrow \theta} \frac{f'(\theta) - \hat{g}'_f(\theta)}{\phi(\theta)} = f'(\theta) - \hat{g}'_f(\theta);
\]
hence $\hat{g}_f'(\theta) = f'(\theta)/2$ as claimed. Then

$$\hat{g}_f'(\theta) = \lim_{\theta \to \Theta} f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} \min \left( \hat{g}_f'(\theta), f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \right) 1_{\hat{g}_f(\theta) = \hat{g}_f(\theta)}$$

$$= f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \lim_{\theta \to \Theta} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} + \min \left( \hat{g}_f'(\theta), f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \right) \lim_{\theta \to \Theta} 1_{\hat{g}_f(\theta) = \hat{g}_f(\theta)}$$

$$= f'(\theta) \lim_{\theta \to \Theta} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} + \min \left( f'(\theta)/2, f'(\theta) \right) \lim_{\theta \to \Theta} 1_{\hat{g}_f(\theta) = \hat{g}_f(\theta)}.$$

Observe that $\hat{g}_f(\theta) = f(\theta)$ and that the slope $\hat{g}_f'(\theta) = f'(\theta)/2$ is strictly smaller than $f'(\theta) > 0$. Therefore, $\hat{g}_f/f$ is strictly decreasing in a neighborhood of $\Theta$, thus $\hat{g}_f = \hat{g}_f$ on $[\Theta, \theta + \varepsilon]$ for some $\varepsilon > 0$. Accordingly, $\lim_{\theta \to \Theta} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} = 0$ and $\lim_{\theta \to \Theta} 1_{\hat{g}_f(\theta) = \hat{g}_f(\theta)} = 1$. From there we conclude that

$$\hat{g}_f'(\theta) = \min \left( f'(\theta)/2, f'(\theta) \right) = f'(\theta)/2,$$

finishing the proof.

\[\square\]

### III.2 Best Response of Firm ODE

For a given $g \in \mathcal{S}_g^C$ and $\theta \in \Theta$, consider the free boundary value problem given in (II.1-II.5) to characterize the value function $v(\cdot, \cdot; \theta, g)$. The boundary $\partial C^{(\theta,g)}$ of the continuation region $C^{(\theta,g)}$ can be described by a boundary function $b(\cdot, \theta; g)$ with the following properties.

**Lemma 3.** For given $g \in \mathcal{S}_g^C$ and $\theta \in \Theta$, the boundary $\partial C^{(\theta,g)}$ is characterized by a function $b : [0, f(\theta; g)] \times \Theta \times \mathcal{S}_g^C : \mathbb{R}_0^+ \times (e, \theta; g) \mapsto b(e, \theta; g)$, that is,

$$\partial C^{(\theta,g)} = \{(b(e, \theta; g), e) : 0 \leq e \leq f(\theta; g)\},$$

which is non-decreasing and continuous with terminal value $b(f(\theta; g), \theta; g) = f(\theta; g)$. The restriction of $b(\cdot, \theta; g)$ to $[\xi, f(\theta; g)]$, with $\xi = \theta f$, is Lipschitz continuous with constant $L_b = \bar{f}/\xi$. 

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Proof of Lemma 3. Lemma 6 part 4 implies that \( b(\cdot, \theta; g) \) is non-decreasing. Part 5 implies that the slope in \( e \) is bounded by \( f(\theta; g)/e \), which also implies continuity. The terminal value \( b(f(\theta; g), \theta; g) = f(\theta; g) \) follows from Lemma 7. The continuity of \( b \) is uniformly in \( \Theta \) with maximum slope \( \tilde{f}/\tilde{\xi} \), where \( \tilde{f} \) is given in Lemma 2 and \( \tilde{\xi} = \theta \tilde{f} \); see discussion around (II.11). □

Proposition 6. For a given continuously differentiable strategy \( g \in \mathcal{A}_g^C \), suppose that the collection of solutions \( \{v(\cdot, \cdot; \theta, g)\}_{\theta \in \Theta} \) of the boundary value problem (II.1-II.5) is sufficiently differentiable. Then the firm’s best response \( f(\cdot; g) \) given in Proposition 2 satisfies

\[
f'(\theta; g) = \frac{(1 + \eta)\sigma^2}{2(r - \mu)} \frac{g(\theta; g) - f(\theta; g)}{C(f(\theta; g)/g(\theta; g)) - f(\theta; g)} \frac{1}{1 - \frac{\partial b}{\partial e}(f(\theta; g), \theta; g)},
\]

for \( \theta \in (\bar{\theta}, \tilde{\theta}) \), where \( \eta = \frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2} + \sqrt{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0 \), and the partial derivative of the boundary describing function \( b \) with respect to \( e \) in \( (f(\theta; g), \theta) \) is a function of \( f(\theta), g(f(\theta)), g'(f(\theta)), \theta \), that is,

\[
\frac{\partial b}{\partial e}(f(\theta; g), \theta; g) = h(f(\theta), g(f(\theta)), g'(f(\theta)), \theta),
\]

for \( \theta \in (\bar{\theta}, \tilde{\theta}) \), and some function \( h(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}_+^+ \times \mathbb{R}_+^+ \times \mathbb{R}_+ \times \Theta \to \mathbb{R}_0^+ \).

Proof of Proposition 6. In the following, the value function \( v(\cdot, \cdot; \cdot, g), g \in \mathcal{A}_g^C \), is frequently differentiated on the boundary \( \partial C(\theta, g) \). Without stating this explicitly, we assume that the differential is calculated in the interior of \( C(\theta, g) \) and by continuity of \( v(\cdot, \cdot; \cdot, g) \), the limit to the boundary is taken. In order to show the claim, we proceed in four steps. First, we differentiate the smooth fit condition in (II.5) with respect to \( \theta \) to obtain

\[
0 = \frac{d}{d\theta} \frac{\partial v}{\partial \theta}(b(e, \theta; g), e, \theta; g)
\]
\[
= \frac{\partial^2 v}{\partial d^2}(b(e, \theta; g), e, \theta; g) \frac{\partial b}{\partial \theta}(e, \theta; g) + \frac{\partial^2 v}{\partial \theta \partial d}(b(e, \theta; g), e, \theta; g),
\]

(III.8)
for \( e \leq f(\theta; g) \). In the second step, we identify \( \frac{\partial^2 v}{\partial d^2} \) at the boundary using (II.2-II.4). In the third step, we characterize \( \frac{\partial^2 v}{\partial \theta \partial d} \) in order to calculate \( \frac{\partial^2 v}{\partial \theta \partial d} \). In the fourth and final step, we use results of the third step and the normal reflection condition in (II.5) to determine \( \frac{\partial b}{\partial \theta} \).

For the second step, observe that
\[
v = \frac{\partial v}{\partial d} = 0 \text{ on } \partial C(\theta, g).
\]
From this and (II.2-II.4) it follows that
\[
\frac{1}{2} \sigma^2 b(e, \theta; g)^2 \frac{\partial^2 v}{\partial d^2}(b(e, \theta; g), e, \theta; g) + \frac{b(e, \theta; g)}{\theta} - C(b(e, \theta; g)/g(e)) = 0,
\]
or, equivalently,
\[
\frac{\partial^2 v}{\partial d^2}(b(e, \theta; g), e, \theta; g) = 2 \frac{C(b(e, \theta; g)/g(e)) - \frac{b(e, \theta; g)}{\theta}}{\sigma^2 b(e, \theta; g)^2},
\]
(III.9) for \( e \leq f(\theta; g) \).

Now, the third step is taken. For \( \theta \in (\hat{\theta}, \bar{\theta}) \) denote by \( u \) the differential of \( v \) with respect to \( \theta \), which by assumption exists is continuous in \( \theta \). The assumed continuity of \( b \), the fact that the continuation region \( C(\theta, g) \) is open in \( \mathcal{C} \), and from equations (II.2-II.4), we obtain
\[
\mu d \frac{\partial u}{\partial d} (d, e; \theta, g) + \frac{1}{2} \sigma^2 d^2 \frac{\partial^2 u}{\partial d^2} (d, e; \theta, g) - \frac{d}{\theta^2} - ru(d, e; \theta, g) = 0,
\]
for \( (d, e) \in C(\theta, g) \). Similar reasoning implies for \( (d, e) \) in the interior of \( \mathcal{C} \setminus C(\theta, g) \) that
\[
u(d, e; \theta, g) = 0,
\]
which extends to all \( (d, e) \in \mathcal{C} \setminus C(\theta, g) \), that is, also to \( \partial C(\theta, g) \), since \( v = 0 \) on the boundary \( \partial C(\theta, g) \), and computing the derivative in \( \theta \) gives
\[
\frac{\partial v}{\partial d} (b(e, \theta; g), e; \theta, g) \frac{\partial b}{\partial \theta} (e, \theta; g) + \frac{\partial v}{\partial \theta} (b(e, \theta; g), e; \theta, g) = 0,
\]
and recalling that \( \frac{\partial v}{\partial d} = 0 \) on \( \partial C(\theta, g) \) by the smooth fit condition in (II.5) and \( \frac{\partial v}{\partial \theta} = u \) by definition. This second-order ODE in \( d \) is not dependent explicitly on \( e \). Therefore, we can interpret \( e \) as well
as $\theta$ as fixed parameters, and $u$ is in its general form given by

$$u(d,e;\theta,g) = -\frac{d}{\theta^2(r-\mu)} + d^{-\eta} L(e,\theta,g) + d^{-\tilde{\eta}} \tilde{L}(e,\theta,g),$$

for $(d,e) \in C(\theta,g)$, or, equivalently, for $d \geq b(e,\theta;g)$, in case $e \leq f(\theta;g)$ and $d \geq e$ in case $e > f(\theta;g)$, where $L(e,\theta,g)$ and $\tilde{L}(e,\theta,g)$ are constants taking values in $\mathbb{R}$, and

$$\eta = \frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2} + \sqrt{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$
and $
\tilde{\eta} = \frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2} - \sqrt{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$.\)

From $r > 0$, we see directly that $\eta > 0$ and $\tilde{\eta} < 0$. It can also be shown that $\tilde{\eta} < -1$, since $\mu < r$; hence $x^{-\tilde{\eta}} \tilde{L}(e,\theta,g)$ becomes the dominating expression of $u$ for large $d$. Observe that the value function $v$ for large $d$ is approximately $\frac{d}{\theta^2(r-\mu)}$, and thus its partial derivative in $\theta$ is approximately $-\frac{d}{\theta^2(r-\mu)}$. Accordingly, the weight of the otherwise dominating expression $d^{-\eta}$ has to be zero, that is, $\tilde{L}(e,\theta,g) = 0$; hence

$$u(d,e;\theta,g) = -\frac{d}{\theta^2(r-\mu)} + d^{-\eta} L(e,\theta,g), \text{ for } (d,e) \in C(\theta,g).$$

Fixing $\theta$ and focusing on $\partial C(\theta,g)$, we obtain $L(e,\theta,g)$ in terms of the boundary defining function $b(\cdot,\theta;g)$. On the boundary, that is, $d = b(e,\theta;g)$ with $e \leq f(\theta;g)$, we have $u(b(e,\theta;g),e,\theta;g) = 0$. Thus,

$$L(e,\theta,g) = \frac{b(e,\theta;g)^{1+\eta}}{\theta^2(r-\mu)}, \text{ for } e \leq f(\theta;g),$$

yielding for $e \leq f(\theta;g)$ that

$$u(d,e;\theta,g) = -\frac{d}{\theta^2(r-\mu)} \left(1 - \left[\frac{d}{b(e,\theta;g)}\right]^{-(1+\eta)}\right) \mathbf{1}_{d \geq b(e,\theta;g)}.$$ (III.10)
The partial derivative with respect to $d$ is then

$$
\frac{\partial u}{\partial d}(d, e; \theta, g) = -\frac{1}{\theta^2(r - \mu)} \left( 1 + \eta \left( \frac{d}{b(e, \theta; g)} \right)^{-1+\eta} \right) 1_{d \geq b(e, \theta; g)},
$$

for $e \leq f(\theta; g)$. In particular, we obtain for $d = b(e, \theta; g)$

$$
\frac{\partial^2 v}{\partial \theta \partial d}(b(e, \theta; g), e; \theta, g) = \frac{\partial u}{\partial d}(b(e, \theta; g), e; \theta, g) = -\frac{1 + \eta}{\theta^2(r - \mu)}, \quad \text{(III.11)}
$$

where the derivative in $(b(e, \theta; g), e) \in \partial C(\theta, g)$ is understood in the limit from the interior of $C(\theta, g)$ and the interchange of the order of differentiation follows from the initial assumption.

For the fourth step, note that the boundary function satisfies $f(\theta; g) = b(f(\theta; g), \theta; g)$, and differentiating this expression with respect to $\theta$ gives us

$$
f'(\theta; g) = \frac{\partial b}{\partial e}(f(\theta; g), \theta; g) f'(\theta; g) + \frac{\partial b}{\partial \theta}(f(\theta; g), \theta; g),
$$

where $f'(\cdot; g)$ denotes $\frac{\partial f}{\partial \cdot} (\cdot; g)$. Now, solve for $\frac{\partial b}{\partial \theta}$ to obtain

$$
\frac{\partial b}{\partial \theta}(f(\theta; g), \theta; g) = f'(\theta; g) \left( 1 - \frac{\partial b}{\partial e}(f(\theta; g), \theta; g) \right). \quad \text{(III.12)}
$$

Finally, we set $e = f(\theta; g)$, hence $b(f(\theta; g), \theta; g) = f(\theta; g)$, and plug (III.9), (III.11) and (III.12) in (III.8), to see that

$$
0 = 2 \frac{C(f(\theta; g) / g(f(\theta; g))) - \frac{f(\theta; g)}{\theta}}{\sigma^2 f(\theta; g)^2} f'(\theta; g) \left( 1 - \frac{\partial b}{\partial e}(f(\theta; g), \theta; g) \right) - \frac{1 + \eta}{\theta^2(r - \mu)},
$$

and solving for $f'(\theta; g)$ gives (III.6).
The partial derivative $\frac{\partial b}{\partial e}(f(\theta; g), \theta; g)$ in (III.6) is now analyzed. For doing so, define the function $\hat{v}: \mathbb{R}^+ \times \Theta \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $(d; \theta, g) \mapsto \hat{v}(d; \theta, g)$ by

$$
\hat{v}(d; \theta, g) = \sup_{\tau \in \mathcal{T}_d} \mathbb{E}_d \left[ \int_0^\tau e^{-rt} \left( d_t / \theta - C(d_t / g) \right) dt \right],
$$

where $\mathcal{T}_d$ is the set of all stopping times with respect to the information generated by $D$ with starting value $d$, and $\mathbb{E}_d$ is the corresponding expectation. In contrast to the function $v$ defined in (13), the direct dependence on the minimum observed cash flow is eliminated. Instead, an optimal stopping problem in the observed cash flow $D$ is given, parameterized by $g \in \mathbb{R}^+$ and $\theta \in \Theta$. However, for $e \leq f(\theta; g)$, the connection between the coordinates $d$ and $e$ of $v$ is dissolved and we have

$$
v(d, e; \theta, g) = \hat{v}(d; \theta, g(e)), \text{ for } (d, e) \in \mathcal{C}, e \leq f(\theta; g), \theta \in \Theta.
$$

Accordingly, we can use $\hat{v}$ instead of $v$ in order to characterize the partial derivative $\frac{\partial b}{\partial e}(f(\theta; g), \theta; g)$. Using the differentiability assumptions on the value function $v(\cdot, \cdot; g)$ and on $g$, we can write

$$
\frac{\partial b}{\partial e}(f(\theta; g), \theta; g) = \lim_{\Delta e \searrow 0, \Delta e} \frac{1}{\Delta e} \left( b(f(\theta; g), \theta; g) - b(f(\theta; g) - \Delta e, \theta; g) \right)
= \lim_{\Delta e \searrow 0, \Delta e} \frac{1}{\Delta e} \left( \sup\{d \geq e = f(\theta; g) : v(d, e; \theta, g) = 0\} \right.
- \sup\{d \geq e = f(\theta; g) - \Delta e : v(d, e; \theta, g) = 0\})
= \lim_{\Delta e \searrow 0, \Delta e} \frac{1}{\Delta e} \left( \sup\{d > 0 : \hat{v}(d; \theta, g(f(\theta; g))) = 0\} \right.
- \sup\{d > 0 : \hat{v}(d; \theta, g(f(\theta; g) - \Delta e)) = 0\}.
$$
Noting that \( f(\theta; g) = b(f(\theta; g), \theta; g) = \sup\{d > 0 : \hat{v}(d; \theta, g(f(\theta; g))) = 0\} \) gives

\[
\frac{\partial b}{\partial e}(f(\theta; g), \theta; g) = \lim_{\Delta e \downarrow 0} \frac{1}{\Delta e} \left( f(\theta; g) - \sup\{d > 0 : \hat{v}(d; \theta, g(f(\theta; g)) - g'(f(\theta; g))\Delta e) = 0\} \right).
\]

\[= h(f(\theta; g), g(f(\theta; g)), g'(f(\theta; g)), \theta),\]

defining the function \( h \) and thus proving (III.7).

\[\square\]

**Remark 1.** The function \( h(f(\theta), \hat{g}(\theta), f'(\theta), \hat{g}'(\theta), \theta) = \frac{\partial b}{\partial e}(f(\theta), \theta; \hat{g} \circ f) \) needs to be computed to calculate the equilibrium \((f^*, g^*)\). The proof of Proposition 6 shows how to obtain this quantity numerically. Define the function \( \hat{v} : \mathbb{R}_+ \times \Theta \times \mathbb{R}_+ \rightarrow \mathbb{R}, (d; \theta, g) \mapsto \hat{v}(d; \theta, g) \) by

\[
\hat{v}(d; \theta, g) = \sup_{\tau \in \mathcal{T}_d} \mathbb{E}_d \left[ \int_0^\tau e^{-rt} (d_t / \theta - C(d_t / g)) \, dt \right],
\]

where \( \mathcal{T}_d \) is the set of all stopping times with respect to the information generated by \( D \) with starting value \( d \) and \( \mathbb{E}_d \) is the corresponding expectation. In contrast to the function \( v \) defined in (13), the direct dependence on the minimum observed cash flow \( E \) is eliminated. Instead, a conventional optimal stopping problem in the observed cash flow \( D \) is given, parameterized by \( \theta \in \Theta \) and \( g \in \mathbb{R}_+ \).

However, for \( e \leq f(\theta; g) \), we have \( v(d, e; \theta, g) = \hat{v}(d; \theta, g(e)) \), for \((d, e) \in C\) and \( \theta \in \Theta \). From the arguments in the proof of Proposition 6, \( g(f(\theta)) = \hat{g}(\theta) \) and \((g^*)'(f(\theta))) = \hat{g}'(\theta) / f'(\theta) \), it follows that

\[
\frac{\partial b}{\partial e}(f(\theta), \theta; g^*) = \lim_{\Delta e \downarrow 0} \frac{1}{\Delta e} \left( f(\theta) - \sup\{d > 0 : \hat{v}(d; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)}\Delta e) = 0\} \right)
\]

\[
\approx \frac{1}{\Delta e} \left( f(\theta) - \sup\{d > 0 : \hat{v}(d; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)}\Delta e) = 0\} \right).
\]

for sufficiently small \( \Delta e \), where the critical level \( \sup\{d > 0 : \hat{v}(d; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)}\Delta e) = 0\} \) is obtained by solving the free-boundary value problem associated optimal stopping problem with value function \( \hat{v}(\cdot; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)}\Delta e) \) numerically.
### III.3 ODE Characterization of Equilibrium

**Proposition 7.** Given the setting of Proposition 3, denote by \((f^*, g^*)\) a fixed point of \(T\). Suppose \(f^*, g^*, \) and \(\phi\) are continuously differentiable, as well as the collection of solutions \((v(\cdot, \cdot; \theta, g^*))_{\theta \in \Theta}\) of the boundary value problem (II.1-II.5) is sufficiently differentiable. Then \((f, \hat{g}, \hat{g}) = (f^*, g^* \circ f^*, g(\cdot; f^*) \circ f^*)\) satisfies

\[
\begin{pmatrix}
  f'(	heta) \\
  \hat{g}'(\theta) \\
  \hat{g}'(\theta)
\end{pmatrix} = \begin{pmatrix}
  \frac{(1+\eta) \sigma^2}{2(r-\mu)} f(\theta)^2 - \frac{f(\theta)}{\theta} \\
  f'(	heta) \frac{\hat{g}(\theta)}{f(\theta)} \mathbf{1}_{\hat{g}(\theta) < \hat{g}(\theta)} + \min \left( \hat{g}'(\theta), f'(	heta) \frac{\hat{g}(\theta)}{f(\theta)} \right) \mathbf{1}_{\hat{g}(\theta) = \hat{g}(\theta)} \\
  \frac{\phi(\theta)}{\tilde{f}(\theta)} (f(\theta) - \hat{g}(\theta))
\end{pmatrix},
\]

(III.13)
on \((\bar{\theta}, \bar{\theta})\), with initial condition

\[
\begin{pmatrix}
  f(\bar{\theta}) \\
  \hat{g}(\bar{\theta}) \\
  \hat{g}(\bar{\theta})
\end{pmatrix} = \begin{pmatrix}
  f^*_1 \\
  f^*_1 \\
  f^*_1
\end{pmatrix},
\]

(III.14)

where the partial derivative of the boundary describing function \(b(\cdot; \cdot; g^*)\) with respect to \(e\) in \((f(\theta), \theta)\) is a function of \(f(\theta), \hat{g}(\theta), f'(\theta), \hat{g}'(\theta)\) and \(\theta\), that is,

\[
\frac{\partial b}{\partial e}(f(\theta), \theta; g^*) = \tilde{h}(f(\theta), \hat{g}(\theta), f'(\theta), \hat{g}'(\theta), \theta),
\]

for some function \(\tilde{h}\), \((f^*_1, g^*_1) \in \mathbb{R}^2\) denotes the equilibrium of the perfect information case, that is, \(\Theta_1 = \{1\}\); hence \(D = X\), with \(f^*_1 = g^*_1\), which exists and is unique under the given assumptions, and \(\Phi(\theta) = \int_0^\theta \phi(t) \, dt, \theta \in \Theta\).

**Proof of Proposition 7.** A fixed point \((f^*, g^*)\) of \(T\), which exists by Proposition 3, satisfies by its very definition that \(g^* = \mathcal{R} \cdot g(\cdot; f^*)\) and \(f^* = f(\cdot; g^*)\), that is, both strategies are their mutual (transformed) best responses. Corollary 1 of Proposition 5 yields the description of \(\hat{g}\) and \(\hat{g}\), as well as Proposition 6 that of \(f\), respectively. When looking at \(\frac{\partial b}{\partial e}\), as given in Proposition 6,
we see that the function $h$ describing $\frac{\partial h}{\partial \epsilon}$ in $(f(\theta), \theta)$ depends now on $\tilde{g}(\theta) = g^*(f(\theta))$ and $\tilde{g}'(\theta) = (g^*)'(f(\theta)) f'(\theta)$, where the latter is equivalent to $\frac{\tilde{g}'(\theta)}{f'(\theta)} = (g^*)'(f(\theta))$, for $\theta \in \Theta$, and thus $\tilde{h}$ defined by

$$\tilde{h}(f(\theta), \tilde{g}(\theta), f'(\theta), \tilde{g}'(\theta), \theta) = h(f(\theta), \tilde{g}(\theta), \frac{\tilde{g}'(\theta)}{f'(\theta)}, \theta),$$

is a function of $f(\theta), \tilde{g}(\theta), f'(\theta), \tilde{g}'(\theta)$ and $\theta$ as claimed.

It remains to verify that the initial condition. Therefore, we focus on $\tilde{g}_f$ and $\hat{g}_f$ around the starting value $\theta$. By assumption, $f^*$ and $\phi$ are continuously differentiable; hence, by Corollary 1 of Proposition 5, we have $\tilde{g}_f'(\theta) = \hat{g}_f'(\theta) = \frac{1}{2} f'(\theta)$. Accordingly, $\tilde{g}_f'(\theta) < f'(\theta) \frac{\tilde{g}(\theta)}{f'(\theta)} = f'(\theta)$, where the strict inequality follows from Lemma 2, implying that $f' \geq l_f > 0$. By the assumed continuity, there exists an $\epsilon > 0$ such that $\tilde{g}' = \tilde{g}'$ on $[\theta, \theta + \epsilon]$, where $\epsilon \leq \Theta - \theta$. It follows that $\hat{g} = \tilde{g}$ on $[\theta, \theta + \epsilon]$, since also $\tilde{g}(\theta) = \hat{g}(\theta)$ by definition. This implies that $\hat{g} \circ f^{-1} = \tilde{g} \circ f^{-1}$ on $f([\theta, \theta + \epsilon])$. Using Proposition 5, the best response $g(\cdot, f^*)$ and its transformation $\mathcal{R}(g(\cdot, f^*)) = g^*$ coincide on $f([\theta, \theta + \epsilon])$. Denote by $\mathbb{P}_{\hat{\theta}}$ a modified prior, which is given by $\mathbb{P}_{\hat{\theta}}(\cdot) = \mathbb{P}_{\Theta}(\cdot | \hat{\theta} \leq \theta + \epsilon')$, for $0 < \epsilon' \leq \epsilon$. Then $(f^*, g^*)$ restricted to $f^*([\theta, \theta + \epsilon])$ and $[\theta, \theta + \epsilon]$, respectively, is an equilibrium for the prior $\mathbb{P}_{\hat{\theta}}$, for all $0 < \epsilon' \leq \epsilon$. Now, the best response operators are continuous in the sup-norm, as shown in Proposition 3. As $\epsilon' \searrow 0$, we are tending to the perfect information case, here with known type $\theta$: hence, the limit $(f^*(\theta), g^*(\theta))$ is an equilibrium of the perfect information case, here scaled by $\theta$. The existence and uniqueness of the equilibrium in the perfect information case, that is, $\Theta_1 = \{1\}$ and $D = X$, follows from Theorem 1 and the working in Appendix C of Manso (2013). Note that Manso (2013) specifies the coupon function as step function, whereas our framework allows for a continuous coupon function $C$, which satisfies Assumption 1. However, using, for example, an approximating sequence of step functions $(C_n)_{n \geq 1}$ to our coupon function $C$, the results carry over. Further, denote by $f^*_{1\epsilon}$ the firm’s default threshold in equilibrium. In order to transfer the result from the firm scale to the rating agency scale, we multiply the equilibrium $f^*_{1\epsilon}$ by $\theta$ and the initial condition follows as claimed.

\qed
The differential equation satisfied by \((f, \check{g}, \hat{g}) = (f^*, g^* \circ f^*, g(\cdot; f^*) \circ f^*)\) in (III.13) and (III.14) of Proposition 7 can be used to obtain the fixed point \((f^*, g^*)\) constructively. The suggested ODE structure is somewhat more complicated, since characterization of the partial derivative \(f'\) on the left-hand side of (III.13) also involves the term \(\check{h}(f(\theta), \check{g}(\theta), f'(\theta), \hat{g}'(\theta), \theta)\) on the right-hand side of (III.13), which depends on \(f'\). Rewriting the first line of (III.13) as \(\check{H}(f(\theta), \hat{g}(\theta), f'(\theta), \hat{g}'(\theta), \theta) = 0\) is an implicit characterisation of \(f'\). To ensure that this implicit characterization of \(f'\) is well-defined, it is required that the partial derivative \(H\) with respect to \(f'\) is nonzero, such that we can invert this relation locally to back out \(f'\).

**Proof of Proposition 4.** Denote \((f^*, g^*)\) a fixed point of \(T\), given in Proposition 3. If \(g(\cdot; f^*) = \mathcal{R} \circ g(\cdot; f^*)\) holds, then \((f^*, g^*)\) is an equilibrium. By Proposition 7, the latter is equivalent to \(\check{g} = \hat{g}\) which is implied by \(\hat{g}' \leq f' \frac{\hat{g}}{f}\), on \((\theta, \bar{\theta})\).

### Appendix IV  Equilibrium for Worst and Best Case of Rating Attitude

**Proposition 8** (Worst Case Specification, \(\alpha = 0\)). Let a firm strategy \(\tau\) be given by a function \(f \in \mathcal{A}_f\), with \(\tau(\theta) = \inf \{t \geq 0 : D_t \leq f(\theta)\}, \theta \in \Theta\). Then the rating agency’s consistent belief is given by (11). The rating agency’s corresponding best response to \(f\) is given by \(\hat{D}^0_0 = g_0(E; f)\), with

\[
g_0(e; f) = \min\{e, \sup f(\Theta)\}, \quad (IV.1)
\]

and \(g_0 \in \mathcal{A}_g\) and satisfies (9). For \(g \in \mathcal{A}_g\) satisfying (9), the firm’s best response \(f(\cdot; g)\) is given according to Proposition 2. Suppose Assumption 1 holds. Then \(T_0 : (f, g) \mapsto (f(\cdot; g), g_0(\cdot; f))\) has at least one fixed point in \(\mathcal{A}_f \times \mathcal{A}_g\). Let \((f^*, g^*_0)\) be such a fixed point, then \((f^*, g^*_0)\) is an equilibrium. Suppose \(f^*\) and \(g^*_0\) are continuously differentiable, and the collection of solutions \((v(\cdot, \cdot; \theta, g^*_0))_{\theta \in \Theta}\)
of the boundary value problem (II.1-II.5) is sufficiently differentiable. Then $f^*$ satisfies

$$f''(\theta) = \frac{(1 + \eta) \sigma^2}{2(r - \mu)} \frac{f^*(\theta)^2 / \theta^2}{C(1) - f^*(\theta) / \theta} \frac{1}{1 - \frac{\partial b}{\partial e}(f^*(\theta), \theta; \text{Id})},$$

(IV.2)
on $(\theta, \bar{\theta})$ with initial condition $f(\theta) = \theta f_1^*$, where $f_1^*$ denotes the unique equilibrium of the perfect information case, that is, $\Theta_1 = \{1\}$, and $\eta = \frac{1}{\sigma^2} \left( \mu - \frac{1}{2} \sigma^2 + \sqrt{\left( \mu - \frac{1}{2} \sigma^2 \right)^2 + 2r \sigma^2} \right) > 0$.

**Proof of Proposition 8.** The rating agency’s belief consistent with a firm strategy $f$ is not depending on a particular choice of the cost rate, and thus, not on $k\pi(0)$, and follows as in Proposition 1. The limiting case $\alpha \downarrow 0$ implies that the rating agency leans more and more to the right end point of potential default thresholds that are not yet ruled out, that is, $g_0(e; f) = \min\{e, \sup f(\Theta)\}$, for $e \geq 0$. Note that $g_0(\cdot; f) \in \mathcal{A}_f$ and it is straight forward to check that (9) holds. The proof on the existence of an equilibrium is based on Schauder fixed-point theorem applied to $T_0$ similar to Proposition 3. It only remains to check that $g_0(\cdot; f)$ is uniformly continuous and thus $T_0$ is uniformly continuous as well. For $f, \tilde{f} \in \mathcal{A}_f$, then

$$|g(e; f) - g(e; \tilde{f})| = |\min\{e, \sup f(\Theta)\} - \min\{e, \sup \tilde{f}(\Theta)\}|$$

$$\leq |\sup f(\Theta) - \sup \tilde{f}(\Theta)| = \|f - \tilde{f}\|,$$

and hence $\|g(\cdot; f) - g(\cdot; \tilde{f})\| \leq \|f - \tilde{f}\|$ giving the desired continuity. The differential equation describing $f^*$ in (IV.2) follows directly from Proposition 6 using the structure of $g_0(\cdot; f) = \text{Id}$ on $f(\Theta)$ for all $f \in \mathcal{A}_f$ and thus also for $f^*$. \hfill $\Box$

**Proposition 9** (Best Case Specification, $\alpha = 1$). Let a firm strategy $\tau$ be given by a function $f \in \mathcal{A}_f$, with $\tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\}, \theta \in \Theta$. Then the rating agency’s consistent belief is given by (11). The rating agency’s corresponding best response to $f$ is given by $\hat{D}_1^* = g_1(E; f)$, with

$$g_1(e; f) = \min\{e, \inf f(\Theta)\},$$

(IV.3)
and $g_1 \in \mathcal{A}_g$ and satisfies (9). For $g \in \mathcal{A}_g$ satisfying (9), the firm’s best response $f(\cdot; g)$ is given according to Proposition 2. Suppose Assumption 1 holds. Then $T_1 : (f, g) \mapsto (f(\cdot; g), g_1(\cdot; f))$ has at least one fixed point in $\mathcal{A}_f \times \mathcal{A}_g$. Let $(f^*, g^*_1)$ be such a fixed point, then $(f^*, g^*_1)$ is an equilibrium. Suppose $f^*$ and $g^*_1$ are continuously differentiable, and the collection of solutions $(v(\cdot; \cdot; \theta, g^*_1))_{\theta \in \Theta}$ of the boundary value problem (II.1-II.5) is sufficiently differentiable. Then $f^*$ satisfies

$$f^*(\theta) = \theta f^*_1,$$

for $\underline{\theta} \leq \theta \leq \bar{\theta}$, where $f^*_1$ denotes the unique equilibrium of the perfect information case, that is, $\Theta_1 = \{1\}$.

**Proof of Proposition 9.** The proof follows the lines of the proof of Proposition 8. The items that require different arguments are discussed below. The limiting case $\alpha \nearrow 1$ implies that the rating agency leans more and more to the left end point of potential default thresholds that are not yet ruled out, that is, $g_1(e; f) = \min\{e, \inf f(\Theta)\}$, for $e \geq 0$. The equation characterizing $f^*(\theta)$ in (IV.4) follows directly from the fact that prior to default of type $\theta$, the consistent belief converges to the true distribution and the corresponding default threshold is then as given.

**Appendix V Auxiliary Results**

From (12) we see that the best response of the rating agency $g(\cdot; f)$ with respect to a firm strategy $f$ depends on the image of $f$, that is, $f(\Theta) \subseteq \mathbb{R}_0^+$. In the case that $f(\Theta)$ is contained in a compact interval $[e, \bar{e}]$, which is bounded away from zero, that is, $0 < e \leq \bar{e} < \infty$, then $g(\cdot; f) = Id$ on $[0, e]$ and $g(\cdot; f) = g(\bar{e})$ on $[\bar{e}, \infty)$. For this setting, the subsequent lemma shows that convergence in the sup-norm $\|\cdot\|_{\infty}$ is preserved under the functional $\mathcal{R}$. Further, the function $\mathcal{R}(g; f)$ then acts on a compact interval $[e_\mathcal{R}, \bar{e}_\mathcal{R}]$, with standard continuation outside, that is, $\mathcal{R}(g) = Id$ on $[0, e_\mathcal{R}]$ and $\mathcal{R}(g) = g(\bar{e}) = \text{constant}$ on $[\bar{e}_\mathcal{R}, \infty)$. 

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**Lemma 4.** Suppose that $g, g' \in \mathcal{A}_g$ are continuous and non-decreasing, with $0 \leq g, g' \leq Id$, and there exists $0 < \varepsilon \leq \bar{e} < \infty$ such that $g = g' = Id$ on $[0, \varepsilon]$ as well as, $g(e) = g(\bar{e})$ and $g'(e) = g'(\bar{e})$, for $e \in [\bar{e}, \infty)$. Then $\mathcal{A}(g), \mathcal{A}(g') = Id$ on $[0, \varepsilon], \mathcal{A}(g)(e) = g(\bar{e})$, and $\mathcal{A}(g')(e) = g'(\bar{e})$, for $e \in [\bar{e}, \infty)$, where $\varepsilon, \bar{e} = \frac{e^2}{\varepsilon}$, and

\[
\|\mathcal{A}(g) - \mathcal{A}(g')\|_{\infty} \leq 2 \frac{\bar{e}}{\varepsilon} \|g - g'\|_{\infty}. \tag{V.1}
\]

In order to prove Proposition 2, the value function $v(\cdot, \cdot, \theta, g)$ of the optimal stopping problem of the firm in (4), with $\theta \in \Theta$ and $g \in \mathcal{A}_g^C$ such that $g$ is non-decreasing and bounded by $Id$, has to be characterized. Note that (4) is on the rating agency-scale, using the imperfectly observed cash flow $D$ and its running minimum $E$. It is helpful to also consider the firm’s optimal stopping problem also on the firm-scale; that is, $X = E/\theta$ and $Y = D/\theta$, for $\theta > 0$. For $(x, y) \in \mathcal{C}$, define

\[
w(x, y; \theta, g) = \sup_{\tau \in \mathcal{F}_{(x, y)}} E_{(x, y)} \left[ \int_0^\tau e^{-rt} \right] X_t - C(\theta X_t / g(\theta Y_t)) dt \right], \tag{V.2}
\]

where the firm cash flow $X$ follows (1), its running minimum $Y = (Y_t)_{t \geq 0}$ is given by $Y_t = \min(Y_0, \inf_{0 \leq s \leq t} X_s)$, for $t \geq 0$, and $g \in \mathcal{A}_g^C$ is non-decreasing, bounded by $Id$, and $\theta \in \Theta$.

First properties of the value function $w(\cdot, \cdot, \theta, g)$ defined in (V.2) are collected in the following lemma. Therefore, it is useful to define the function $g_{\theta}$ appearing inside the interest payment rate function $C$ in (V.2), by $g_{\theta} : \mathbb{R}_{0}^+ \to \mathbb{R}_{0}^+, y \mapsto g_{\theta}(y) = \frac{1}{\theta} g(\theta y)$.

**Lemma 5.** Denote $v(\cdot, \cdot, \theta, g)$ and $w(\cdot, \cdot, \theta, g)$ the value functions specified in (13) and (V.2), respectively, for $\theta > 0$ and $g \in \mathcal{A}_g^C$, then $g_{\theta} \in \mathcal{A}_g^C$ and

\[
w(x, y; \theta, g) = v(\theta x, \theta y; \theta, g), \quad \text{for } (x, y) \in \mathcal{C}, \tag{V.3}
\]

\[
w(x, y; \theta, g) = v(x, y; 1, g_{\theta}), \quad \text{for } (x, y) \in \mathcal{C}. \tag{V.4}
\]

Moreover, for $g \in \mathcal{A}_g$ and $0 < \theta' \leq \theta$ it holds as follows:

1. If $g$ is non-decreasing, then $g_{\theta}$ is non-decreasing and $\theta' g_{\theta'} \leq \theta g_{\theta}$.
2. If $g/\text{Id}$ is non-increasing, then $g_\theta/\text{Id}$ is non-increasing and $g_{\theta'} \geq g_\theta$.

3. If $g \leq \text{Id}$, then $g_\theta \leq \text{Id}$.

**Proof of Lemma 5.** (V.3) follows directly from the definition in (13), as $(D,E)$ is obtained from $(X,Y)$ by multiplying by $\theta > 0$. (V.4) follows likewise, using the definition of $g_\theta$ above and (V.2). To show part 1, observe that for $0 < y' \leq y$, it follows $\theta y' \leq \theta y$, and by $g$ being non-decreasing, we have

$$g_\theta(y) = \frac{g(\theta y)}{\theta} \geq \frac{g(\theta y')}{\theta} = g_\theta(y');$$

that is, $g_\theta$ is non-decreasing. Now,

$$\theta g_\theta(y) = g(\theta y) \geq g(\theta' y) = \theta' g_{\theta'}(y), \text{ for } y \geq 0,$$

since $g$ is non-decreasing and $\theta' y \leq \theta y$. For part 2, consider that

$$\frac{g_\theta(y')}{y'} = \frac{g(\theta y')}{\theta y'} \geq \frac{g(\theta y)}{\theta y} = \frac{g_\theta(y)}{y}, \text{ for } y > 0,$$

since $g/\text{Id}$ is non-increasing and $\theta y \geq \theta y'$. Thus, $g_\theta/\text{Id}$ is non-increasing. Furthermore,

$$g_{\theta'}(y) = \frac{y g(\theta' y)}{\theta' y} \geq \frac{y g(\theta y)}{\theta y} = g_\theta(y), \text{ for } y > 0,$$

since $g/\text{Id}$ is non-increasing and $\theta' y \leq \theta y$. For part 3, write

$$g_\theta(y) = \frac{g(\theta y)}{\theta} \leq \frac{\theta y}{\theta} = y, \text{ for } y \geq 0,$$

since $g \leq \text{Id}$; hence $g_\theta \leq \text{Id}$.

The value functions $v$ and $w$ given in (13) and (V.2), respectively, are described by expectations conditioning on the starting values $(e,d)$ and $(x,y)$, respectively. For the subsequent analysis, it
is helpful to write the dependence on the starting value directly into the payoff function, which is possible since the driving processes $X$ and $E$, respectively, are geometric Brownian motions, see (1) and (2), respectively. Denote by $(\tilde{X}, \tilde{Y})$ the process $X$ defined in (13) with starting value 1, and $\tilde{Y} = (\tilde{Y}_t)_{t \geq 0}$ its running minimum; that is, $\tilde{Y}_t = \inf_{0 \leq s \leq t} \tilde{X}_s$, for $t \geq 0$. Then

$$v(d, e; \theta, g) = \sup_{\tau \in \mathcal{F}} \mathbb{E} \left[ \int_0^\tau e^{-rt} \left( d \tilde{X}_t / \theta - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))) \right) \, dt \right], \tag{V.5}$$

$$w(x, y; \theta, g) = \sup_{\tau \in \mathcal{F}} \mathbb{E} \left[ \int_0^\tau e^{-rt} \left( x \tilde{X}_t - C(x \tilde{X}_t / g(\min(y, x \tilde{Y}_t))) \right) \, dt \right], \tag{V.6}$$

where $\mathcal{F}$ denotes the set of stopping times w.r.t. to the filtration generated by $(\tilde{X}, \tilde{Y})$. To allow for straightforward calculations subsequently, we extend the interest payment rate function $C$ from $[1, \infty]$ trivially to $[0, \infty]$ by setting $C|_{[0,1)} = C(1)$. This representation allows to establish the following properties of $v(\cdot, \cdot; \theta, g)$ and $w(\cdot, \cdot; \theta, g)$.

**Lemma 6.** Denote by $v(\cdot, \cdot; \theta, g)$ and $w(\cdot, \cdot; \theta, g)$ the value functions specified in (13) and (V.2), respectively, for $\theta > 0$ and $g, g' \in \mathscr{A}_g^C$; then the following holds true:

1. $v(\cdot, \cdot; \theta, g)$ and $w(\cdot, \cdot; \theta, g)$ are non-negative.

2. If $0 < \theta' \leq \theta$, then $v(\cdot, \cdot; \theta', g) \geq v(\cdot, \cdot; \theta, g)$ and $w(\cdot, \cdot; \theta', g) \geq w(\cdot, \cdot; \theta, g)$.

3. If $g' \leq g$, then $v(\cdot, \cdot; g', g) \geq v(\cdot, \cdot; g, g)$ and $w(\cdot, \cdot; g', g) \geq w(\cdot, \cdot; g, g)$.

4. $v(\cdot, \cdot; \theta, g)$ and $w(\cdot, \cdot; \theta, g)$ are non-increasing in $e$ and $y$, respectively; that is,

$$v(d, e'; \theta, g) \geq v(d, e; \theta, g), \text{ for } 0 \leq e' \leq e \leq d < \infty,$$

$$w(x, y'; \theta, g) \geq w(x, y; \theta, g), \text{ for } 0 \leq y' \leq y \leq x < \infty.$$
5. \(v(\cdot, \cdot; \theta, g)\) and \(w(\cdot, \cdot; \theta, g)\) are non-decreasing on rays starting in the origin; that is,

\[
v(d, e; \theta, g) \leq v(\lambda d, \lambda e; \theta, g), \text{ for } (d, e) \in C \text{ and } \lambda \geq 1,
\]
\[
w(x, y; \theta, g) \leq w(\lambda x, \lambda y; \theta, g), \text{ for } (x, y) \in C \text{ and } \lambda \geq 1.
\]

**Proof of Lemma 6.** Part 1 follows from \(\tau = 0\). For the remainder of the proof, recall the convention the interest payment rate function \(C\) is extended from \([1, \infty)\) trivially to \([0, \infty)\) by setting \(C|_{[0,1]} = C(1)\). For part 2, we focus on \(v\) as given in (V.5) and compare the accumulated discounted net income stream until \(\tau \in \tilde{T}\). For \(0 < \theta' \leq \theta\), we have almost surely

\[
d\tilde{X}_t/\theta' - C(d\tilde{X}_t/g(\min(e, d\tilde{Y}_t))) \geq \tilde{X}_t/\theta - C(d\tilde{X}_t/g(\min(e, d\tilde{Y}_t))),
\]

hence almost surely

\[
\int_0^\tau e^{-rt} \left(d\tilde{X}_t/\theta' - C(d\tilde{X}_t/g(\min(e, d\tilde{Y}_t))))\right) \, dt \\
\geq \int_0^\tau e^{-rt} \left(\tilde{X}_t/\theta - C(d\tilde{X}_t/g(\min(e, d\tilde{Y}_t)))\right) \, dt.
\]

The inequality is preserved by taking the expectation and the supremum over all \(\tau \in \tilde{T}\), and the first assertion of part 2 follows. For \(w\), we take (V.2). For \(g/Id\) non-increasing, as assumed, part 2 of Lemma 5 gives that \(0 < \theta' \leq \theta\) implies \(g_{\theta'} \geq g_{\theta}\), and since \(C\) is non-increasing we have

\[
x\tilde{X}_t - C(x\tilde{X}_t/g_{\theta'}(\min(y, x\tilde{Y}_t))) \leq x\tilde{X}_t - C(x\tilde{X}_t/g_{\theta}(\min(y, x\tilde{Y}_t))),
\]

and by similar arguments as before, that is, integrating the discounted payoff stream over \([0, \tau]\) as well as noting the inequality is preserved by taking the expectation and the supremum over all \(\tau \in \tilde{T}\), it follows that \(w(x, y; \theta', g) \leq w(x, y; \theta, g)\), for all \((x, y) \in C\), as claimed. To show part 3,
observe that for \( g' \leq g \), we have

\[
C\left(\frac{a}{g'(b)}\right) \leq C\left(\frac{a}{g(b)}\right), \text{ for } (a,b) \in \mathbb{R}_0^+ \times \mathbb{R}^+,
\]

since \( C \) is non-increasing, and the first assertion follows using similar arguments as for part 2. For \( w \), we calculate

\[
g'_\theta(y) = \frac{g'(\theta y)}{\theta} \leq \frac{g(\theta y)}{\theta} = g_\theta(y), \text{ for } y > 0.
\]

Hence, \( g'_\theta \leq g_\theta \) and the second assertion follows by identical arguments as the first assertion.

To verify assertion 4, observe that \( g \) is non-decreasing by assumption, and thus \( g_\theta \) is also non-decreasing by part 1 of Lemma 5. Therefore, \( g(b') \leq g(b) \) and \( g_\theta(b') \leq g_\theta(b) \), for \( 0 \leq b' \leq b \).

Since \( C \) is non-increasing, it holds that

\[
C\left(\frac{a}{g(b')}\right) \leq C\left(\frac{a}{g(b)}\right) \text{ and } C\left(\frac{a}{g_\theta(b')}\right) \leq C\left(\frac{a}{g_\theta(b)}\right), \text{ for } a \geq 0 \text{ and } 0 < b' \leq b.
\]

Applying the same arguments as in part 3 to the representations in (V.5) and (V.6) gives the claimed result. Now, part 5 is verified. Note that \( g/Id \) is non-increasing by assumption, which implies by part 2 of Lemma 5 that \( g \geq g_\lambda \), where we set \( 1 = \theta' = \theta = \lambda \), and

\[
\frac{a}{g(b)} \leq \frac{a}{g_\lambda(b)} = \frac{\lambda a}{g(\lambda b)}, \text{ for } (a,b) \in \mathbb{R}_0^+ \times \mathbb{R}^+.
\]  (V.7)
Taking a look at the net income rate in (V.5) for \(v\) with starting value \((\lambda d, \lambda e)\), where \((d, e) \in \mathcal{C}\) and \(\lambda \geq 1\), we obtain

\[
\lambda d \tilde{X}_t / \theta - C(\lambda e \tilde{X}_t / g(\min(\lambda e, \lambda d \tilde{Y}_t))) \\
= d \tilde{X}_t / \theta - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))) + (\lambda - 1) d \tilde{X}_t / \theta \\
+ C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))) - C(\lambda d \tilde{X}_t / g(\lambda \min(e, d \tilde{Y}_t))) \\
\geq d \tilde{X}_t / \theta - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))),
\]

since \((\lambda - 1)d \tilde{X}_t / \theta\) is greater equal to zero due to \(\lambda \geq 1\) and \(C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))) - C(\lambda d \tilde{X}_t / g(\lambda \min(e, d \tilde{Y}_t))) \geq 0\) by (V.7), set \(a = d \tilde{X}_t\) and \(b = \min(e, d \tilde{Y}_t)\).

And by similar arguments as before, that is, integrating the discounted payoff stream over \([0, \tau]\) as well as noting the inequality is preserved by taking the expectation and the supremum over all \(\tau \in \hat{T}\), it follows that \(v(d, e; \theta, g) \leq v(\lambda d, \lambda e; \theta, g)\), for all \((d, e) \in \mathcal{C}\), as claimed. When considering \(w\), observe that from Lemma 5 part 2 it follows that \(g_\theta/Id\) is non-increasing. Now, the similar reasoning as for \(v\) applies and the proof is finished.

For a given rating agency strategy \(g \in \mathcal{A}_g^C\) and \(\theta > 0\), the early exercise region

\[
\mathcal{E}(\theta; g) = \{(d, e) \in \mathcal{C} : v(d, e; \theta, g) = 0\}
\]

(V.8)

allows us to characterize the best response of the firm \(\tau(\theta; g)\) as first hitting time. An important subset of \(\mathcal{E}(\theta; g)\) is that on the diagonal, which is identified with \(\mathcal{D}(\theta; g)\) and the corresponding supremum \(f(\theta; g)\); that is,

\[
\mathcal{D}(\theta; g) = \{d \in \mathbb{R}_0^+ : (d, d) \in \mathcal{E}(\theta; g)\}, \text{ and } D(\theta; g) = \sup \mathcal{D}(\theta; g).
\]

(V.9)
Lemma 7. Let \( \mathcal{E}(\theta; g) \), \( \mathcal{D}(\theta; g) \), and \( D(\theta; g) \) be given by (V.8) and (V.9), respectively, for \( \theta > 0 \) and \( g \in \mathcal{A}_g^C \). Then

\[
\mathcal{D}(\theta; g) = [0, D(\theta; g)],
\]

(V.10)

and for \( (d,e) \in \mathcal{E}(\theta; g) \), it holds that

\[
e \leq d \leq D(\theta; g).
\]

(V.11)

Proof of Lemma 7. To see the first assertion, note that \( v(0, 0; \theta, g) = 0 \), and hence \( \mathcal{D}(\theta; g) \) is non-empty. For \( d \in \mathcal{D}(\theta; g) \), we have \( d' \in \mathcal{D}(\theta; g) \), for \( d' \in (0, d] \) by part 5 of Lemma 6 by setting \( \lambda = d/\theta \) \( \geq 1 \). Furthermore, for \( d > \theta \tilde{C} \), we have that \( v(d, d; \theta, g) > 0 \), since then the income stream from not defaulting in \( (d, d) \) is strictly positive. Accordingly, \( \mathcal{D}(\theta; g) \) is a convex and bounded subset of \( \mathbb{R}_0^+ \). Since \( v(\cdot, \cdot; \theta, g) \) is continuous, \( \mathcal{D}(\theta; g) \) is also closed and \( D(\theta; g) \in \mathcal{D}(\theta; g) \), and (V.10) follows. For the second assertion, take \( (d, e) \in \mathcal{E}(\theta; g) \), then \( (d, d) \in \mathcal{E}(\theta; g) \) by part 4 of Lemma 6, what is equivalent to \( d \in \mathcal{D} \). Thus, \( d \leq D(g; h) \). Since \( (d, e) \in \mathcal{E}(\theta; g) \subseteq \mathcal{C} \), we have \( e \leq d \), finishing the proof.

Lemma 8. The set \( \mathcal{K}_f \) is convex and compact in \( (C(\Theta, \mathbb{R}), \| \cdot \|_\infty) \). Moreover, \( \mathcal{K}_f \) is uniformly bounded by \( \overline{\theta} \) and uniformly Lipschitz continuous with Lipschitz \( L_f = \tilde{f} \).

Proof of Lemma 8. To see that \( \mathcal{K}_f \) is convex, take \( f, f' \in \mathcal{K}_f \), \( \lambda \in [0, 1] \) and define \( f^\lambda = \lambda f + (1 - \lambda) f' \). Now, \( f^\lambda \) is continuous, since \( f, f' \) are, hence, \( f^\lambda \in C(\Theta, \mathbb{R}) \). For \( \theta, \theta' \in \Theta \) with \( \theta' \leq \theta \), we have

\[
f^\lambda(\theta) - f^\lambda(\theta') = \lambda f(\theta) + (1 - \lambda) f'(\theta) - \lambda f(\theta') - (1 - \lambda) f'(\theta')
\]

\[
= \lambda (f(\theta) - f(\theta')) + (1 - \lambda) (f'(\theta) - f'(\theta'))
\]

\[
\leq \lambda L_f (\theta - \theta') + (1 - \lambda) L_f (\theta - \theta') = L_f (\theta - \theta').
\]
Using similar reasoning, one verifies that all conditions of the definition of $\mathcal{K}_f$ in (II.10) hold for $f^\lambda$, and thus $f^\lambda \in \mathcal{K}_f$. Accordingly, $\mathcal{K}_f$ is convex. To see that $\mathcal{K}_f$ is compact, it is by Arzela-Ascoli sufficient to show that $\mathcal{K}_f$ is closed, bounded, and equicontinuous. To show that $\mathcal{K}_f$ is closed consider a sequence $(f_n)_{n \geq 1}$ in $\mathcal{K}_f$ that converges to some $f \in C(\Theta, \mathbb{R})$, that is, $\lim_{n \to \infty} \|f_n - f\|_\infty = 0$. For $\theta, \theta' \in \Theta$ with $\theta' \leq \theta$ we have

$$f(\theta) - f(\theta') \leq f_n(\theta) + \|f_n - f\|_\infty - f_n(\theta') + \|f_n - f\|_\infty \leq L_f (\theta - \theta') + 2 \|f_n - f\|_\infty.$$  

This holds for all $n \geq 1$. As $n \to \infty$, we obtain $f(\theta) - f(\theta') \leq L_f (\theta - \theta')$. Using similar reasoning, one verifies that all conditions of the definition of $\mathcal{K}_f$ in (II.10) hold for $f$, and thus $f \in \mathcal{K}_f$. Accordingly, $\mathcal{K}_f$ is closed. That $\mathcal{K}_f$ is bounded follows immediately from the definition with uniform upper bound $\overline{\Theta_f}$. The equicontinuity of $\mathcal{K}_f$ is implied if all $f \in \mathcal{K}_f$ are Lipschitz continuous with a common Lipschitz constant $L_f$, which holds by the very definition of $\mathcal{K}_f$. Note that the common Lipschitz constant is given by $L_f = \overline{f}$.

**Lemma 9.** The set $\mathcal{K}_g$ is convex and compact in $(C(\Xi, \mathbb{R}^+_0), \|\cdot\|_\infty)$. Moreover, $\mathcal{K}_g$ is uniformly bounded by $\overline{\theta}^2 \overline{f}^2 / (\overline{\theta f})$ and uniformly Lipschitz continuous with Lipschitz constant $L_g = 1$.

**Proof of Lemma 9.** The proof follows along the same lines as that of Lemma 8, once it is shown that $\mathcal{K}_g$ is uniformly bounded and uniformly Lipschitz continuous. The uniform bound of $\overline{\theta}^2 \overline{f}^2 / (\overline{\theta f})$ follows from $g \leq Id$, for $g \in \mathcal{K}_h \subseteq (\Xi)$, and $\overline{\zeta} = \overline{\theta}^2 \overline{f}^2 / (\overline{\theta f})$. For the uniform Lipschitz continuity, observe for $g \in \mathcal{K}_g$ and $e, e' \in \Xi$ with $e' \leq e$ that

$$0 \leq g(e) - g(e') = eg(e)/e - g(e') \leq eg(e')/e' - g(e') = (e - e') g(e')/e' \leq (e' - e),$$

where the first step follows since $g \in \mathcal{K}_g$ is non-decreasing, and the second step from $g/Id$ is non-increasing, and the final step from $g \leq Id$. Accordingly, $g$ is Lipschitz continuous with Lipschitz
constant $L_g = 1$, which is common for all $g \in \mathcal{K}_g$. Now, the remaining claims follow by the same arguments as in the proof of Lemma 8.

Proof of Lemma 4. Take $e \in [0, e_\mathcal{R}]$ with $e_\mathcal{R} = e$, then $\mathcal{R}(g)(e) = g(e) = e$ and $\mathcal{R}(g')(e) = g'(e) = e$; hence $|\mathcal{R}(g)(e) - \mathcal{R}(g')(e)| = 0$. Now, take $e \in (e, \bar{e})$. Noting that $g(z)/z = g'(z)/z = 1$ on $(0, e]$ and in general $g/\text{Id}, g'/\text{Id} \leq 1$, we see that

$$\mathcal{R}(g)(e) = e \inf\{g(z)/z : e \leq z \leq e\}$$

and

$$\mathcal{R}(g')(e) = e \inf\{g'(z)/z : e \leq z \leq e\}.$$ 

Without loss of generality, assume $\mathcal{R}(g')(e) \leq \mathcal{R}(g)(e)$ and write

$$\mathcal{R}(g)(e) = e \frac{g(z_0)}{z_0}, \text{ and } \mathcal{R}(g')(e) = e \frac{g'(z'_0)}{z'_0},$$

where $z_0, z'_0 \geq e$ are the respective minimizing arguments of the expressions above. These quantities exist due the continuity of $g$ and $g'$, but are perhaps not unique. Then by assumption and the optimality of $z_0$, we have

$$e \frac{g'(z'_0)}{z'_0} = \mathcal{R}(g')(e) \leq \mathcal{R}(g)(e) = e \frac{g(z_0)}{z_0} \leq e \frac{g'(z'_0)}{z'_0}$$

and

$$0 \leq \mathcal{R}(g)(e) - \mathcal{R}(g')(e) \leq e \frac{g'(z'_0)}{z'_0} - e \frac{g'(z'_0)}{z'_0} \leq e \frac{g - g'\|_{\infty}}{z_0} \leq \frac{\bar{e}}{e} \|g - g'\|_{\infty}.$$
Finally, we have to check the case $e \in (\bar{e}, \infty)$. We see that

$$\mathcal{R}(g)(e) = e \left( \inf \{ g(z) / z : e \leq z \leq \bar{e} \} \wedge \inf \{ g(z) / z : \bar{e} \leq z \leq e \} \right)$$

$$= \left( \frac{e}{\bar{e}} \mathcal{R}(g)(\bar{e}) \right) \wedge \left( e \inf \{ g(z) / z : \bar{e} \leq z \leq e \} \right)$$

$$= \left( \frac{\mathcal{R}(g)(\bar{e})}{\bar{e}} \right) e \wedge g(\bar{e}), \text{ for } e \geq \bar{e},$$

and, analogously,

$$\mathcal{R}(g')(e) = \left( \frac{\mathcal{R}(g')(\bar{e})}{\bar{e}} \right) e \wedge g'(\bar{e}), \text{ for } e \geq \bar{e}.$$

Without loss of generality, assume that $\mathcal{R}(g')(\bar{e}) \leq \mathcal{R}(g)(\bar{e})$. Define

$$e_0 = \inf \{ e \geq \bar{e} : \mathcal{R}(g)(e) = g(\bar{e}) \} = \frac{g(\bar{e}) \bar{e}}{\mathcal{R}(g)(\bar{e})} \text{ and}$$

$$e'_0 = \inf \{ e \geq \bar{e} : \mathcal{R}(g')(e) = g'(\bar{e}) \} = \frac{g'(\bar{e}) \bar{e}}{\mathcal{R}(g')(\bar{e})}.$$

Consider the case $e \geq e'_0$; then $\mathcal{R}(g')(e) = \mathcal{R}(g')(e'_0) = g'(\bar{e})$ and

$$|\mathcal{R}(g)(e) - \mathcal{R}(g')(e)| = \left\{ \begin{array}{ll} |g(\bar{e}) - g'(\bar{e})|, & \text{for } e \geq (e_0 \vee e'_0), \\ \left| \frac{\mathcal{R}(g)(\bar{e})}{\bar{e}} e - \frac{\mathcal{R}(g')(\bar{e})}{\bar{e}} e_0' \right|, & \text{for } e'_0 \leq e < (e_0 \vee e'_0). \end{array} \right.$$
Focusing on \( e_0^* \leq e < (e_0 \lor e_0') \), recalling \( \mathcal{R}(g')(\bar{e}) \leq \mathcal{R}(g)(\bar{e}) \), and using that the assumption \( g, g' \) are non-decreasing implies \( \mathcal{R}(g)(\bar{e}) \geq e \) and \( \mathcal{R}(g')(\bar{e}) \geq e \), respectively, gives us

\[
\left| \mathcal{R}(g)(e) - \mathcal{R}(g')(e) \right| = \left| \frac{\mathcal{R}(g)(\bar{e})}{\bar{e}} e - \frac{\mathcal{R}(g')(\bar{e})}{\bar{e}} e_0' \right|
= \frac{1}{\bar{e}} \left( \mathcal{R}(g)(\bar{e}) (e - e_0') + (\mathcal{R}(g)(\bar{e}) - \mathcal{R}(g')(\bar{e})) e_0' \right)
\leq \frac{1}{\bar{e}} \left( (e_0 \lor e_0') - e_0' \right) + \frac{e^2}{\bar{e}} \| g - g' \|_{\infty}
\leq \frac{\bar{e}}{\mathcal{R}(g')(\bar{e})} \left( \left( \frac{g(\bar{e})}{\mathcal{R}(g')(\bar{e})} - \frac{g'(\bar{e})}{\mathcal{R}(g')(\bar{e})} \right) \lor 0 \right) + \frac{e^2}{\bar{e}} \| g - g' \|_{\infty}
\leq \frac{\bar{e}}{\mathcal{R}(g')(\bar{e})} \left( \left( \frac{\mathcal{R}(g')(\bar{e})}{\mathcal{R}(g')(\bar{e})} g(\bar{e}) - g'(\bar{e}) \right) \lor 0 \right) + \frac{e^2}{\bar{e}} \| g - g' \|_{\infty}
\leq \frac{\bar{e}}{\mathcal{R}(g')(\bar{e})} (g(\bar{e}) - g'(\bar{e})) \lor 0 + \frac{e^2}{\bar{e}} \| g - g' \|_{\infty}
\leq 2 \frac{\bar{e}^2}{\bar{e}} \| g - g' \|_{\infty}.
\]

Thus, for all \( e \geq e_0' \) we have

\[
\left| \mathcal{R}(g)(e) - \mathcal{R}(g')(e) \right| \leq 2 \frac{\bar{e}^2}{\bar{e}} \| g - g' \|_{\infty}.
\]

For \( \bar{e} \leq e \leq e_0' \), compute

\[
\left| \mathcal{R}(h)(y) - \mathcal{R}(h')(y) \right| \leq \left( \mathcal{R}(g)(\bar{e}) - \mathcal{R}(g')(\bar{e}) \right) \frac{e}{\bar{e}} \leq \frac{\bar{e}}{\bar{e}} \| g - g' \|_{\infty} \frac{e_0'}{\bar{e}}
\leq \frac{\bar{e}}{\bar{e}} \| g - g' \|_{\infty} \frac{1}{\bar{e}} \frac{g'(\bar{e})}{\mathcal{R}(g')(\bar{e})} \leq \frac{\bar{e}}{\bar{e}} \| g - g' \|_{\infty} \frac{1}{\bar{e}} \frac{\bar{e}^2}{\bar{e}}
= \frac{\bar{e}^2}{\bar{e}} \| g - g' \|_{\infty}.
\]

Noting that \( e_0, e_0' \leq \bar{e} \mathcal{R} = \frac{\bar{e}^2}{\bar{e}} \) and verifying that \( \mathcal{R}(g) \) and \( \mathcal{R}(g') \) are constant and equal to \( g(\bar{e}) \) and \( g'(\bar{e}) \), respectively, finishes the proof. \( \square \)