We construct a dynamic model of a multi-asset over-the-counter (OTC) market that operates via search and bargaining and empirically test its implications using data from the US corporate bond market. The key novelty in our model is that investors can hold and manage portfolios of OTC-traded assets. We characterize the stationary equilibrium in closed form which includes investors’ valuations, terms of trade, and the characteristic function of the distribution of investors’ states. Tractability of the model allows us to derive natural proxies for important measures of market liquidity such as trading volume, price dispersion, and price impact. Among other within-market and cross-market comparative statics, we find that the alleviation of search frictions in one market may lead to opposite observations regarding liquidity in other markets depending on which liquidity measure is used. For example, a reduction of search frictions in one market decreases trade volume in other markets implying lower liquidity. At the same time, it decreases price dispersion and price impact implying higher liquidity. We test empirically these key liquidity equations that come out of our model. Our regressions indicate significant support for the search-and-bargaining framework in uncovering the determinants of endogenous liquidity differentials across OTC assets. Finally, we argue that, among the liquidity measures we analyze, price impact can serve as a good measure of welfare loss caused by OTC frictions, while other measures overstate or understate this deadweight loss.
1 Introduction

Many financial assets are traded over the counter (OTC) including fixed income securities, derivatives, and some stocks. For investors who invest in these OTC-traded securities, portfolio management is more complicated than analyzing only the dynamics of payoff risk because OTC trades in practice involve search and bargaining. Whenever they intend to update their portfolios, investors must go through a process which includes locating a suitable counterparty and negotiating over the terms of trade. And yet, perhaps surprisingly, optimal portfolio management by considering the risks associated with search and bargaining and its equilibrium implications for prices, liquidity, and welfare remain open questions. In this paper, we contribute to the search-theoretic OTC market literature by exploring this understudied area of managing portfolios containing multiple OTC assets.

More precisely, we construct a dynamic equilibrium model in which investors can invest in portfolios of OTC assets. Each OTC asset is traded in a segmented, fully decentralized market that operates via search and bilateral bargaining. The assets differ from each other in their exposure to an aggregate risk factor and in the severity of search frictions in the particular segment of the OTC market where they are traded (i.e., different assets have different contact rates). This heterogeneous exposure to search frictions in the cross section of assets conveniently represents liquidity differentials unrelated to the payoff risk heterogeneity. We show that studying a model with arbitrary joint distribution of payoff risk and contact rates in the cross section of assets is crucial to rationalize puzzling empirical observations such as the apparent flees from quality in the Euro-area government bond market and the non-monotonicity of liquidity in credit rating in the US corporate bond market.

In our model, a continuum of risk-averse investors with stochastic hedging needs contact one another pairwise in different segments of the OTC market and bargain bilaterally over the terms of trade including price and quantity of the asset that is traded in that particular market segment. The continuum population, together with the pairwise matching assumption, allows us to formulate the model as an exact mean field game.\(^1\) When negotiating over the terms of trade, investors take as given the equilibrium distribution of asset positions in order to evaluate the value of their outside option, i.e., the value of continuing search. In turn, these negotiated terms generate the distribution of asset positions. Thus, the distribution of investors’ positions and their strategies must be jointly pinned down as a fixed point of the mean field game, which complicates the equilibrium analysis. However, employing the characteristic function

\(^1\)See Lasry and Lions (2007).
techniques and focusing on an asymptotic case in which investors are averse to systematic risk only, we show that the model is fully tractable.\textsuperscript{2}

The presence of search frictions in our model in the sense of inability to instantly access a competing counterparty makes investors’ current state a determinant of their marginal valuation. This means that when bargaining over the terms of trade for an asset, an investor will take into account her current state including her hedging need and her positions in all other assets, unlike all other multi-asset OTC market models which allow investors to hold only one of the many assets at a time. In the characterization of equilibrium, we show that an investor’s current state can be summarized by a sufficient statistic which equals her hedging need type plus the weighted sum of her (excess) inventories in all assets with weights being the assets’ exposure to the aggregate risk factor. We term this sufficient statistic “excess risk exposure” because it is equal to the difference between the investor’s current exposure to systematic risk and the per capita endowment of systematic risk in the economy at large. We derive all the stationary equilibrium objects in closed form including investors’ valuations, terms of trade, and the characteristic function of the distribution of investors’ excess risk exposures.

When each pair of buyer and seller contact, their negotiated trade quantity is determined such that their excess risk exposures are pair-wise equalized. This implies that investors tend to trade safer assets in larger quantities, and vice-versa, riskier assets are traded in smaller quantities. Thus, controlling for the contact rate (i.e., the inverse of the exposure to search frictions), safer assets have larger trade volume than riskier assets. However, high contact rate in the market for a particular asset allows investors to have more frequent opportunities to exchange that asset, and so tends to increase the trade volume. As a result, we show that upward-sloping iso-trade-volume curves arise on the plane of systematic risk and contact rate because systematic risk and contact rate have an opposite impact on equilibrium trade volume. This is consistent with the flight to quality and flight to liquidity phenomena observed in various OTC markets in practice. In addition, it can rationalize more puzzling observations such as flies from quality or non-monotonicity of trade volume in credit rating in markets where quality and liquidity are (locally) inversely related in the cross section of assets.\textsuperscript{3}

\textsuperscript{2}Praz (2014) and ¨Usl¨u (2019) also use the same “source-dependent” risk aversion approach combined with characteristic functions (Fourier transform) in their respective single-asset models. Although we have a multi-asset model, our characterization is even more explicit than theirs because we do not assume any \textit{ex ante} heterogeneity in investors’ characteristics or asymmetric information. As a result, we are able to obtain an explicit expression for the characteristic function of the distribution of investors’ excess risk exposures, while Praz (2014) and ¨Usl¨u (2019) can only characterize the moments explicitly.

\textsuperscript{3}See Beber, Brandt, and Kavajecz (2009) and Geromichalos, Herrenbrueck, and Lee (2018) for further details about these puzzling empirical facts.
In addition to the intuitive trade volume results explained above, we also obtain a general substitutability result regarding trade volume as a cross-market comparative static. We show that while an increase in the contact rate in a market increases the equilibrium trade volume in that market, it decreases the volume in all other markets. As the contact rate in a certain market increases, investors have more frequent opportunities in that market to equalize their excess risk exposures, which means that there will be less misallocation in their excess risk exposures when they meet in other markets. This depresses the volume they trade in other markets. This is an important cross-market comparative static that could not be deduced from single-asset models, whose comparative statics typically imply a positive relationship between contact rate and trade volume.\footnote{See Hugonnier, Lester, and Weill (2014) and Üslü (2019) for example.}

As is the case with negotiated quantity, when each pair of buyer and seller contact, their negotiated price also depends on their current excess risk exposures. This gives rise to equilibrium price dispersion. We calculate two price-related measures of liquidity: price dispersion and price impact. Price dispersion is defined to be the standard deviation of the equilibrium price distribution, while price impact is defined to be price dispersion divided by the standard deviation of the equilibrium quantity distribution. While the former is a natural definition for price dispersion, the latter is a model-informed measure for price impact. We show that the negotiated prices in equilibrium are equal to the mid-point of the negotiating parties’ marginal valuations. Thus, a natural measure of price impact in a certain market is (half of) the sensitivity of an investor’s marginal valuation to her position in the asset which is traded in that market. We show that, in equilibrium, this sensitivity measure coincides with the ratio of price dispersion to quantity dispersion, and hence, we define it to be price impact.

We find that, while a general substitutability holds regarding the effect of an increase in the contact rate of an asset on the trade volume of other assets, a general complementarity holds regarding price dispersion and price impact. This is due to investors’ ability to hold multiple OTC assets at the same time, which is a unique feature of our model. As a result of this feature, investors recognize different markets as perfectly substitutable venues in terms of the opportunity to pair-wise equalize their excess risk exposures. In turn, the sensitivity of their marginal valuations to excess risk exposures turn out to depend on the total contact rate of all markets. Accordingly, price dispersion and price impact in an individual market also depend on the total contact rate of all markets, instead of the contact rate of that particular market only. Consequently, we find that while the cross-sectional trade volume patterns are determined by
both risk and contact rate differentials, the cross-sectional differences in price dispersion and price impact are solely determined by risk differentials across assets. That is to say, an increase in the contact rate in a market decreases the price dispersion and price impact in all markets by the same factor.

To understand the extent to which the within-market and cross-market comparative statics results summarized above hold in real-world OTC markets, we test our theoretical formulas for trading volume, price dispersion, and price impact in the cross section of bonds traded in the US corporate bond market. The corporate bond market is a textbook example of an OTC market where majority of trades are purely bilateral and subject to significant search frictions. Overall, our results from empirical tests of liquidity are mostly consistent with the implications of the theoretical model. We interpret this as pointing to the success and usefulness of the search-theoretic approach in uncovering the determinants of endogenous liquidity differentials across OTC assets, especially considering its ability to lead to parsimonious and tractable models as exemplified by our theoretical model.

As is typical in the dynamic search models with price dispersion, the equilibrium of our model exhibits *endogenous intermediation*. Not only do investors trade to share risk with other investors, but they also trade to profit from price dispersion, i.e., they buy assets from those who are willing to sell at relatively low prices and later sell these assets to those who are willing to buy at higher prices. As a result, investors’ gross trade volume exceeds their net trade volume, where the difference is their intermediation volume. This allows us to calculate the aggregate length of intermediation chains as an equilibrium result as the ratio of intermediation volume to net volume. Similar to the gross volume patterns, we find that upward-sloping iso-net-volume and iso-intermediation-volume curves arise on the plane of systematic risk and contact rate. More interestingly, we find that the cross-sectional systematic risk and contact rate differentials scale up or down all volume measures in the same proportions, which imply that the aggregate length of intermediation chains is the same across all markets. This is another result that would not obtain as a comparative static result of a single-asset model.

We show that the sole determinant of the aggregate length of intermediation chains is the equilibrium level of misallocation, which is the ratio of the speed at which investors’ hedging needs change to the speed at which investors can trade in some market. If the equilibrium misallocation is smaller, intermediation chains are longer. In an equilibrium with large misallocation, investors use their trading opportunities mostly to correct their risk exposure. As misallocation gets smaller, investors use their trading opportunities mostly to facilitate the flow of assets be-
tween investors with extreme risk exposures, i.e., from one end of the market to the other end. There is a literature that has established the efficiency-enhancing role of long intermediation chains in markets with adverse selection between ultimate buyers and ultimate sellers.\(^5\) Our model shows that the bilateral trading alone, abstracting from informational frictions, can lead to the positive link between long intermediation chains and more efficient allocation.

In the last part of the paper, we analyze the extent to which our liquidity measures can serve as a proxy for the welfare impact of OTC market frictions. To do so, we look at the asymptotic effect of OTC frictions on trade volume, price dispersion, price impact, and welfare. We show that trade volume overstates and price dispersion understates the welfare impact of frictions, while price impact can serve as a good measure to quantify the welfare loss.

The remainder of the paper is organized as follows. We next discuss how our paper relates to the existing literature. Section 2 describes the model environment. Section 3 studies the stationary equilibrium in this environment, while Section 4 discusses the main results about the various endogenous measures of liquidity. Section 5 analyzes the extent to which our theoretical findings are consistent with liquidity differentials across corporate bonds in practice. Section 6 presents further qualitative results regarding intermediation chains and welfare. Section 7 concludes.

### 1.1 Related literature

Search-theoretic approach to OTC market structure, spurred by Duffie, Gårleanu, and Pedersen (2005), has proven very useful in analyzing the determinants and various measures of market liquidity and become the dominant approach in modelling OTC markets. Our paper contributes to this literature by considering a multi-asset trading model, where investors are allowed to hold portfolios of OTC assets. In particular, our model follows the approach of having only search frictions like the single-asset models of Gårleanu (2009), Afonso and Lagos (2015), and Üslü (2019) and does not impose any restrictions on portfolio holdings. This sets apart our model from all the existing search-theoretic multi-asset OTC models such as Vayanos and Wang (2007), Vayanos and Weill (2008), Weill (2008), Sambalaibat (2015), Milbradt (2017), and An (2019), whose investors can only hold an indivisible position in one of the assets. Thus, to our knowledge, our model is the first to analyze investors’ optimal portfolio management strategies in OTC markets and the resulting effect of asset characteristics on equilibrium terms of trade.

\(^5\)See, for example, Glode and Opp (2016) and Glode, Opp, and Zhang (2019).
Because of the rich heterogeneity in investors’ hedging needs and portfolios, the equilibrium of our model exhibits price dispersion and, accordingly, endogenous intermediation. Thus, our model is also related to the single-asset endogenous intermediation models of Afonso and Lagos (2015), Hugonnier et al. (2014), Üslü (2019), Farboodi, Jarosch, and Shimer (2015), Farboodi, Jarosch, and Menzio (2016), Shen, Wei, and Yan (2018), and Bethune, Sultanum, and Trachter (2018), for example. Compared to these papers, our contribution is to obtain cross-market comparative statics regarding market liquidity, which are not possible to obtain in single-asset models. Similar to Hugonnier, Lester, and Weill (2019), Hendershott, Li, Livdan, and Schürhoff (2015), Shen et al. (2018), Bethune et al. (2018), and An (2019), not only do we construct an OTC trading model, but we also empirically test our model’s key implications. While these papers only test market-wide implications or test cross-sectional implications via comparative statics of model parameters, our theoretical portfolio choice model allows us to formulate precise cross-sectional hypotheses and directly test them.

Malamud and Rostek (2017) and Aymanns, Georg, and Golub (2018) study static network-based models of multi-asset OTC markets, where investors engage in one-shot trading game in multiple segmented markets at the same time. Our dynamic model, instead, analyzes how investors optimally manage their portfolios over time by fully internalizing the option value of waiting and continuing search. There are also multi-asset models in the search-theoretic literature on monetary economics. See, among others, Rocheteau (2011), Li, Rocheteau, and Weill (2012), Hu (2013), Lagos (2013), Geromichalos and Herrenbrueck (2016), Geromichalos et al. (2018), and Hu, In, Lebeau, and Rocheteau (2018). These papers also focus on analyzing liquidity differentials across assets. However, the concept of liquidity they analyze is mainly the assets’ ability to serve as medium of exchange, while we focus on market liquidity, i.e., the ease of sale and purchase.

Finally, our paper is also related to the literature that analyzes the efficiency-enhancing role of intermediation chains in markets with bilateral trade and information frictions with papers by Glode and Opp (2016) and Glode et al. (2019), for example. While these papers are partial equilibrium models which study the trade of an indivisible unit of an asset along a chain of bilateral trades, we establish that long intermediation chains are an indication of higher efficiency in our general equilibrium setup even in the absence of information frictions.
2 Environment

Time is continuous and has an infinite horizon. We fix a probability space \((\Omega, \mathcal{F}, \Pr)\) and a filtration \(\{\mathcal{F}_t, t \geq 0\}\) of sub-\(\sigma\)-algebras satisfying the usual conditions (see Protter, 2004). An economy is populated by a continuum of investors with a normalized mass of 1. Investors are von Neumann-Morgenstern expected utility maximizers with a constant absolute risk aversion (CARA) coefficient of \(\gamma > 0\). They discount the future at rate \(r > 0\) and are also able to borrow and lend frictionlessly at this exogenous rate \(r\).

There are \(J \in \mathbb{Z}_+\) risky assets, which are indexed by \(j \in \mathcal{J} \equiv \{1, 2, \ldots, J\}\) and in zero net supply. Investors can trade these assets over the counter. The assets’ cumulative dividend flows, \(D_j\), evolve according to

\[
dD_{jt} = m_j dt + \sigma_j \psi_j dB_t + \nu_j dB_{jt}
\]

for \(j \in \mathcal{J}\), where \(B_t\) is a standard Brownian motion. The first term of (1) captures the expected dividend flow. The second term captures the systematic risk and depends on the aggregate volatility parameter \(\sigma\). The last term captures the asset-specific risk; i.e., \(B_{jt}\)'s are i.i.d. standard Brownian motion processes, which are also independent of \(B_t\).

Investor \(i \in [0, 1]\) has a cumulative background income process \(Z^i\):

\[
dZ^i_t = m^i Z^i dt + \eta^i_t \sigma dB_t,
\]

where

\[
d\eta^i_t = \sigma \eta^i dB^i_t.
\]

The exogenous object \(\eta^i_t \sigma^2 \psi_j\) captures the instantaneous covariance between the payoff of OTC asset \(j\) and the investor \(i\)'s background income for all \(j \in \mathcal{J}\). This covariance is time-varying and heterogeneous across investors. Thus, this heterogeneity creates the fundamental gains from trade. We interpret this heterogeneity as instantaneous hedging need differentials across investors.

Importantly, the heterogeneity-driving coefficient \(\eta^i_t\) is stochastic itself. Investors continuously receive idiosyncratic shocks to the covariance between the asset payoffs and their background risk, which creates the motive to trade even in steady state.\(^6\) Arrival of these shocks is

\(^6\)To generate trade volume, Lo, Mamaysky, and Wang (2004), Chapter III of Praz (2014, co-authored with Julien Cujean), and Sannikov and Skrzypacz (2016) also utilize hedging needs that follow diffusion processes. Antill and Duffie (2018) allow for Lévy processes which include pure jump, pure diffusion, and jump-diffusion processes.
governed as a diffusion by the standard Brownian motion processes $B^i_t$, which are i.i.d. in the cross section of investors and independent of $B_t$ and $B_j$s, as well. Since the assets are in zero net supply and (2) does not have a drift term, there is no aggregate traded or non-traded risk.

Trades are fully bilateral and take place in segmented markets for assets 1, 2, and so on. Each investor is endowed with $J$ trading specialists, indexed by $j$. A specialist indexed by $j$ has specialization in trading in market $j$. In each market, pair-wise meetings among trading specialists follow standard random search and matching dynamics. A given specialist in market $j$ meets another specialist at Poisson arrival times with intensity $\lambda_j > 0$, reflecting the overall search efficiency of the market $j$ for $j \in J$. Conditional on a meeting in market $j$, the counterparty is drawn randomly and uniformly from the pool of all specialists operating in market $j$.

Let $a_{-j}$ refer to a $J - 1$-dimensional vector that represents an investor’s asset positions in all markets except for $j$. A meeting in market $j$ between the specialist who serves investor $(\eta, a_j, a_{-j})$ and another specialist who serves investor $(\eta', a'_j, a'_{-j})$ is followed by a bargaining process over quantity $q$ and unit price $P$. The specific bargaining protocol we employ is the axiomatic bargaining à la Nash (1950) in which investors are symmetric in their bargaining powers. The resulting number of assets that the investor $(\eta, a_j, a_{-j})$ purchases is denoted by $q_j [(\eta, a_j, a_{-j}), (\eta', a'_j, a'_{-j})]$. Thus, her position in asset $j$ will become $a_j + q_j [(\eta, a_j, a_{-j}), (\eta', a'_j, a'_{-j})]$ after this trade, while her counterparty’s position in asset $j$ will become $a'_j - q_j [(\eta, a_j, a_{-j}), (\eta', a'_j, a'_{-j})]$. The per unit price, the investor $(\rho, a_j, a_{-j})$ will pay, is denoted by $P_j [(\eta, a_j, a_{-j}), (\eta', a'_j, a'_{-j})]$.

3 Equilibrium

We solve for the equilibrium of the economy described in the previous section in two steps. First, we study a “partial equilibrium” determination of investors’ stationary trading rules by taking as given the dynamics of a joint distribution of investors’ types and asset positions, denoted by $\Phi$. In the second part, we endogenize the dynamics of the equilibrium joint distribution generated by investors’ stationary optimal trading rules.

\footnote{See Duffie, Qiao, and Sun (2017) for a formal treatment of the existence of continuous-time independent random matching in a continuum population.}
3.1 Investor’s problem

Let $a \in \mathbb{R}^J$ denote the vector of asset positions. Let both $U (W, \eta, a)$ and $U (W, \eta, a_j, a_{-j})$, that we use interchangeably, refer to the maximum attainable continuation utility of an investor of type $(\eta, a)$ with current wealth $W$. They satisfy

$$U (W, \eta, a) = \sup_c \mathbb{E}_t \left[ -\int_t^\infty e^{-r(s-t)} e^{-\gamma c_s} ds \right| W_t = W, \eta_t = \eta, a_t = a],$$

subject to

$$dW_t = (rW_t - c_t + m_Z) dt + \eta_t \sigma dB_t + \sum_{j=1}^J \{a_{jt} - dD_{jt} - P_j [(\eta_{t-}, a_{t-}), (\eta'_t, a'_t)] \} da_{jt},$$

$$da_{jt} = \begin{cases} q_j [(\eta_{t-}, a_{t-}), (\eta'_t, a'_t)] & \text{if } (\eta'_t, a'_t) \text{ is contacted in market } j \\ 0 & \text{if no contact in market } j, \end{cases}$$

where

$$\{q_j [(\eta, a), (\eta', a')], P_j [(\eta, a), (\eta', a')])\} = \arg\max_{q,P} \left[ U (W - Pq, \eta, a_j + q, a_{-j}) - U (W, \eta, a_j, a_{-j}) \right]^\frac{1}{2}$$

subject to

$$U (W - Pq, \eta, a_j + q, a_{-j}) \geq U (W, \eta, a_j, a_{-j}),$$

$$U (W' + Pq, \eta', a'_j - q, a'_{-j}) \geq U (W', \eta', a'_j, a'_{-j}).$$

Since CARA preferences imply zero wealth effects, terms of trade are independent of wealth levels as will be clear shortly. To prevent Ponzi schemes from arising in the optimal solution, we impose the transversality condition

$$\lim_{T \to \infty} e^{-r(T-t)} \mathbb{E}_t \left[ e^{-r\gamma W_T} \right] = 0.$$

We use the technique of the stochastic dynamic programming to derive the optimal rules. Assuming sufficient differentiability and applying the Ito’s lemma for Lévy processes, the investor’s
value function \( U(W, \eta, a) \) satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

\[
0 = \sup_c \left\{ -e^{-\gamma c} + U_W(W, \eta, a) \left( rW - c + m_Z + \sum_{j=1}^J a_j m_j \right) \right. \\
+ \frac{1}{2} U_{WW}(W, \eta, a) \left( \eta^2 \sigma^2 + 2\eta \sigma^2 \sum_{j=1}^J \psi_j a_j + 2\sigma^2 \sum_{j=1}^J \sum_{j<k} \psi_j \psi_k a_j a_k + \sigma^2 \sum_{j=1}^J \psi_j a_j^2 + \sum_{j=1}^J \nu_j a_j^2 \right) \\
+ \frac{1}{2} U_{\eta\eta}(W, \eta, a) \sigma^2 - r U(W, \eta, a) + \dot{U}(W, \eta, a) \\
+ \sum_{j=1}^J \left( 2\lambda_j \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ U(W - q_j(\mu, \mu'), P_j(\mu, \mu'), \eta, a_j + q_j(\mu, \mu'), a_{-j}) - U(W, \eta, a_j, a_{-j}) \right] \right. \\
\left. \left. \Phi(da', d\eta') \right) \right\}, \tag{4}
\]

where \( \mu \equiv (\eta, a) \) and \( \mu' \equiv (\eta', a') \).

We solve this HJB equation by making the standard Ansatz

\[
U(W, \eta, a) = -e^{-r\gamma(W + V(\eta, a) + V)},
\]

where

\[
V = \frac{1}{r} \left( m_Z + \frac{\log r}{\gamma} \right)
\]
is a constant and \( V(\eta, a) \) is the consumption-equivalent value function that will determine the terms of trade. Using the Ansatz, we find that the optimal consumption is

\[
c = -\frac{\log r}{\gamma} + r \left( W + V(\eta, a) + V \right).
\]

Substituting \( c \) into (4) and dividing by \( r\gamma U(W, \eta, a) \), we find at steady state that (4) is satisfied if and only if

\[
rV(\eta, a) = \sum_{j=1}^J m_j a_j - \frac{1}{2} r\gamma \sigma^2 \left( \eta^2 + 2\eta \sum_{j=1}^J \psi_j a_j + 2 \sum_{j=1}^J \sum_{j<k} \psi_j \psi_k a_j a_k + \sum_{j=1}^J \psi_j a_j^2 \right) \\
- \frac{1}{2} r\gamma \sum_{j=1}^J \nu_j a_j^2 - \frac{1}{2} \sigma^2 \left[ r\gamma (V(\eta, a))^2 - V_{\eta\eta}(\eta, a) \right] \\
+ \sum_{j=1}^J \left( 2\lambda_j \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ 1 - e^{-r\gamma[V(\eta, a_j + q_j(\mu, \mu'), a_{-j}) - V(\eta, a_j, a_{-j}) - q_j(\mu, \mu')P_j(\mu, \mu')] / r\gamma} \Phi(da', d\eta') \right) \right. \\
\left. \left. \right) \right). \tag{5}
\]
Terms of bilateral trades, \( q_j (\mu, \mu') \) and \( P_j (\mu, \mu') \), maximize the Nash product (3). By dividing by \( U (W, \eta, a)^{\frac{1}{2}} U (W', \eta', a')^{\frac{1}{2}} \), we simplify (3) as

\[
\{ q_j [(\eta, a), (\eta', a')] , P_j [(\eta, a), (\eta', a')] \}
\]

subject to

\[
1 - e^{-r\gamma [V(\eta, a_j + q, a_{-j}) - V(\eta, a_j, a_{-j})] - qP]} \geq 0,
\]
\[
1 - e^{-r\gamma [V(\eta', a'_j - q, a'_{-j}) - V(\eta', a'_j, a'_{-j})] + qP] } \geq 0,
\]

which verifies that there are no wealth effects. Solving this problem is relatively straightforward: We set up the Lagrangian of this problem. Then using the first-order and Kuhn-Tucker conditions, the trade quantity \( q_j [(\eta, a), (\eta', a')] \) solves

\[
V^{(j)} (\eta, a_j + q, a_{-j}) = V^{(j)} (\eta', a'_j - q, a'_{-j}) ,
\]

where \( V^{(j)} \) stands for the partial derivative with respect to the argument representing the position in asset \( j \). And, the negotiated price \( P_j [(\eta, a), (\eta', a')] \) is determined such that the joint trade surplus is split equally between the negotiating parties

\[
P = \frac{V (\eta, a_j + q, a_{-j}) - V (\eta, a_j, a_{-j}) - (V (\eta', a'_j - q, a'_{-j}) - V (\eta', a'_j, a'_{-j}))}{2q},
\]

if \( V^{(j)} (\eta, a_j, a_{-j}) \neq V^{(j)} (\eta', a'_j, a'_{-j}) \); and \( P = V^{(j)} (\eta, a_j, a_{-j}) \) if \( V^{(j)} (\eta, a_j, a_{-j}) = V^{(j)} (\eta', a'_j, a'_{-j}) \).

Substituting the pricing function into (5), we get

\[
rV (\eta, a) = \sum_{j=1}^{J} m_j a_j - \frac{1}{2} r\gamma \sigma^2 \left( \eta^2 + 2\eta \sum_{j=1}^{J} \psi_j a_j + 2 \sum_{j=1}^{J} \sum_{k>j} \psi_j \psi_k a_j a_k + \sum_{j=1}^{J} \psi_j^2 a_j^2 \right) - \frac{1}{2} \sigma^2 \eta \left[ r\gamma (V(\eta, a) - V_{\eta\eta}(\eta, a)) \right] - r\gamma \sum_{j=1}^{J} \psi_j^2 a_j^2 - \frac{1}{2} \sigma^2 \eta \left[ r\gamma (V(\eta, a) - V_{\eta\eta}(\eta, a)) \right] - V_{\eta\eta}(\eta, a) \]
\]

subject to (6).

\[
\Phi (da', d\eta'),
\]

12
In order to obtain an asymptotic solution of Equation (8) in closed form, we follow Üslü (2019) and calculate the limit as the CARA coefficient vanishes and the aggregate volatility goes to infinity at the same speed. Mathematically, this leads to the first-order linear approximation \( \frac{1-e^{-x}}{r\gamma} \approx x \) that ignores terms of order higher than 1 in \([V (\eta, a_j + q, a_{-j}) - V (\eta, a_j, a_{-j})]\).\(^8\)

Economically, this approximation can be understood in terms of source-dependent risk aversion; i.e., we assume that investors are averse towards systematic risk (risk generated by \(B_t\) in the model), while they are neutral towards other types of risk. The assumption does not suppress the impact of risk aversion because the instantaneous mean-variance benefit function (11) associated with asset holdings possesses a negative definite quadratic part. Thus, as is formally stated in the lemma below, this assumption only linearizes the preferences of investors over jumps in the continuation values created by idiosyncratic risk or jump risk of trade opportunities.

**Lemma 1.** Fix parameters \(\gamma\) and \(\sigma\) and let \(\bar{\sigma} = \sigma \sqrt{\gamma}/\gamma\). In the stationary equilibrium, investors’ value functions solve the following HJB equation in the limit as \(\gamma \to 0\):

\[
\begin{align*}
& rV (\eta, a) = \sum_{j=1}^J m_j a_j - \frac{1}{2} r\gamma \bar{\sigma}^2 \left( \eta^2 + 2\eta \sum_{j=1}^J \psi_j a_j + 2 \sum_{j=1}^J \sum_{k>j}^J \psi_j \psi_k a_j a_k + \sum_{j=1}^J \psi_j^2 a_j^2 \right) \\
& + \frac{1}{2} \sigma^2 \eta V_{\eta\eta} (\eta, a) + \sum_{j=1}^J \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ V (\eta, a_j + q (\mu, \mu'), a_{-j}) - V (\eta, a_j, a_{-j}) \right. \\
& \quad \left. + V (\eta', a_j' - q (\mu, \mu'), a_{-j}') - V (\eta', a_j', a_{-j}') \right] \Phi (da', d\eta') .
\end{align*}
\]

subject to (6).

Notice that the quantity which solves (6) is also the maximizer of the joint trade surplus; i.e.,

\[
q_j [ (\eta, a), (\eta', a') ] = \arg \max_q \left\{ V (\eta, a_j + q, a_{-j}) - V (\eta, a_j, a_{-j}) + V (\eta', a_j' - q, a_{-j}') - V (\eta', a_j', a_{-j}') \right\} .
\]

\(^8\)The same approximation is also used by Biais (1993), Duffie, Gărleanu, and Pedersen (2007), Vayanos and Weill (2008), Gărleanu (2009), and Praz (2014).
Using this and ignoring bars, (9) can be written as

\[
\begin{align*}
\r V (\eta, a) &= u (\eta, a) + \frac{1}{2} \sigma^2 \eta \eta \left( \eta, a \right) \\
+ \sum_{j=1}^{J} \left( \lambda_j \int_{\mathbb{R}} \int_{\mathbb{R}} \max_{q} \left[ V (\eta, a_j + q, a_{-j}) - V (\eta, a_j, a_{-j}) + V (\eta', a'_j - q, a'_{-j}) \
- V (\eta', a'_j, a'_{-j}) \right] \phi (da', d\eta') \right),
\end{align*}
\]

(10)

where

\[
\begin{align*}
u (\eta, a) &\equiv m^T a - \frac{1}{2} r \gamma \sigma^2 (\eta^2 + 2 \eta \psi^T a + a^T \Psi a) \\
\Psi &\equiv (\psi_1, \psi_2, ..., \psi_J)^T,
\end{align*}
\]

(11)
is the instantaneous mean-variance benefit to the investor from holding the portfolio \( a \) when she is of type \( \eta \),

\[
m \equiv [m_1, m_2, ..., m_J]^T,
\]

\[
\psi \equiv [\psi_1, \psi_2, ..., \psi_J]^T,
\]

and

\[
\Psi \equiv \psi^T \psi.
\]

In order to solve for \( V (\eta, a) \), we follow the method of undetermined coefficients. The complete solution is given in Proposition 1. Since (10) is a flow Bellman equation with a negative definite linear-quadratic return function, the solution \( V (\eta, a) \) itself inherits the negative definite linear-quadratic functional form as well. As a result, to find the stationary equilibrium value of \( V (\eta, a) \), we are required to use the cross-sectional mean of the linear part and of the quadratic part of the return function, i.e., \( \mathbb{E} [a'] \) and \( \mathbb{E} [(\eta')^2 + 2 \eta' \psi^T a' + (a')^T \Psi a'] \), respectively, instead of the entire joint distribution \( \phi (a', \eta') \).

What is more striking is that determining the investors’ equilibrium trading behavior does not require calculating any moment of the endogenous distribution of asset positions. To see this, one can take the derivative of (10) with respect to \( a \) by applying the envelope theorem and arrive at the following vector of partial derivatives:

\[
\r \frac{\partial V}{\partial a} (\eta, a) = m - r \gamma \sigma^2 (\eta \psi + \Psi a)
\]

\[
+ \sum_{k=1}^{J} \left( \lambda_k \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ \frac{\partial V}{\partial a} (\eta, a_k + q_k [(\eta, a), (\eta', a')], a_{-k}) - \frac{\partial V}{\partial a} (\eta, a) \right] \phi (da', d\eta') \right).
\]

(12)
Equation (12) provides us with a flow Bellman equation for the vector of marginal valuations, where the $j^{th}$ column of $\frac{\partial V}{\partial a}(\eta, a)$ is the marginal valuation for asset $j$, $V^{(j)}(\eta, a)$. As can be seen, the flow Bellman equations for the marginal valuation have a return function that is linear and separable in $\eta$ and all $a_j$s for $j \in J$. In turn, the FOC (6) for Nash bargaining and (12) imply that $V^{(j)}(\eta, a)$ is itself linear and separable in all of its arguments. Thus, calculating the equilibrium value of the second line of (12) requires only the first moment of the asset holding distribution for assets $j \in J$, which equal the exogenous supply of those assets by market clearing: $E[a'] = 0$. The following proposition establishes the optimal trading behavior of investors at steady state.

**Proposition 1.** Let $\lambda = \sum_{k=1}^{J} \lambda_k$ and

$$\theta(\eta, a) = \eta + \psi^T(a - E[a']).$$

The unique quadratic stationary value function that solves (10) is

$$V(\eta, a) = \frac{\gamma \sigma^2}{2r + \lambda} \left( -\sigma^2_{\eta} + \frac{\lambda}{2} \mathbb{E} \left[ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' \right] \right) - \frac{\lambda \gamma \sigma^2}{2r + \lambda} \psi^T \mathbb{E} [a'] \eta$$

$$+ \left( \frac{1}{r} - \frac{\gamma \sigma^2}{2r + \lambda} \psi^T \mathbb{E} [a'] \psi \right)^T a - \frac{\gamma \sigma^2}{2r + \lambda} \left( \eta^2 + 2\eta \psi^T a + a^T \Psi a \right), \quad (13)$$

Thus, at steady state, investors’ marginal valuations, individual trade sizes, and transaction prices are given by:

$$\frac{\partial V}{\partial a}(\eta, a) = \frac{1}{r} \frac{\partial u}{\partial a} (0, \mathbb{E} [a']) - \frac{r \gamma \sigma^2}{r + \lambda/2} \theta(\eta, a) \psi, \quad (14)$$

$$q_j[(\eta, a), (\eta', a')] = \frac{\theta(\eta', a') - \theta(\eta, a)}{2 \psi_j}, \quad (15)$$

and

$$P_j[(\eta, a), (\eta', a')] = V^{(j)} \left( \frac{\eta + \eta'}{2}, \frac{a + a'}{2} \right) = \frac{V^{(j)}(\eta, a) + V^{(j)}(\eta', a')}{2}, \quad (16)$$

respectively.

Equation (14) reveals important information about the effect of OTC frictions. In a frictionless market, the equilibrium marginal valuation would not depend on the current state as investors would equalize their marginal valuation instantly. The frictionless case is achieved in the limit as $\lambda \rightarrow \infty$. When all $\lambda_k$s are finite, investors’ marginal valuation is dependent on their current state as well. This essentially reflects the time cost of search. When negotiating a trade,
investors rationally expect that they will spend some time with their post-trade portfolio as a result of limited trading opportunities, even if their preferred portfolio becomes very different. Therefore, this situation creates deviation of the marginal valuation from what would obtain in a frictionless benchmark case.

Combining (14) with the FOC (6) for Nash bargaining, one sees that investors’ bilateral trade quantities are determined such that their \( \theta \)s are pair-wise equalized. Thus, the composite type \( \theta \) serves as a sufficient statistic for investors’ optimal trading behavior. We name \( \theta \) an investor’s \textit{excess risk exposure} because it is equal to the difference between the investor’s exposure to systematic risk, \( \eta + \psi^T a \), and the total systematic risk in the economy at large, \( \psi^T E [a'] \). (15) and (16) provide us with explicit expression for the bilateral trade sizes and prices. One sees from (15) that the larger the difference between the bargaining parties’ excess risk exposures, the larger the trade size as their post-trade excess risk exposures are equalized. In addition, the larger the systematic risk exposure of the traded asset, the smaller the trade size. As expected, investors must exchange smaller quantity of the asset to equalize their excess risk exposures if per-unit systematic risk content of the asset is larger. Finally, (16) tells us that the bilateral trade price is equal to their post-trade marginal valuation, which equals the midpoint of investors’ initial marginal valuations due to symmetric bargaining powers.

3.2 Dynamics of the distribution of investors’ states

Proposition 1 shows that the excess risk exposure \( \theta \) is a sufficient statistic for investors’ equilibrium trading behavior. Furthermore, as mentioned above, the equilibrium value function \( V(\eta, a) \) depends only on two particular moments calculated from the equilibrium distribution: \( E[a'] \) and \( E[(\eta')^2 + 2\eta'\psi^T a' + (a')^T \Psi a'] \). The former is totally pinned down by the market-clearing conditions \( E[a'] = 0 \) and, accordingly, the latter is equal to \( E[(\theta')^2] \). Thus, determining the equilibrium dynamics of \( \theta \) is sufficient to analyze the investor’s optimal trading and their equilibrium value functions. Accordingly, what we do next is to calculate the distribution of \( \theta \) across investors instead of the joint distribution of their hedging need type \( \eta \) and their portfolios \( a \).

Lemma 2. Let \( \lambda = \sum_{k=1}^{J} \lambda_k \). The pdf \( g(\theta) \) of excess risk exposures satisfies the following Kolmogorov Forward Equation:

\[
\dot{g}(\theta) = \frac{1}{2}g''(\theta) \sigma^2_{\eta} + 4\lambda \int_{R} g(\theta') g(2\theta - \theta') d\theta' - 2\lambda g(\theta) \tag{17}
\]
for all $\theta \in \mathbb{R}$,

$$\int_{\mathbb{R}} g(\theta) \, d\theta = 1,$$

(18)

and

$$\int_{\mathbb{R}} \theta g(\theta) \, d\theta = 0.$$  

(19)

Equation (18) implies that $g(\theta)$ is a pdf. Equation (19) is implied by the market-clearing conditions and the fact that $\eta$ does not have a drift. Equation (17) has the usual inflow-outflow interpretation. The first term represents the net inflow due to the diffusion process that $\eta$ follows. The second and third terms represent the inflow and the outflow due to trading, respectively. The second term is a convolution integral because any investor of type $\theta'$ can become of type $\theta$ following a trade with the “right” counterparty. It is easy to see from Proposition 1 that $\theta' + \psi_j q_j (\theta', 2\theta - \theta') = \theta$, and hence, the right counterparty in this context is a counterparty of type $2\theta - \theta'$. Since the convolution integral complicates the computation of the pdf, we will make use of the characteristic function (Lukacs, 1970, p. 5):\(^9\)

$$\hat{g}(z) = \int_{\mathbb{R}} e^{iz\theta} g(\theta) \, d\theta.$$  

Proposition 2. Let $\lambda = \sum_{k=1}^{J} \lambda_k$ and let $\hat{g}(\cdot)$ be the characteristic function of the equilibrium pdf $g(\cdot)$ of excess risk exposures. Then, $\hat{g}(\cdot)$ satisfies the system

$$\dot{\hat{g}}(z) = -\left( \frac{1}{2} \sigma^2 \eta^2 z^2 + 2\lambda \right) \hat{g}(z) + 2\lambda \left[ \hat{g}\left(\frac{z}{2}\right) \right]^2$$

(20)

for all $z \in \mathbb{R}$,

$$\hat{g}(0) = 1,$$

(21)

and

$$\frac{d}{dz} \hat{g}(0) = 0.$$  

(22)

At steady state, the characteristic function has the following explicit expression:

$$\hat{g}(z) = \prod_{k=0}^{\infty} \left( \frac{1}{1 + \frac{\sigma^2 \eta^2 z^2}{4k+1} \lambda} \right)^{2^k}.$$  

(23)

\(^9\)Duffie and Manso (2007), Praz (2014), Üslü (2019), and Andrei and Cujean (2017), among others, also made use of characteristic functions or Fourier transforms to deal with the convolution integral in the context of search and matching models.
From (23), one sees that as $\frac{\sigma}{\sqrt{\lambda}}$ goes to zero, $\hat{g}(z)$ approaches 1, which is the characteristic function of the degenerate distribution with the mass point at $\theta = 0$. This degenerate distribution would obtain if investors were to trade in a continuous Walrasian market. Thus, $\frac{\sigma}{\sqrt{\lambda}}$ can be understood as a measure of misallocation resulting from the frictional structure of OTC trading. Indeed, if $\sigma$ is larger, this means that at any instant the exogenous stochastic process of $\eta$ makes $\eta$s more dispersed in the cross section, which in turn leads to a larger cross-sectional dispersion of investors’ excess risk exposures, $\theta$. On the other hand, if $\lambda$ is larger, investors have more frequent opportunities to make their $\theta$s closer together, which implies a lower dispersion of $\theta$s. In the limit as $\lambda$ goes to infinity, investors enjoy infinitely frequent opportunities to make their $\theta$s closer together so they successfully equalize them at $\theta = 0$, which coincides with the frictionless benchmark allocation.

Using the system (20)-(22), together with $\dot{\hat{g}}(z) = 0$, it is possible to derive recursively all moments of the stationary excess risk exposure distribution (Lukacs, 1970, p. 21):

$$E[\theta^n] = i^{-n} \left[ \frac{d^n}{dz^n} \hat{g}(z) \right]_{z=0}. \quad (24)$$

Using this technique, we derive in closed form proxies for important dimensions of market (il)liquidity including price dispersion, price impact, and sharp bounds for trading volume.

4 Results

In this section, we derive certain endogenous equilibrium objects that are related to market liquidity and have direct counterparts easily calculated from transaction-level data.

4.1 Trade volume

In the previous section, we have established that, as a result of search frictions, there is a non-degenerate distribution of investors’ cross-sectional excess risk exposures. According to our bilateral matching protocol, there is a measure $\lambda_j$ meetings among these investors at any instance in market $j$, in which each pair of investors bilaterally equalize their excess risk exposures by trading the quantity (15) stated in Proposition 1. Thus, instantaneous aggregate trading volume in market $j$ can be calculated as

$$V_j = \lambda_j \int_{\mathbb{R}} \int_{\mathbb{R}} |q_j(\theta, \theta')| g(\theta') g(\theta) d\theta' d\theta. \quad (25)$$

By using Proposition 1 and Proposition 2, we arrive at the following proposition, which provides us with a closed-form formula for equilibrium trade volume.
Proposition 3. Trade volume in market $j$ in the stationary equilibrium is

\[ V_j = \frac{\lambda_j}{|\psi_j|} \frac{1}{\pi} \int_{\mathbb{R}_+} \frac{1}{z^2} \left[ 1 - \prod_{k=0}^{\infty} \left( 1 + \frac{1}{\frac{\sigma^2}{\lambda^2} + 1} \frac{\sigma^2}{\lambda^2} z^2 \right)^{2k+1} \right] dz. \]  

Equation (26) shows that four factors, $\sigma_\eta$, $\lambda$, $\lambda_j$, and $|\psi_j|$, together determine the trade volume in market $j$. The integral term is a measure of equilibrium misallocation, which is increasing in $\frac{\sigma^2}{\lambda}$, i.e., how intensely investors’ hedging need changes relative to how frequently they can trade in some market. The rate of meetings in market $j$, $\lambda_j$, has two opposing effects on trading volume. First, it has a direct positive impact. Second, it has an indirect negative impact through $\lambda$, i.e., as $\lambda_j$ increases, the misallocation decreases and this depresses the trade volume. However, the former effect dominates, and $\lambda_j$ correlates positively with trading volume in market $j$. The systematic risk of asset $j$, $|\psi_j|$, has a negative impact on trade volume by decreasing the individual trade sizes. Indeed, trading an asset with large systematic risk leads to a large movement of the excess risk exposures, and hence, investors trade these assets in smaller quantities while trying to equalize their excess risk exposures through bilateral trade.

Equation (27) provides intuitive and empirically accurate cross-market comparative statics. An increase in $|\psi_A|$, for example, leads to a decline in the relative volume in market $A$, which is consistent with flight-to-quality observations in OTC markets. An increase in $\lambda_A$, on the other hand, leads to an increase in the relative volume in market $A$, which is consistent with flight-to-liquidity observations in OTC markets. Thus, one virtue of our model is to demonstrate how the cross section of trade volume is determined jointly by arbitrary combinations of asset quality and liquidity and also to shed light on some of the puzzling evidence documented in the empirical literature. For example, from (27), we see that if the asset quality and liquidity are negatively correlated and if the asset liquidity is more cross-sectionally dispersed, it would be possible to observe apparent flees from quality, as Beber, Brandt, and Kavajecz (2009) document in the Euro-area government bond market, which features a unique negative correlation between credit quality and liquidity across countries.

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11See, for example, Longstaff (2004), Beber, Brandt, and Kavajecz (2009), Ben-Rephael (2017), and Rzeźnik (2017).
Although (26) is an explicit expression for trade volume, it is not straightforward to study its limiting properties, especially for $\lambda_j$, because the integral cannot be computed exactly. To overcome this difficulty, we calculate sharp lower and upper bounds for trade volume using results from the recent probability theory literature.

**Proposition 4.** Trade volume in market $j$ in the stationary equilibrium satisfies the following inequalities:

\[
\frac{1}{4} \sqrt{\frac{7}{3}} \frac{\lambda_j}{|\psi_j|} \frac{\sigma_\eta}{\sqrt{\lambda}} \leq V_j \leq \frac{2}{\pi} \frac{\lambda_j}{|\psi_j|} \frac{\sigma_\eta}{\sqrt{\lambda}}.
\]

(28)

Thus,

\[
\lim_{\lambda_j \to \infty} V_j = \lim_{|\psi_j| \to 0} V_j = \lim_{\sigma_\eta \to \infty} V_j = \infty,
\]

\[
\lim_{\lambda_j \to 0} V_j = \lim_{|\psi_j| \to \infty} V_j = \lim_{\sigma_\eta \to 0} V_j = 0,
\]

and

\[
\lim_{\lambda_k \to \infty} V_j = 0
\]

for all $k \in J$ such that $k \neq j$.

Proposition 28 gives us interesting limiting results. As the systematic risk of asset $j$ approaches zero, investors trade it in increasingly larger quantities to equalize their excess risk exposures, and hence, the trading volume of asset $j$ approaches infinity. Vice-versa, as the systematic risk of asset $j$ approaches infinity, its trading volume approaches zero because trading even a small quantity leads to a large change in investors’ excess risk exposures. More interestingly, the effect of $\lambda_j$ and $\lambda_k$ for $k \neq j$ on trade volume in market $j$ in the limit are the opposite. As $\lambda_k$ for $k \neq j$ approaches infinity, investors’ valuations approach the frictionless benchmark valuations and the distribution of excess risk exposures approaches the degenerate distribution in which there is no misallocation. As a result, investors do not trade asset $j$ in the limit. If $\lambda_j$ approaches infinity, the same effect is observed on the equilibrium level of misallocation. However, this does not dry up the trading in market $j$. On the contrary, the reason why investors can achieve the degenerate distribution of excess risk exposures, they trade in market $j$ with infinite intensity, which implies that trading volume in market $j$ goes to infinity while volume in all other markets go to zero. This is an interesting cross-market implication of decline of search frictions in one market that would not obtain in single-asset OTC models. This is one of the implications of our model that we test in Section 5.
4.2 Price dispersion

As is typical in this class of models, different investor pairs trade at different prices because the lack of immediate access to a competing counterparty is reflected as a discount or premium in the bilaterally negotiated prices. Thus, the law of one price does not apply in the frictional OTC market equilibrium. An interesting equilibrium object to calculate is the price dispersion. As the measure of price dispersion, we calculate in closed form the cross-sectional standard deviation $\sigma_P$ of the equilibrium price distribution.

**Proposition 5.** *Price dispersion in market $j$ measured by the standard deviation of the stationary equilibrium price distribution is*

$$\sigma_{P_j} = \frac{1}{\sqrt{2}} \frac{r \gamma \sigma^2 |\psi_j|}{r + \lambda/2} \frac{\sigma_\eta}{\sqrt{\lambda}}. \quad (29)$$

*The price dispersion in market $A$ relative to the price dispersion in market $B$ is*

$$\frac{\sigma_{P_A}}{\sigma_{P_B}} = \frac{|\psi_A|}{|\psi_B|}. \quad (30)$$

One advantage of our model relative to the models that restrict investors asset positions to $\{0, 1\}$ such as Hugonnier et al. (2014) and Shen et al. (2018) is the following. In those models, the standard deviation of price is not available in closed form, but the difference between the maximum and the minimum price. From an econometric point of view, one would like a measure that takes into account the distributional effect, i.e. trades that are more likely to happen should have higher weight than trades that are less likely, in the calculation of price dispersion. Our price dispersion measure (29) takes into account the impact of this distributional consideration.

Our price dispersion measure (29) is the product of two factors. The first factor captures the sensitivity of transaction prices in market $j$ to investors’ excess risk exposures, which decreases with $\lambda$. The fact that $\lambda$ is finite is the reason why there is a deviation from the law of one price. The second factor, common with the trade volume (28), captures the misallocation. An increase in $\lambda$ reduces the equilibrium level of misallocation so investors’ marginal valuations become less dispersed, so does price dispersion.

The relative price dispersion measure (30) provides very interesting cross-market comparative statics. Intuitively, an increase in the systematic risk $|\psi_A|$ of asset $A$ increases the relative price dispersion in market $A$ because the price of asset $A$ becomes more sensitive to excess risk exposures when it contains more systematic risk. More surprisingly, an increase in the liquidity $\lambda_A$ does not affect the relative price dispersion in any market. This is an interesting result.
that could not be obtained in the comparative statics of single-asset models. In a single-asset model typically an increase in $\lambda_A$ will lead to a decline in price dispersion because distortions on extensive and intensive margins alleviate. Here, these effects are present as well, but the difference is that an increase in $\lambda_A$ leads to a decline in the price dispersion in both market $A$ and market $B$ by reducing misallocation and by reducing the sensitivity of prices to excess risk exposures. This happens because when trading in market $B$, an investor takes into account how her position in asset $A$ can expose her to the risk of being stuck with a suboptimal portfolio due to the search frictions in market $A$, and vice-versa. However, when we look at the relative price dispersion, we see that the effect of an increase in $\lambda_A$ work in the same way in both markets so the relative price dispersion stays unaffected.

4.3 Price impact

In search models, equilibrium price dispersion arises because investors with different marginal valuations bilaterally negotiate and then their valuation differentials translate into different realized prices. It is possible to interpret this as price impact due to illiquidity. Price impact arises for various reasons in market microstructure models such as strategic interaction\textsuperscript{12} or a combination of strategic interaction and adverse selection.\textsuperscript{13} In our model, it arises due to search frictions.

To understand the way we quantify the price impact in the equilibrium of our model, one must inspect the Nash bargained price (16). Using (14) and (15), one sees that in order to buy $q$ units of asset $j$ from a counterparty with current excess risk exposure of $\theta'$, an investor pays

$$P_j (q \mid \theta') = \frac{u^{(j)}(0, \mathbb{E}[a])}{r} - \frac{r\gamma\sigma^2\psi_j}{r + \lambda/2} (\theta' - \psi_j q).$$

As can be seen, the sensitivity of the transaction price to the traded quantity is

$$\left| \frac{\partial P_j (q \mid \theta')}{\partial q} \right| = \frac{r\gamma\sigma^2 |\psi_j|^2}{r + \lambda/2}.$$

The following proposition establishes that this sensitivity is equal to (a normalized version of) the ratio of price dispersion to quantity dispersion. Thus, we quantify the price impact in the cross-section of equilibrium trades as the ratio of price dispersion to quantity dispersion.

**Proposition 6.** Price impact in market $j$ in the stationary equilibrium is

$$\delta_j \equiv \frac{2\sigma_{P_j}}{\sigma_{q_j}} = \frac{r\gamma\sigma^2 |\psi_j|^2}{r + \lambda/2}. \quad (31)$$

\textsuperscript{12}See, for example, Vayanos (1999), Rostek and Weretka (2015), and Antill and Duffie (2018).
\textsuperscript{13}See, for example, Kyle (1985), Kyle (1989), Sannikov and Skrzypacz (2016), and Du and Zhu (2017).
The price impact in market $A$ relative to the price impact in market $B$ is

$$\frac{\delta_A}{\delta_B} = \left| \frac{\psi_A}{\psi_B} \right|^2. \quad (32)$$

The price impact (31) is calculated using the second moment of equilibrium price and quantity distributions, but the rationale behind it being a measure of price impact comes from the investor's problem. In particular, $\delta_j$ is equal to the half of (the absolute value of) the sensitivity of an investor's marginal valuation for asset $j$ to her position in asset $j$. Thus, on the margin, it measures how much the investor's marginal valuation changes as she changes infinitesimally her position in asset $j$. Equation (31) shows that price impact is present due to search frictions. As search frictions vanish (i.e., $\lambda_j \to 0$ for any $j$), $\delta_j$ goes to 0. Importantly, the price impact in one market is affected exactly the same way by the illiquidity of either markets. It is because investors use any asset to satisfy the same type of hedging need and if one market becomes more or less liquid, their reliance on that market adjust accordingly. In the end, what determines the price impact is the overall illiquidity of the markets rather than the illiquidity of an individual market. As a result, (32) shows that the relative price impact is affected only by systematic risks of the asset and not by their illiquidity.

An interesting comparative statics revealed by (32) is that as the systematic risk $|\psi_A|$ of asset $A$ increases, the relative price impact in market $A$ increases in a convex way. Convexity arises because the systematic risk increases both the price dispersion and the reciprocal of quantity dispersion linearly. Thus, it enters the relative price impact with an exponent of 2.

## 5 Testing the model’s implications

In this section, we empirically test the model’s implications in the US corporate bond market. Corporate bonds are traded over the counter with majority of these trades being purely bilateral and subject to significant search frictions.

### 5.1 Data

The data used in this study come from several sources. We obtain our bond transactions data from the enhanced version of Trade Reporting and Compliance Engine (TRACE), for the sample period from July 1, 2002 to June 30, 2017. TRACE dataset covers virtually all transactions of the US corporate bond market, and reports trade price, volume, buy/sell indicator, as well as the type of the counterparty (dealer vs. customer). We use the data filters proposed by
Dick-Nielsen (2014) to eliminate erroneous entries, reversals as well as canceled, corrected, and commissioned trades. We further remove the non-secondary market transactions and the transactions that are labeled as when-issued, have special sales conditions, or have more than five-day settlement.

We merge the cleaned TRACE data with Mergent Fixed Income Securities Database (FISD), to bring bond characteristics such as security type, offering amount, offering date, maturity date, and coupon rate. We eliminate bonds that are asset-backed, government-backed, agency-backed, or equity-linked, bonds that are putable, convertible, or part of unit deals and preferred shares, bonds with unusual coupons (floating rate, pay-in-kind, or split coupons) or are issued in non-USD currencies. We also remove the transactions that are priced below $5 or above $1,000 and the transactions that occur within less than one year remaining to maturity date. We next bring the macroeconomic indicators to our data, such as the GDP forecast dispersion, treasury rate, and volatility index (VIX), obtained from the Federal Reserve website.

After merging and cleaning the data, we calculate the liquidity measures and the predictors. Our objective is to construct the variables as close as possible to the variables in the theoretical model, while keeping in mind the properties of transaction-based data. Although the bond transactions data is intraday, most bonds do not trade daily. In addition, the cross section of bonds that are traded rapidly changes over time. If the liquidity measures were calculated at a high frequency (e.g. daily), we might lose the illiquid bonds from our sample. If we instead calculated the liquidity measures in a lower frequency (e.g. monthly), our econometric specification could be too sluggish to capture the sensitivity of liquidity to systematic risk and to the shifts in time-varying cross section. We therefore calculate our liquidity measures at a weekly level to capture the cross section of bonds both completely and dynamically.

We next construct our measures of liquidity, trade volume ($V_j$), price dispersion ($\sigma_{P_j}$), and price impact ($\delta_j$). These measures are directly based on the model and calculated for each bond-week. The predictors are similarly based on the model, calculated prior to the beginning of the same bond-week to avoid any time overlap between a dependent variable and a predictor.\footnote{For instance, we use the number of trades as a control for the variations in the liquidity measures due to firm-specific news events. Without making sure that the number of trades is calculated for a non-overlapping period, it could be problematic to use it as a predictor in regressions in which trade volume or price dispersion is the dependent variable. The detailed variable definitions and methodology of calculations are included in Appendix B.} We require non-missing observations for liquidity measures and the predictor variables to be included in our final sample. Our sampling procedure results in 3,565,689 bond-week observations of 26,583 bonds issued by 3,764 firms over the sample period from July 1,
Table 1: Descriptive statistics

This table presents the descriptive statistics of the sample used in this study. The sample period is from July 1, 2002 to June 30, 2017. The sample includes 26,583 bonds of 3,764 firms, and the observation unit is bond-week. The dependent variables, trade volume, price dispersion, and price impact, are calculated at weekly frequency for each bond. The predictors are calculated within the most recent quarter prior to beginning of the week. For the readily available time-series variables (e.g., VIX), we bring the most recent observation prior to beginning of the week. The table reports mean, standard deviation, 1st, 25th, 50th, 75th, and 99th percentile observations for each variable. Detailed variable definitions and sources of data are provided in Appendix B.

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<th>Mean</th>
<th>St. dev.</th>
<th>1st</th>
<th>25th</th>
<th>50th</th>
<th>75th</th>
<th>99th</th>
<th>Obs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trade volume ($mm)</td>
<td>11.57</td>
<td>34.10</td>
<td>0.01</td>
<td>0.24</td>
<td>2.03</td>
<td>10.50</td>
<td>128.78</td>
<td>3,565,689</td>
</tr>
<tr>
<td>Price dispersion</td>
<td>0.53</td>
<td>1.01</td>
<td>0.00</td>
<td>0.14</td>
<td>0.39</td>
<td>0.76</td>
<td>2.43</td>
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<tr>
<td>Price impact</td>
<td>13.47</td>
<td>61.07</td>
<td>0.00</td>
<td>0.12</td>
<td>0.73</td>
<td>6.94</td>
<td>171.08</td>
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</tr>
<tr>
<td>Offering amount ($bn)</td>
<td>0.67</td>
<td>3.46</td>
<td>0.01</td>
<td>0.25</td>
<td>0.45</td>
<td>0.75</td>
<td>3.00</td>
<td>3,565,689</td>
</tr>
<tr>
<td>Offering amt., other bonds ($bn)</td>
<td>3,838</td>
<td>1,359</td>
<td>1,853</td>
<td>2,475</td>
<td>3,763</td>
<td>4,864</td>
<td>6,246</td>
<td>3,565,689</td>
</tr>
<tr>
<td>Volatility beta</td>
<td>0.72</td>
<td>28.26</td>
<td>0.00</td>
<td>0.03</td>
<td>0.23</td>
<td>5.96</td>
<td>5.96</td>
<td>3,565,689</td>
</tr>
<tr>
<td>Average number of trades</td>
<td>22.19</td>
<td>40.17</td>
<td>1.42</td>
<td>5.50</td>
<td>10.42</td>
<td>23.00</td>
<td>180.50</td>
<td>3,565,689</td>
</tr>
<tr>
<td>GDP forecast dispersion</td>
<td>0.36</td>
<td>0.12</td>
<td>0.19</td>
<td>0.27</td>
<td>0.32</td>
<td>0.42</td>
<td>0.74</td>
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</tr>
<tr>
<td>Treasury rate, 1 mo. (%)</td>
<td>0.95</td>
<td>1.49</td>
<td>0.00</td>
<td>0.03</td>
<td>0.15</td>
<td>1.14</td>
<td>5.18</td>
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<tr>
<td>Volatility index</td>
<td>18.66</td>
<td>8.07</td>
<td>10.35</td>
<td>16.29</td>
<td>21.00</td>
<td>50.13</td>
<td>50.13</td>
<td>3,565,689</td>
</tr>
</tbody>
</table>

Table 1 presents the sample summary. The mean weekly trade volume is $11.57 million, and median trade volume is $2.03 million. Similarly, we see that the cross-sectional mean of the weekly average number of trades is 22.19, and median average number of trades is 10.42. Inspection of quartile observations reveals that the distributions of trade volume and average number of trades are right skewed. We similarly observe highly skewed distributions in other liquidity measures and several control variables.

5.2 Empirical results

In this section, we test the implications of our model for each endogenous liquidity measure. The theoretical results regarding the determinants of trade volume ($V_j$), price dispersion ($\sigma_{P_j}$), and price impact ($\delta_j$), from Equations (28), (29), and (31), respectively, are as follows:

\[
V_j \propto \frac{\lambda_j}{|\psi_j|} \frac{\sigma_\eta}{\sqrt{\lambda_j + \lambda_{-j}}},
\]

\[
\sigma_{P_j} = \frac{1}{\sqrt{2}} \frac{r \gamma \sigma^2 |\psi_j|}{r + (\lambda_j + \lambda_{-j})/2} \frac{\sigma_\eta}{\sqrt{\lambda_j + \lambda_{-j}}},
\]

\[
\delta_j = \frac{r \gamma \sigma^2 |\psi_j|^2}{r + (\lambda_j + \lambda_{-j})/2}.
\]

The theoretical model suggests a clear relationship for each pair of liquidity measure and predictor. For instance, it suggests that trade volume ($V_j$) of a particular bond $j$ is increasing.
Table 2: Determinants of liquidity in the cross section, baseline model

This table presents determinants of liquidity in the cross section of OTC-traded corporate bonds under a log-linear functional form assumption. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript \( j \) refers to bond \( j \), and the subscript \(-j\) refers to all other bonds except bond \( j \). Detailed variable definitions are provided in Appendix B. The standard errors are double clustered by bond and week, and the \( t \)-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10\%, 5\%, and 1\% levels, respectively.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \text{Trade volume (} V_j \text{)} )</td>
<td>( \text{Price dispersion (} \sigma_{P_j} \text{)} )</td>
<td>( \text{Price impact (} \delta_j \text{)} )</td>
</tr>
<tr>
<td>Offering amount ( (\lambda_j) )</td>
<td>1.017***</td>
<td>-0.225***</td>
<td>-1.209***</td>
</tr>
<tr>
<td></td>
<td>(120.58)</td>
<td>(-47.67)</td>
<td>(-106.88)</td>
</tr>
<tr>
<td>Offering amount, other bonds ( (\lambda_{-j}) )</td>
<td>85.222***</td>
<td>-14.275***</td>
<td>-97.425***</td>
</tr>
<tr>
<td></td>
<td>(4.24)</td>
<td>(-4.92)</td>
<td>(-4.33)</td>
</tr>
<tr>
<td>Volatility beta ( (\psi_j) )</td>
<td>-0.023***</td>
<td>0.064***</td>
<td>0.066***</td>
</tr>
<tr>
<td></td>
<td>(-6.81)</td>
<td>(30.83)</td>
<td>(14.05)</td>
</tr>
<tr>
<td>Average number of trades ( (\text{ANT}_j) )</td>
<td>0.599***</td>
<td>0.749***</td>
<td>1.056***</td>
</tr>
<tr>
<td></td>
<td>(83.45)</td>
<td>(128.19)</td>
<td>(95.56)</td>
</tr>
<tr>
<td>Year-week FE</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Observations</td>
<td>3,565,689</td>
<td>3,565,689</td>
<td>3,565,689</td>
</tr>
<tr>
<td>Adjusted ( R^2 )</td>
<td>0.512</td>
<td>0.223</td>
<td>0.214</td>
</tr>
</tbody>
</table>

with its offering amount \( (\lambda_j) \) and decreasing with volatility beta \( (\psi_j) \) and the total offering amount of other bonds \( (\lambda_{-j}) \).\(^\text{15}\) Therefore, these equations provide the expected coefficient signs for each predictor.

Importantly, Equations (28), (29), and (31) of liquidity measures all have multiplicative functional forms. In order to test their implications accurately, we take the natural logarithm of each variable.\(^\text{16}\) We thus run the linear regression specified below with the logged version of variables to better capture the true functional form of our theoretical equations.\(^\text{17}\) We start

\(^{15}\)Because asset-specific contact rates, \( \lambda_j \), are deep parameters that govern the exogenous liquidity differentials among the assets, one ideally needs to use a proxy that comes outside the sample from which the endogenous liquidity proxies are calculated. Considering this, we think offering amount is a sensible choice. Furthermore, there are theoretical and practical motivation for this choice. Theoretically, Weill (2008) show that assets with high offering amount end up having high endogenous contact rates when \textit{ex ante} identical agents allocate their search budget across assets. In practice, shorting a corporate bond is a complicated process which involves borrowing the bond before being able to short, while there are no short-sale costs or restrictions in our model. This may give an advantage to bonds with high offering amount in the market for borrowing corporate bonds. Thus, one may argue that interpreting the offering amount as a measure of \( \lambda_j \) makes sense in a model without short sale restrictions like ours because the offering amount has the role of alleviating frictions in practice, which is captured by \( \lambda_j \) in the model.

\(^{16}\)More specifically, we multiply each dependent variable and predictor with 100, add one, and then take its natural logarithm. Adding one is the standard practice in log-linear regressions when some observations have zero values. We multiply variables with 100 to make elasticity interpretations more accurate—if the variable’s values are very small then adding one distorts the elasticities.

\(^{17}\)Appendix Table C.2 presents our findings under linear functional form. Note the significant increase in the model fit for all specifications. The adjusted \( R^2 \)s increases from 0.135, 0.031, and 0.011 in Appendix Table C.2 to 0.486, 0.213, and 0.208 in corresponding Table 3, respectively.
our tests with a simple specification:

$$\text{Liquidity}_{j,t} = \beta_1 \lambda_{j,t} + \beta_2 \lambda_{-j,t} + \beta_3 \psi_{j,t} + \beta_4 \text{ANT}_{j,t} + \tau_t + \varepsilon_{j,t},$$

where “\text{Liquidity}_{j,t}” of bond \(j\) in week \(t\) denotes the liquidity measure; trade volume (\(V_j\)), price dispersion (\(\sigma_{P_j}\)), or price impact (\(\delta_j\)), and \(\tau_t\) denotes the time-specific intercepts for year-weeks. We run this regression separately for each measure.

Table 2 presents our findings under the baseline model. We directly control for the time-fixed effects in this table to isolate and focus on the cross-sectional relation of liquidity measures with the predictors. Column (1) shows our findings for trade volume (\(V_j\)). We find that trade volume increases with the offering amount of the bond (\(\lambda_j\)), as suggested by the theoretical model. Specifically, a percentage increase in offering amount of bond \(j\) leads to a percentage increase in trade volume of the same bond. Consistent with the theoretical results, we find that trade volume decreases with volatility beta (\(\psi_j\), sensitivity of bond volatility to aggregate volatility). The only failure of our test is the sign of the coefficient of the offering amount of other bonds (\(\lambda_{-j}\)). While our model predicts that the sign must be negative, Table 2 presents a positive coefficient. We suspect that this inconsistency might be because empirical time fixed effects are too strong in isolating the cross-sectional relation of liquidity measures with the predictors compared to the isolation power of the non-cross-sectional variables in the model. Indeed, when we use a model-informed set of time-varying variables in Table 3 instead of time fixed effects, the sign of the coefficient of \(\lambda_{-j}\) becomes consistent with the model.

In addition to our main predictors and time fixed effects, we also include the average number of trades on the bond to control for the variation in liquidity measures due to firm-specific news events. Although investors in our model trade only because of changes in their idiosyncratic hedging needs, the firm-specific news events absent in our model trigger trading activity in practice and this activity in turn affects bond’s liquidity measures (e.g., rating changes in Jankowitsch, Ottonello, and Subrahmanyam, 2018 and earnings announcements in Wei and Zhou, 2016). Consistent with earlier work, Table 2 presents positive and significant relation between this control variable and the liquidity measures.

In Column (2) of Table 2, we repeat our estimations for price dispersion (\(\sigma_{P_j}\)). We find that price dispersion is decreasing with offering amount of bond \(j\) (\(\lambda_j\)), and with the total offering amount of other bonds (\(\lambda_{-j}\)), exactly as predicted by the theoretical model. A percentage increase in offering amount of bond \(j\) leads to a 22.5 basis points decrease in price dispersion of the same bond. We further find that price dispersion is increasing with volatility beta (\(\psi_j\)), consistent with the model.
Table 3: Determinants of liquidity in the cross section

This table presents determinants of liquidity in the cross section of OTC-traded corporate bonds, using a log-linear specification that accurately reflects the multiplicative relation of predictors in the theoretical model. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript \( j \) refers to bond \( j \), and the subscript \(-j\) refers to all other bonds except bond \( j \). Detailed variable definitions are provided in Appendix B. The standard errors are double clustered by bond and week, and the \( t \)-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
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<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( V_j )</td>
<td>( \sigma_{P_j} )</td>
<td>( \delta_j )</td>
</tr>
<tr>
<td>Offering amount (( \lambda_j ))</td>
<td>0.990***</td>
<td>-0.224***</td>
<td>-1.186***</td>
</tr>
<tr>
<td></td>
<td>(97.03)</td>
<td>(-45.83)</td>
<td>(-92.28)</td>
</tr>
<tr>
<td>Offering amount, other bonds (( \lambda_{-j} ))</td>
<td>-0.198***</td>
<td>-0.009</td>
<td>0.443***</td>
</tr>
<tr>
<td></td>
<td>(-2.70)</td>
<td>(-0.29)</td>
<td>(12.63)</td>
</tr>
<tr>
<td>Volatility beta (( \psi_j ))</td>
<td>-0.020***</td>
<td>0.060***</td>
<td>0.060***</td>
</tr>
<tr>
<td></td>
<td>(-4.75)</td>
<td>(27.28)</td>
<td>(13.01)</td>
</tr>
<tr>
<td>Average number of trades (( \text{ANT}_j ))</td>
<td>0.592***</td>
<td>0.742***</td>
<td>1.049***</td>
</tr>
<tr>
<td></td>
<td>(74.76)</td>
<td>(128.55)</td>
<td>(91.11)</td>
</tr>
<tr>
<td>GDP forecast dispersion (( \sigma_{\eta} ))</td>
<td>0.089*</td>
<td>0.171***</td>
<td>0.163***</td>
</tr>
<tr>
<td></td>
<td>(1.93)</td>
<td>(8.16)</td>
<td>(5.20)</td>
</tr>
<tr>
<td>Treasury rate (( r ))</td>
<td>0.054***</td>
<td>0.030***</td>
<td>0.006</td>
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<tr>
<td></td>
<td>(3.90)</td>
<td>(5.39)</td>
<td>(0.87)</td>
</tr>
<tr>
<td>Volatility index (( \sigma ))</td>
<td>-0.077*</td>
<td>0.471***</td>
<td>0.657***</td>
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<tr>
<td></td>
<td>(-1.70)</td>
<td>(22.56)</td>
<td>(23.62)</td>
</tr>
<tr>
<td>Observations</td>
<td>3,565,689</td>
<td>3,565,689</td>
<td>3,565,689</td>
</tr>
<tr>
<td>Adjusted ( R^2 )</td>
<td>0.486</td>
<td>0.213</td>
<td>0.208</td>
</tr>
</tbody>
</table>

In Column (3) of Table 2, we estimate the specification for price impact (\( \delta_j \)). As predicted by the model, we find that price impact is decreasing with the offering amount of bond \( j \) \( (\lambda_j) \) as well as with the total offering amount of other bonds \( (\lambda_{-j}) \). A percentage increase in offering amount of bond \( j \) leads to approximately 1.2 percent decrease in price impact. Finally, consistent with the model, we find that price impact is increasing with volatility beta \( (\psi_j) \). Overall, our empirical results for the cross-sectional analysis in Table 2 are consistent with the theoretical model.

We next extend our cross-sectional analysis to a test that incorporates model-informed time-series factors. Instead of using time fixed-effects for each year-week, we directly include macroeconomic indicators. Specifically, we extend our regression equation to:

\[
\text{Liquidity}_{j,t} = \alpha + \beta_1 \lambda_{j,t} + \beta_2 \lambda_{-j,t} + \beta_3 \psi_{j,t} + \beta_4 \text{ANT}_{j,t} + \beta_5 \sigma_{\eta,t} + \beta_6 r_t + \beta_7 \sigma_t + \varepsilon_{j,t},
\]

where \( \sigma_{\eta,t} \), \( r_t \), and \( \sigma_t \) denote GDP forecast dispersion, treasury rate, and VIX volatility index, respectively. This allows the empirical specification to better correspond to the theoretical model in explaining the relation between macro time-series and our liquidity measures.

Table 3 presents our main results. Column (1) shows our findings for trade volume \( (V_j) \).
Similar to Table 2, we find here that trade volume decreases with volatility beta ($\psi_j$, sensitivity of bond volatility to aggregate volatility) as suggested by the theoretical model. Consistent with the theoretical results, trade volume increases with the offering amount of the bond ($\lambda_j$), while it decreases with the total offering amount of other bonds ($\lambda_{-j}$). Specifically, a percentage increase in the offering amount of bond $j$ leads to a 99 basis points increase in trade volume of the same bond. Similarly, a percentage increase in the total offering amount of other bonds ($-j$) leads to a 19.8 basis points decrease in trade volume of bond $j$. We further find that an increase in GDP forecast dispersion ($\sigma_\eta$) leads to an increase in trade volume, consistent with the theoretical results. Finally, we find a positive relation between trade volume and treasury rate ($r$) and a negative relation between trade volume and volatility index ($\sigma$). Our theoretical model does not suggest a sign in either direction for the relation between trade volume and these two variables.

In Column (2) of Table 3, we repeat our estimations for price dispersion ($\sigma_{P_j}$). Consistent with the theoretical model, we find that price dispersion is increasing with volatility beta ($\psi_j$). Specifically, a percentage increase in volatility beta leads to a 6 basis points increase in price dispersion. We again find negative relation between price dispersion and offering amount of bond $j$, similar to our baseline regression result in both economic and statistical significance. Furthermore, consistent with the model, we find that an increase in GDP forecast dispersion ($\sigma_\eta$) leads to an increase in price dispersion. Finally, we find a positive significant relation between price dispersion and treasury rate ($r$), as well as between price dispersion and volatility index ($\sigma$), both consistent with the model.

In Column (3) of Table 3, we estimate the specification for price impact ($\delta_j$). As predicted by the model, we indeed find a positive significant relation between price impact and volatility beta ($\psi_j$). A percentage increase in volatility beta leads to a 6 basis points increase in price impact. Price impact also increases with the offering amount of bond $j$. Finally, we find a positive significant relation between price impact and volatility index ($\sigma$), and we find an insignificant relation between price impact and treasury rate ($r$). Volatility index, in particular, has a strong relation with price impact, consistent with our model. A percentage increase in volatility index leads to 65.7 basis points increase in price impact.

Overall, our results from empirical tests of liquidity are mostly consistent with the implications of the theoretical model. We interpret this as pointing to the success and usefulness of the search-theoretic approach in uncovering the determinants of endogenous liquidity differentials across OTC assets, especially considering its ability to lead to parsimonious and tractable
models as exemplified by our theoretical model.

6 Intermediation chains and welfare

6.1 Length of intermediation chains

Note that, in our model, bilateral trade quantities reflect both parties’ trading needs. Indeed, the trade quantity (15) is determined such that the investor of type $\theta$ partially satisfies her own trading need and partially provides intermediation to her counterparty of type $\theta'$. As a result, the expression (26) for gross volume can be decomposed into two components: net volume and intermediation volume. This allows us to calculate the aggregate length of intermediation chains as

$$\frac{\text{intermediation volume}}{\text{net volume}} = \frac{\text{gross volume}}{\text{net volume}} - 1.$$

We already have an expression for gross volume at (26). Now we define the individual instantaneous expected net trade volume of an investor of type $\theta$ as:

$$\mathcal{N}V_j(\theta) = 2\lambda_j \left| \int_{\mathbb{R}} q_j(\theta, \theta') g(\theta') d\theta' \right|.$$

Aggregated over all investors,

$$\mathcal{N}V_j = \frac{1}{2} \int_{\mathbb{R}} \mathcal{N}V_j(\theta) g(\theta) d\theta = \lambda_j \left| \int_{\mathbb{R}} q_j(\theta, \theta') g(\theta') d\theta' \right| g(\theta) d\theta.$$

Using Proposition 1 and 2, we obtain the following proposition.

**Proposition 7.** Aggregate length of intermediation chains in market $j$ in the stationary equilibrium is

$$LIC_j \equiv \frac{\mathcal{N}V_j}{\mathcal{N}V_j} - 1 = \frac{\int_{\mathbb{R}^+} \hat{g}(z)[1-\hat{g}(z)]dz}{\int_{\mathbb{R}^+} \frac{1-\hat{g}(z)}{z^2}dz}.$$

Thus, two model parameters $\sigma_\eta$ and $\lambda$ alone determine the aggregate length of intermediation chains in any market:

$$\frac{\partial LIC_j}{\partial \sigma_\eta} < 0 \quad \text{and} \quad \frac{\partial LIC_j}{\partial \lambda} > 0.$$

Proposition 7 has important implications about the role of long intermediation chains in OTC markets. First, intermediation chains are longer if investors can trade very frequently but they do not have large trading needs. Thus, intermediation chains are longer if misallocation is
low. This shows us the efficiency enhancing role of long intermediation chains in OTC markets. Second, the aggregate length of intermediation chains is the same across all markets. Thus, it is an aggregate liquidity phenomenon.

6.2 Welfare cost of frictions

The next thing we do is to analyze the welfare loss due to OTC frictions. To this end, we will look at the difference between the welfare in a continuous frictionless Walrasian benchmark and the welfare in the OTC market equilibrium. Because $V(\eta, a)$ is a consumption-equivalent value function, following Gârleanu (2009), we obtain the social welfare in the OTC market equilibrium by aggregating all investors’ value functions:

$$W = \int \int_{\mathbb{R}^2} V(\eta, a) \Phi(d a, d \eta).$$

Proposition 1 shows that the continuation utilities are available in closed form and quadratic in excess risk exposures. Therefore, using the first two moments of the equilibrium excess risk exposure distribution from Proposition 2, the welfare can be derived in closed form, which allows us to arrive at the following proposition.

**Proposition 8.** In the stationary equilibrium, the deadweight loss caused by the frictions characteristic of OTC market structure is

$$DWL = -\frac{\gamma \sigma^2 \eta_2}{2\lambda}.$$

One immediate observation we make is that as $\lambda_j$ for any $j \in J$ approaches infinity, OTC frictions vanish and the social welfare approaches the frictionless welfare.

Proposition 8, together with Proposition 4, 5, and 6, allows us to evaluate the extent to which various liquidity measures reflect the welfare loss from illiquidity. One easily sees from Proposition 4 that as $\lambda_j$ approaches infinity, trading volume in market $j$ approaches infinity with the speed of convergence of $\sqrt{\lambda_j}$ and trading volume in other markets approaches zero with the speed of convergence of $1/\sqrt{\lambda_j}$. However, the speed of convergence of social welfare to the frictionless welfare is $1/\lambda_j$. Thus, trading volume cannot be an accurate measure of welfare loss created by frictions.

Propositions 5 and 6 imply that as $\lambda_j$ approaches infinity, both price dispersion and price impact vanishes with the speeds of convergence of $1/(\lambda_j \sqrt{\lambda_j})$ and $1/\lambda_j$, respectively. Consistent with the findings of Hugonnier et al. (2014), price dispersion vanishes faster than the
welfare loss, and hence, it cannot be used as a measure of welfare loss. However, we find that our price impact measure, which is the ratio of price dispersion to quantity dispersion, can serve as a good measure of welfare loss. Note that allowing investors to bargain over trade quantities is an important feature of our model that allows us to derive price impact. In the model of Hugonnier et al. (2014) with the \{0, 1\} restriction on asset positions, the quantity distribution over investors’ trades is 1 and −1 with equal probability, which results in a constant quantity dispersion. Thus, in their model, price dispersion and price impact vanish at the same speed.

7 Conclusion

We develop a search-theoretic model to study the impact of heterogeneity in asset characteristics on their endogenous liquidity differentials. Thanks to the tractability of our model, we derive natural theoretical counterparts for various measures of market liquidity easily calculated from transaction-level data. We find that the alleviation of search frictions in one market may lead to opposite observations regarding liquidity in other markets depending on which liquidity measure is used. We argue that, among the liquidity measures we analyze, price impact can serve as a good measure to quantify welfare loss caused by OTC frictions. Our empirical tests indicate significant support for the search-and-bargaining framework in uncovering the determinants of endogenous liquidity differentials across OTC assets.

References


A Proofs

A.1 Proof of Proposition 1

Conjecture

\[ V (\eta, a) = D + E^T a + F (\eta^2 + 2\eta \psi^T a + a^T \Psi a) + M \eta \]  

(34)

for \( D, E, F, \) and \( M \) to be determined. Take the derivative with respect to \( a \):

\[ \frac{\partial V}{\partial a} (\eta, a) = E + 2F (\eta \psi + \Psi a) . \]

The marginal valuation for asset \( j \) is, then,

\[ V^j (\eta, a) = E_j + 2F \psi_j \left( \eta + \sum_{k=1}^{J} \psi_k a_k \right) . \]

Using the FOC (6) for Nash bargaining,

\[ q_j [(\eta, a), (\eta', a')] = \frac{\eta' - \eta + \sum_{k=1}^{J} \psi_k (a'_k - a_k)}{2 \psi_j} . \]  

(35)

\[ (34) \text{ implies} \]

\[ V (\eta, a_j + q_j (\mu, \mu'), a_{-j}) - V (\eta, a_j, a_{-j}) + V (\eta', a'_j - q_j (\mu, \mu'), a'_{-j}) - V (\eta', a'_j, a'_{-j}) \]

\[ = -2qF \psi_j \left[ \eta' - \eta - \psi_j q + \sum_{k=1}^{J} \psi_k (a'_k - a_k) \right] . \]

Using (35),

\[ V (\eta, a_j + q_j (\mu, \mu'), a_{-j}) - V (\eta, a_j, a_{-j}) + V (\eta', a'_j - q_j (\mu, \mu'), a'_{-j}) - V (\eta', a'_j, a'_{-j}) \]

\[ = -\frac{1}{2} F \left[ (\eta' + \psi^T a')^2 - 2 (\eta' + \psi^T a') (\eta + \psi^T a) + (\eta + \psi^T a)^2 \right] \]

\[ = -\frac{1}{2} F \left[ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' - 2 (\eta' + \psi^T a') (\eta + \psi^T a) + \eta^2 + 2\eta \psi^T a + a^T \Psi a \right] . \]
Then, we are ready to set up the equation that will determine the undetermined coefficients using the HJB (10):

\[ r \left[ D + E^T a + F (\eta^2 + 2\eta \psi^T a + a^T \Psi a) + M \eta \right] = m^T a - \frac{1}{2} r \gamma \sigma^2 (\eta^2 + 2\eta \psi^T a + a^T \Psi a) + \sigma^2 a^T F \\
- \frac{1}{2} \lambda F \int \int_{\mathbb{R} \times \mathbb{R}} \left[ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' \right] (\eta + \psi^T a) \\
+ \eta^2 + 2\eta \psi^T a + a^T \Psi a \right] \Phi (da', d\eta'). \]

Letting \( E [\cdot] \) denote the cross-sectional mean and noticing that \( E [\eta'] = 0, \)

\[ r \left[ D + E^T a + F (\eta^2 + 2\eta \psi^T a + a^T \Psi a) + M \eta \right] = m^T a - \frac{1}{2} r \gamma \sigma^2 (\eta^2 + 2\eta \psi^T a + a^T \Psi a) + \sigma^2 a^T F \\
- \frac{1}{2} \lambda F \left\{ E \left[ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' \right] - 2 (\eta + \psi^T a) \psi^T E [a'] + \eta^2 + 2\eta \psi^T a + a^T \Psi a \right\}. \]

Thus, the coefficients solve

\[
\begin{align*}
rD &= \sigma^2 F - \frac{1}{2} \lambda F E [ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' ] \\
rE &= m + \lambda F \psi^T E [a'] \psi \\
rF &= -\frac{1}{2} r \gamma \sigma^2 - \frac{1}{2} \lambda F \\
rM &= \lambda F \psi^T E [a'],
\end{align*}
\]

which implies that

\[
\begin{align*}
D &= \frac{\gamma \sigma^2}{2r + \lambda} \left( -\sigma^2 + \frac{\lambda}{2} E [ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' ] \right) \\
E &= \frac{1}{r} m - \lambda \frac{\gamma \sigma^2}{2r + \lambda} \psi^T E [a'] \psi \\
F &= -\frac{r \gamma \sigma^2}{2r + \lambda} \\
M &= -\lambda \frac{\gamma \sigma^2}{2r + \lambda} \psi^T E [a'].
\end{align*}
\]

Putting together,

\[
\begin{align*}
V (\eta, a) &= \frac{\gamma \sigma^2}{2r + \lambda} \left( -\sigma^2 + \frac{\lambda}{2} E [ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' ] \right) \\
&\quad + \left( \frac{1}{r} m - \lambda \frac{\gamma \sigma^2}{2r + \lambda} \psi^T E [a'] \psi \right)^T a - \frac{r \gamma \sigma^2}{2r + \lambda} (\eta^2 + 2\eta \psi^T a + a^T \Psi a) - \lambda \frac{\gamma \sigma^2}{2r + \lambda} \psi^T E [a'] \eta.
\end{align*}
\]
which is Equation (13) of Proposition 1. By taking the derivative with respect to $a$, one obtains (14). (35) is equal to (15). Substituting into the Nash bargaining price (7), one obtains (16). Since $V(\eta, \cdot)$ stated above is negative definite for all $\eta \in \mathbb{R}$, (15) and (16) constitute the unique solution to the Nash bargaining problem. By construction, $V(\eta, a)$ given by (13) is the unique quadratic solution to the HJB equation (10).

### A.2 Proof of Lemma 2

The dynamics of the composite type $\theta$ for a given investor $i$ is

$$d\theta_t = \sigma_\eta dB^i_t + \sum_{j=1}^{J} \left[ \theta_{t-} + q_j \left( \theta_{t-}, \theta'_t \right) \psi_j \right] dN^j_t - \sum_{j=1}^{J} \theta_{t-} dN^j_t,$$

where $N^j$ is an independent Poisson process with jump intensity $2\lambda_j$ for $j \in \mathcal{J}$ and $\theta'_t$, the counterparty’s composite type, is a random draw from the pdf $g(t, \theta')$.

Define

$$H(t, \theta_0, \theta) \equiv \Pr \left[ \theta_t \leq \theta \mid \theta_0 \right]$$

and

$$h(t, \theta_0, \theta) \equiv \frac{\partial}{\partial \theta} H(t, \theta_0, \theta).$$

In equilibrium, the dynamics of the cross-sectional pdf of composite types, $g(t, \theta)$, is generated by (36):

$$g(t+s, \theta) = \int_{\mathbb{R}} g(t, \xi) h(s, \xi, \theta) \, d\xi.$$ 

It follows that for any $s > 0$,

$$\frac{1}{s} [g(t+s, \theta) - g(t, \theta)] = \frac{1}{s} \int_{\mathbb{R}} [g(t, \xi) - g(t, \theta)] h(s, \xi, \theta) \, d\xi.$$ 

Taking the limit in (37) as $s \to 0$ and applying the Ito’s lemma for Lévy processes on the RHS leads to

$$\frac{\partial g(t, \theta)}{\partial t} = \frac{1}{2} \sigma^2_\eta \frac{\partial^2 g(t, \theta)}{\partial \theta^2} + \sum_{j=1}^{J} 2\lambda_j \left[ \frac{\partial}{\partial \theta} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{q_j(\tilde{\theta}, \theta') \psi_j \leq \theta - \tilde{\theta} \}} g(t, \theta') g(t, \tilde{\theta}) \, d\theta' d\tilde{\theta} \right] - \sum_{j=1}^{J} 2\lambda_j g(t, \theta).$$
This second-order partial differential equation (PDE) satisfied by the densities at dates \( t > 0 \) generated by Lévy processes is called the \textit{Kolmogorov forward equation}.\footnote{For a reference, see Guttorp (1995, p. 133) or Stokey (2009, p. 50).}

Since (15) provides us with explicit expression for trade sizes, we can get rid of indicator function inside the integral:

\[
\frac{\partial g(t, \theta)}{\partial t} = \frac{1}{2} \sigma_{\eta}^2 \frac{\partial^2 g(t, \theta)}{\partial \theta^2} + \sum_{j=1}^{J} 2 \lambda_j \left[ \frac{\partial}{\partial \theta} \int_{-\infty}^{2 \theta - \theta'} g(t, \theta') \, d\theta' \right] - \sum_{j=1}^{J} 2 \lambda_j g(t, \theta).
\]

One can calculate the derivate inside the square bracket using Leibniz rule:

\[
\frac{\partial g(t, \theta)}{\partial t} = \frac{1}{2} \sigma_{\eta}^2 \frac{\partial^2 g(t, \theta)}{\partial \theta^2} + \sum_{j=1}^{J} 4 \lambda_j \left[ \int_{\mathbb{R}} g(t, \theta') \int_{\mathbb{R}} g(t, 2 \theta - \theta') \, d\theta' \right] - \sum_{j=1}^{J} 2 \lambda_j g(t, \theta).
\]

By defining \( \lambda \equiv \sum_{j=1}^{J} \lambda_j \) and suppressing \( ts \), one obtains Equation (17) of the lemma. Equation (18) obtains because \( g(\theta) \) is a pdf. Equation (19) is implied by the market-clearing conditions and the fact that \( \eta \) does not have a drift.

### A.3 Proof of Proposition 2

We first calculate the characteristic function of the second term on the RHS of (17):

\[
\int_{\mathbb{R}} 4 \lambda \left[ \int_{\mathbb{R}} g(\theta') g(2 \theta - \theta') \, d\theta' \right] e^{iz \theta} \, d\theta' = 4 \lambda \int_{\mathbb{R}} \left[ g(\theta') \int_{\mathbb{R}} g(2 \theta - \theta') e^{iz \theta} \, d\theta' \right] \frac{1}{2} e^{i z \theta'} \, d\theta' = 4 \lambda \int_{\mathbb{R}} \left[ g(\theta') \hat{g} \left( \frac{z}{2} \right) \right] \frac{1}{2} e^{i z \theta'} \, d\theta' = 2 \lambda \hat{g} \left( \frac{z}{2} \right) \int_{\mathbb{R}} g(\theta') e^{i z \theta'} \, d\theta' = 2 \lambda \left[ \hat{g} \left( \frac{z}{2} \right) \right]^2.
\]

That if \( \hat{g}(z) \) is the characteristic function of \( g(\theta) \), \((-iz)^n \hat{g}(z)\) is the characteristic function of \( \frac{\partial^n}{\partial \theta^n} g(\theta) \) implies that the characteristic function of the first term on the RHS of (17) is

\[
-\frac{1}{2} \sigma_{\eta}^2 z^2 \hat{g}(z).
\]

Putting together and using the linearity, differentiability, and integrability of the characteristic function, Equation (20) of the proposition obtains.
To obtain Equation (21) and (22), we apply the identities satisfied by all characteristic functions

\[ \hat{g}(0) = \int_{\mathbb{R}} g(\theta) \, d\theta \]

and

\[ \hat{g}'(0) = i \int_{\mathbb{R}} \theta g(\theta) \, d\theta \]

to Equation (18) and (19), respectively.

To derive the last equation of the proposition, note that, at steady state, (20) implies

\[ \hat{g}(z) = \frac{1}{1 + \sigma^2 \eta z^2} \left[ \hat{g} \left( \frac{z}{2} \right) \right]^2, \tag{38} \]

which also implies

\[ \hat{g} \left( \frac{z}{2} \right) = \frac{1}{1 + \sigma^2 \eta \left( \frac{z}{2} \right)^2} \left[ \hat{g} \left( \frac{z}{4} \right) \right]^2. \]

Substituting into (38),

\[ \hat{g}(z) = \frac{1}{1 + \sigma^2 \eta z^2} \left( \frac{1}{1 + \sigma^2 \eta z^2} \right)^2 \left( \frac{1}{1 + \sigma^2 \eta z^2} \right)^4 \left[ \hat{g} \left( \frac{z}{8} \right) \right]^8. \]

Evaluating (38) at \( \frac{z}{4} \) and substituting into the previous equality,

\[ \hat{g}(z) = \lim_{K \to \infty} \prod_{k=0}^{K} \left[ m(k, z) \right]^{2^k}, \tag{39} \]

where

\[ m(k, z) \equiv \frac{1}{1 + \sigma^2 \eta \left( \frac{z}{4^k} \right)^2}. \]

Note that \( m(k, \cdot) \) is the characteristic function of a Laplace distribution for all \( k \in \{0, 1, 2, \ldots\} \), which means it is an infinitely divisible characteristic function (Lukacs, 1970, p. 109). Then,
Corollary to Theorem 5.3.3 of Lukacs (1970) implies that \([m(k, \cdot)]^{2^k}\) is an infinitely divisible characteristic function as well for all \(k\) because \(2^k\) is a positive real number (p. 111). Theorem 5.3.2 of Lukacs (1970) states that the product of a finite number of infinitely divisible characteristic functions is an infinitely divisible characteristic function (p. 109). Thus,

\[
\prod_{k=0}^{K} [m(k, z)]^{2^k}
\]

is an infinitely divisible characteristic function. Then, from Theorem 5.3.3 of Lukacs (1970), the limit (39) is an infinitely divisible characteristic function because it is the limit of a sequence of infinitely divisible characteristic functions (p. 110). Finally, Theorem 5.3.1 of Lukacs (1970) implies that the RHS of (23) does not vanish because \(\hat{g}(z) \neq 0\) for all \(z \in \mathbb{R}\) holds for any infinitely divisible characteristic function (p. 108).

### A.4 Proof of Proposition 3

Substituting (15) into (25),

\[ V_j = \frac{\lambda_j}{2 |\psi_j|} \int \int_{\mathbb{R}} |\theta' - \theta| g(\theta') g(\theta) d\theta' d\theta. \]

Written in a more compact way,

\[ V_j = \frac{\lambda_j}{2 |\psi_j|} \mathbb{E} [||\theta' - \theta||]. \quad (40) \]

Thus, we need to calculate the first absolute moment of \(\theta' - \theta\). Note that the characteristic function of \(\theta' - \theta\) is \(\hat{g}(z) \hat{g}(-z)\) because \(\theta'\) and \(\theta\) are independently distributed due to random matching. Also, using the fact that \(\hat{g}(\cdot)\) is an even function, the characteristic function of \(\theta' - \theta\) is \([\hat{g}(z)]^2\).

Corollary 3.3 of Pinelis (2016) implies that

\[ \mathbb{E} [||\theta' - \theta||] = \frac{2}{\pi} \int_{0+}^{\infty} \frac{1 - [\hat{g}(z)]^2}{z^2} dz. \]

Substituting into (40) and using (23), one obtains Equation (26) of the proposition. It is straightforward to obtain (27) from (26).

### A.5 Proof of Proposition 4

In the probability theory literature, some sharper bounds for first absolute moments have recently been developed than usual Hölder-Lyapunov inequalities could provide. For the upper
bound, we use Theorem 6 of Ushakov (2011):
\[
\mathbb{E} \left[ | \theta' - \theta | \right] \leq \frac{4}{\pi} \sqrt{\text{var} [\theta]}
\]
(41)
because $\theta'$ and $\theta$ are independently distributed due to random matching. And, for the lower bound we use Corollary 2.3 of Berger (1997):
\[
\frac{\{ \mathbb{E} \left[ (\theta' - \theta)^2 \right] \}^{\frac{3}{2}}}{\{ \mathbb{E} \left[ (\theta' - \theta)^4 \right] \}^{\frac{1}{2}}} \leq \mathbb{E} \left[ | \theta' - \theta | \right].
\]
Again using the fact that $\theta$ and $\theta'$ are independently distributed and $\mathbb{E} [\theta] = 0$, this can be re-written as
\[
\frac{2 (\mathbb{E} [\theta^2])^{\frac{3}{2}}}{\{ \mathbb{E} [\theta^4] + 3 (\mathbb{E} [\theta^2])^2 \}^{\frac{1}{2}}} \leq \mathbb{E} \left[ | \theta' - \theta | \right].
\]
(42)
Thus, we need higher order usual moments of $\theta$ to be able to calculate the bounds for trade volume. Using (20) and (24) and equating $\dot{\hat{g}}(z) = 0$, one easily obtains
\[
\mathbb{E} [\theta^2] = \frac{\sigma_\eta^2}{\lambda}
\]
\[
\mathbb{E} [\theta^3] = 0
\]
\[
\mathbb{E} [\theta^4] = \frac{27 \sigma_\eta^4}{7 \lambda^2}.
\]
Substituting into (42) and (41),
\[
\frac{1}{2} \sqrt{\frac{7}{3}} \frac{\sigma_\eta}{\sqrt{\lambda}} \leq \mathbb{E} \left[ | \theta' - \theta | \right] \leq \frac{4}{\pi} \frac{\sigma_\eta}{\sqrt{\lambda}}.
\]
Combining with (40), one obtains Equation (28) of the proposition. Then, the limiting results follow by Squeeze Theorem.

A.6 Proof of Proposition 5

\[
\sigma_{P_j}^2 \equiv \int_{\mathbb{R}} \int_{\mathbb{R}} \left\{ P_j (\theta, \theta') - \mathbb{E} [P_j (\theta'', \theta''')] \right\}^2 g (\theta') g (\theta) d\theta' d\theta,
\]
where
\[
\mathbb{E} [p_j (\theta'', \theta''')] = \int_{\mathbb{R}} \int_{\mathbb{R}} P_j (\theta'', \theta''') g (\theta''') g (\theta'') d\theta''' d\theta'' = \frac{1}{r} \frac{\partial u}{\partial a_j} (0, \mathbb{E} [a'])
\]
The last equality follows from (16) and $\mathbb{E} [\theta] = 0$. Thus,
\[
\sigma_{P_j}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left\{ P_j (\theta, \theta') - \frac{1}{r} \frac{\partial u}{\partial a_j} (0, \mathbb{E} [a']) \right\}^2 g (\theta') g (\theta) d\theta' d\theta.
\]
Using (16),

\[
\sigma_{P_j}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( - \frac{r \gamma \sigma^2 \psi_j}{r + \lambda/2} \right)^2 g(\theta') g(\theta) \, d\theta' d\theta \\
= \left( \frac{1}{2} \frac{r \gamma \sigma^2 \psi_j}{r + \lambda/2} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} (\theta + \theta')^2 g(\theta') g(\theta) \, d\theta' d\theta \\
= 2 \left( \frac{1}{2} \frac{r \gamma \sigma^2 \psi_j}{r + \lambda/2} \right)^2 E[\theta^2].
\]

Using the second moment derived in the earlier proof A.5 and taking the square-root of both sides, Equation (29) of the proposition follows. It is straightforward to obtain (30) from (29).

### A.7 Proof of Proposition 6

\[
\sigma_{q_j}^2 \equiv \int_{\mathbb{R}} \int_{\mathbb{R}} \{ q_j(\theta, \theta') - E[q_j(\theta'', \theta''')] \}^2 g(\theta') g(\theta) \, d\theta' d\theta,
\]

where

\[
E[q_j(\theta'', \theta''')] = \int_{\mathbb{R}} \int_{\mathbb{R}} q_j(\theta'', \theta''') g(\theta''') g(\theta'') \, d\theta'' d\theta' = 0.
\]

The last equality follows from (15) and $E[\theta] = 0$. Thus,

\[
\sigma_{q_j}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} [q_j(\theta, \theta')]^2 g(\theta') g(\theta) \, d\theta' d\theta.
\]

Using (15),

\[
\sigma_{q_j}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \frac{\theta' - \theta}{2\psi_j} \right)^2 g(\theta') g(\theta) \, d\theta' d\theta \\
= \left( \frac{1}{2\psi_j} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} (\theta' - \theta)^2 g(\theta') g(\theta) \, d\theta' d\theta \\
= 2 \left( \frac{1}{2\psi_j} \right)^2 E[\theta^2].
\]

Using the second moment derived in the earlier proof A.5 and taking the square-root of both sides,

\[
\sigma_{q_j} = \sqrt{2} \frac{\sigma_n}{|\psi_j| \sqrt{\lambda}}.
\]

Combining with (29), one obtains Equation (31) of the proposition. It is straightforward to obtain (32) from (31).
A.8 Proof of Proposition 7

\[ NV_j = \lambda_j \int_{\mathbb{R}} \left| \int \frac{\theta' - \theta}{2\psi_j} g(\theta') \, d\theta' \right| g(\theta) \, d\theta \]

\[ = \lambda_j \int_{\mathbb{R}} \left| \int \frac{\theta' - \theta}{2\psi_j} g(\theta') \, d\theta' \right| g(\theta) \, d\theta \]

\[ = \frac{\lambda_j}{2|\psi_j|} \int |\theta| g(\theta) \, d\theta \]

\[ = \frac{\lambda_j}{2|\psi_j|} \mathbb{E} [ |\theta| ] . \]

Corollary 3.3 of Pinelis (2016) implies that

\[ \mathbb{E}_g [ |\theta| ] = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \hat{g}(z)}{z^2} \, dz . \]

Thus,

\[ NV_j = \frac{\lambda_j}{\pi |\psi_j|} \int_0^{\infty} \frac{1 - \hat{g}(z)}{z^2} \, dz . \]

Note from the earlier proof A.5 that

\[ V_j = \frac{\lambda_j}{\pi |\psi_j|} \int_0^{\infty} \frac{1 - [\hat{g}(z)]^2}{z^2} \, dz . \]

Thus,

\[ LIC_j = \frac{V_j}{NV_j} - 1 \]

\[ = \frac{\lambda_j}{\pi |\psi_j|} \int_0^{\infty} \frac{1 - [\hat{g}(z)]^2}{z^2} \, dz - 1 \]

\[ = \frac{\int_0^{\infty} \frac{1 - \hat{g}(z)}{z^2} \, dz}{0^{\infty}} - 1 \]

\[ = \frac{\int_0^{\infty} \frac{\hat{g}(z)[1 - \hat{g}(z)]}{z^2} \, dz}{0^{\infty}} , \]
which is equal to Equation (33) of the proposition.

Next, we derive the comparative statics results stated in the proposition. As an intermediate step, we define

$$m \equiv \frac{\sigma^2}{\lambda}.$$  

Then, we derive the comparative statics using

$$\frac{\partial \text{LIC}_j}{\partial \sigma_\eta} = \frac{\partial \text{LIC}_j}{\partial m} \frac{\partial m}{\partial \sigma_\eta} = \frac{2\sigma_\eta}{\lambda} \frac{\partial \text{LIC}_j}{\partial m}$$

and

$$\frac{\partial \text{LIC}_j}{\partial \lambda} = \frac{\partial \text{LIC}_j}{\partial m} \frac{\partial m}{\partial \lambda} = -\frac{\sigma^2}{\lambda^2} \frac{\partial \text{LIC}_j}{\partial m}.$$  

(33) implies

$$\frac{\partial \text{LIC}_j}{\partial m} = \left(\int_{0^+}^{\infty} \frac{\partial \hat{g}(z)}{\partial m} \frac{1 - \hat{g}(z)}{z^2} dz - \int_{0^+}^{\infty} \frac{\hat{g}(z) \partial \hat{g}(z)}{\partial m} dz\right) \int_{0^+}^{\infty} \frac{1 - \hat{g}(z)}{z^2} dz + \int_{0^+}^{\infty} \hat{g}(z) \frac{\partial \hat{g}(z)}{\partial m} dz \int_{0^+}^{\infty} \frac{1}{z^2} \frac{\partial \hat{g}(z)}{\partial m} dz$$

$$= \frac{\left(\int_{0^+}^{\infty} \frac{1 - \hat{g}(z)}{z^2} dz\right)^2}{\int_{0^+}^{\infty} \frac{1 - \hat{g}(z)}{z^2} dz}$$

$$\left[\int_{0^+}^{\infty} \frac{1 - \hat{g}(z)}{z^2} dz\right] < 0.$$  

The last inequality follows from the fact that \( \hat{g}(z) \in (0, 1] \) because \( \hat{g}(\cdot) \) is a non-vanishing characteristic function of a symmetric distribution and that

$$\frac{\partial \hat{g}(z)}{\partial m} = -\hat{g}(z) \sum_{k=0}^{\infty} \frac{z^2}{2^k} \frac{z^{2k}}{2^k} < 0.$$  

Thus,

$$\frac{\partial \text{LIC}_j}{\partial \sigma_\eta} < 0 \quad \text{and} \quad \frac{\partial \text{LIC}_j}{\partial \lambda} > 0.$$
A.9 Proof of Proposition 8

\[
\mathbb{W} = \int \int_{\mathbb{R} \times \mathbb{R}^J} \frac{\gamma \sigma^2}{2r + \lambda} \left( -\sigma^2 + \frac{\lambda}{2} \mathbb{E} \left[ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' \right] \right) - \frac{\lambda \gamma \sigma^2}{2r + \lambda} \psi^T \mathbb{E} [a'] \eta \\
+ \left( \frac{-m}{r} - \frac{\gamma \sigma^2}{2r + \lambda} \psi^T \mathbb{E} [a'] \psi \right)^T a - \frac{r \gamma \sigma^2}{2r + \lambda} \left( \eta^2 + 2\eta \psi^T a + a^T \Psi a \right) \Phi (da, d\eta) \\
= \frac{\gamma \sigma^2}{2r + \lambda} \left( -\sigma^2 + \frac{\lambda}{2} \mathbb{E} \left[ (\eta')^2 + 2\eta' \psi^T a' + (a')^T \Psi a' \right] \right) - \frac{\lambda \gamma \sigma^2}{2r + \lambda} \psi^T \mathbb{E} [a'] \mathbb{E} [\eta] \\
+ \left( \frac{-m}{r} - \frac{\gamma \sigma^2}{2r + \lambda} \psi^T \mathbb{E} [a'] \psi \right)^T \mathbb{E} [a] - \frac{r \gamma \sigma^2}{2r + \lambda} \mathbb{E} \left[ \eta^2 + 2\eta \psi^T a + a^T \Psi a \right],
\]

where the first equality follows from (13). Note that because (2) does not have a drift, \( \mathbb{E} [\eta] = 0 \).

And, in equilibrium,

\[
\mathbb{E} [a] = 0, \\
\mathbb{E} \left[ \eta^2 + 2\eta \psi^T a + a^T \Psi a \right] = \mathbb{E} \left[ (\theta + \psi^T \mathbb{E} [a])^2 \right].
\]

Thus,

\[
\mathbb{W} = \frac{\gamma \sigma^2}{2r + \lambda} \left( -\sigma^2 + \frac{\lambda}{2} \mathbb{E} [\theta^2] \right) - \frac{r \gamma \sigma^2}{2r + \lambda} \mathbb{E} [\theta^2].
\]

Using the second moment derived in the earlier proof A.5, one obtains

\[
\mathbb{W} = -\frac{\gamma \sigma^2 \sigma^2}{2\lambda}.
\]

What remains to show is that the welfare in the Walrasian benchmark is 0. To calculate the Walrasian welfare, we resort to the First Welfare Theorem, solve the problem of an unconstrained planner. Because we are in a transferable utility environment, the planner’s problem is:

\[
\mathcal{W}^W = \max \phi \frac{1}{r} \int \int_{\mathbb{R} \times \mathbb{R}^J} u(\eta, a) \phi (a, \eta) d\eta d\eta
\]

subject to

\[
\int \int_{\mathbb{R} \times \mathbb{R}^J} \phi (a, \eta) d\eta = 1, \\
\int \int_{\mathbb{R} \times \mathbb{R}^J} a_j \phi (a, \eta) d\eta = 0 \quad \text{for all} \ j \in J, \\
\phi (a, \eta) \geq 0 \quad \text{for all} \ (a, \eta) \in \mathbb{R}^{J+1}.
\]

48
Since the planner has a continuum of control variables as in Üslü (2019) and Farboodi et al. (2015), we follow these papers in appealing to van Imhoff (1982) and interpret the integrals in the objective function as summation over discrete intervals with both lengths $da$ and $d\eta$ approaching zero. Then, the standard Lagrangian techniques imply the following FOCs:

$$\frac{u(\eta, a)}{r} - \sum_{j=1}^{J} \mu_j a_j \leq 0 \quad \text{with equality if } \phi(a, \eta) > 0,$$

where $\mu_j$ is the Lagrange multiplier in front of the resource constraint on asset $j$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^J} a_j \phi(a, \eta) \, da \, d\eta = 0.$$

Substituting these FOCs to (43),

$$\mathbb{W}^W = \int_{\mathbb{R}} \int_{\mathbb{R}^J} \sum_{j=1}^{J} \mu_j a_j \phi(a, \eta) \, da \, d\eta.$$

Thus, the resource constraint implies that $\mathbb{W}^W = 0$ and the proof is complete.
# B Variable definitions

<table>
<thead>
<tr>
<th>Variable</th>
<th>Type</th>
<th>Description</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dependent variables</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trade volume ($V_j$)</td>
<td>$\text{mm}$</td>
<td>Weekly total of trade sizes of the bond, in million dollars.</td>
<td>TRACE</td>
</tr>
<tr>
<td>Price dispersion ($\sigma_{P_j}$)</td>
<td>Decimal</td>
<td>The square root of weekly second moment of demeaned bond prices. Demeaned prices for each bond-day are calculated as the difference between bond price and daily mean bond price. We follow Jankowitsch, Nashikkar, and Subrahmanyam (2011) and Feldhüter (2012) in our definition of price dispersion.</td>
<td>TRACE</td>
</tr>
<tr>
<td>Price impact ($\delta_j$)</td>
<td>Decimal</td>
<td>The ratio of weekly price dispersion to square root of weekly second moment of trade sizes of the bond, multiplied by two.</td>
<td>TRACE</td>
</tr>
<tr>
<td><strong>Predictors</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Offering amount, $\lambda_j$</td>
<td>$\text{bn}$</td>
<td>Offering amount of the bond $j$, in billion dollars.</td>
<td>FISD, TRACE</td>
</tr>
<tr>
<td>Offering amount $-j$, $\lambda_{-j}$</td>
<td>$\text{bn}$</td>
<td>Total offering amount of the other bonds, in billion dollars. Calculated as the summation of the offering amount of the unique bonds that have been traded other than bond $j$, over the quarter prior to the beginning of current week.</td>
<td>FISD, TRACE</td>
</tr>
<tr>
<td>Volatility beta ($\psi_j$)</td>
<td>Decimal</td>
<td>The sensitivity of weekly volatility of bond returns to VIX rate. For each bond, we first calculate the weekly standard deviation of daily returns, where daily returns are calculated based on trade size weighted average of bond prices (clean price + accrued interest). Then, we regress weekly volatility of bond returns on VIX rate, over the quarter prior to the beginning of current week. We then take the absolute value of the coefficient estimate of this regression, to more accurately test the theoretical model. This volatility beta, is the sensitivity measure of bond volatility to systematic volatility.</td>
<td>FISD, TRACE</td>
</tr>
<tr>
<td>Average number of trades</td>
<td>Decimal</td>
<td>The average of the weekly number of trades of bond $j$, calculated over the quarter prior to the beginning of current week.</td>
<td>TRACE</td>
</tr>
<tr>
<td>GDP forecast dispersion ($\sigma_\eta$)</td>
<td>Decimal</td>
<td>The difference between the 75$^{\text{th}}$ percentile and the 25$^{\text{th}}$ percentile of the one quarter ahead forecasts of real GDP growth.</td>
<td>FED</td>
</tr>
<tr>
<td>Treasury rate ($r$)</td>
<td>Pct.</td>
<td>One-month treasury bill rate.</td>
<td>FED</td>
</tr>
<tr>
<td>Volatility index ($\sigma$)</td>
<td>Decimal</td>
<td>Chicago Board of Options Exchange (CBOE) volatility index (VIX).</td>
<td>FED</td>
</tr>
</tbody>
</table>
## C Additional results

### C.1 Interdealer trades

**Table C.1:** Determinants of liquidity in the cross section, interdealer trades

This table presents our main findings when the sample of transactions are based on interdealer trades only. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript $j$ refers to bond $j$, and the subscript $-j$ refers to all other bonds except bond $j$. Detailed variable definitions are provided in Appendix B. The standard errors are double clustered by bond and week, and the $t$-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th></th>
<th>(1) Trade volume ($V_j$)</th>
<th>(2) Price dispersion ($\sigma_{P_j}$)</th>
<th>(3) Price impact ($\delta_j$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Offering amount ($\lambda_j$)</td>
<td>0.558***</td>
<td>-0.169***</td>
<td>-0.775***</td>
</tr>
<tr>
<td></td>
<td>(72.28)</td>
<td>(-36.86)</td>
<td>(-73.27)</td>
</tr>
<tr>
<td>Offering amount, other bonds ($\lambda_{-j}$)</td>
<td>-0.258***</td>
<td>-0.178***</td>
<td>0.315***</td>
</tr>
<tr>
<td></td>
<td>(-4.84)</td>
<td>(-7.37)</td>
<td>(8.61)</td>
</tr>
<tr>
<td>Volatility beta ($\psi_j$)</td>
<td>0.016***</td>
<td>0.094***</td>
<td>0.089***</td>
</tr>
<tr>
<td></td>
<td>(4.77)</td>
<td>(42.88)</td>
<td>(21.74)</td>
</tr>
<tr>
<td>Average number of trades ($\text{ANT}_j$)</td>
<td>0.792***</td>
<td>0.476***</td>
<td>0.474***</td>
</tr>
<tr>
<td></td>
<td>(107.67)</td>
<td>(108.73)</td>
<td>(53.69)</td>
</tr>
<tr>
<td>GDP forecast dispersion ($\sigma_{\eta}$)</td>
<td>0.008</td>
<td>0.200***</td>
<td>0.235***</td>
</tr>
<tr>
<td></td>
<td>(0.21)</td>
<td>(10.11)</td>
<td>(7.37)</td>
</tr>
<tr>
<td>Treasury rate ($r$)</td>
<td>0.042***</td>
<td>0.034***</td>
<td>0.011*</td>
</tr>
<tr>
<td></td>
<td>(4.21)</td>
<td>(7.90)</td>
<td>(1.74)</td>
</tr>
<tr>
<td>Volatility index ($\sigma$)</td>
<td>-0.013</td>
<td>0.518***</td>
<td>0.648***</td>
</tr>
<tr>
<td></td>
<td>(-0.37)</td>
<td>(28.82)</td>
<td>(22.87)</td>
</tr>
<tr>
<td>Observations</td>
<td>2,965,789</td>
<td>2,965,789</td>
<td>2,965,789</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.383</td>
<td>0.157</td>
<td>0.111</td>
</tr>
</tbody>
</table>
C.2 Linear functional form

Table C.2: Determinants of liquidity in the cross section, linear functional form

This table presents determinants of liquidity in the cross section of OTC-traded corporate bonds under a linear functional form assumption. The single-letter name of each variable, as used in the theoretical model, is provided in the parenthesis adjacent to the variable. The subscript $j$ refers to bond $j$, and the subscript $-j$ refers to all other bonds except bond $j$. Detailed variable definitions are provided in Appendix B. The standard errors are double clustered by bond and week, and the $t$-statistics are reported in parentheses. *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th></th>
<th>(1) Trade volume ($V_j$)</th>
<th>(2) Price dispersion ($\sigma_{P_j}$)</th>
<th>(3) Price impact ($\delta_j$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Offering amount ($\lambda_j$)</td>
<td>0.5034</td>
<td>-0.0035***</td>
<td>-0.8300*</td>
</tr>
<tr>
<td></td>
<td>(1.56)</td>
<td>(-2.40)</td>
<td>(-1.81)</td>
</tr>
<tr>
<td>Offering amount, other bonds ($\lambda_{-j}$)</td>
<td>0.0001</td>
<td>-0.0001***</td>
<td>-0.0056***</td>
</tr>
<tr>
<td></td>
<td>(0.67)</td>
<td>(-16.99)</td>
<td>(-21.04)</td>
</tr>
<tr>
<td>Volatility beta ($\psi_j$)</td>
<td>-0.0019**</td>
<td>-0.0000</td>
<td>0.0155*</td>
</tr>
<tr>
<td></td>
<td>(-2.21)</td>
<td>(-0.35)</td>
<td>(1.80)</td>
</tr>
<tr>
<td>Average number of trades ($\text{ANT}_j$)</td>
<td>0.3062***</td>
<td>0.0023***</td>
<td>-0.1521***</td>
</tr>
<tr>
<td></td>
<td>(18.78)</td>
<td>(22.06)</td>
<td>(-21.18)</td>
</tr>
<tr>
<td>GDP forecast dispersion ($\sigma_\eta$)</td>
<td>-0.2336</td>
<td>0.3155***</td>
<td>18.8553***</td>
</tr>
<tr>
<td></td>
<td>(-0.20)</td>
<td>(8.77)</td>
<td>(9.40)</td>
</tr>
<tr>
<td>Treasury rate ($r$)</td>
<td>0.5323***</td>
<td>-0.0063***</td>
<td>0.1443</td>
</tr>
<tr>
<td></td>
<td>(4.70)</td>
<td>(-2.63)</td>
<td>(0.70)</td>
</tr>
<tr>
<td>Volatility index ($\sigma$)</td>
<td>-0.0061</td>
<td>0.0100***</td>
<td>0.3895***</td>
</tr>
<tr>
<td></td>
<td>(-0.34)</td>
<td>(17.69)</td>
<td>(10.77)</td>
</tr>
<tr>
<td>Observations</td>
<td>3,565,689</td>
<td>3,565,689</td>
<td>3,565,689</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.135</td>
<td>0.031</td>
<td>0.011</td>
</tr>
</tbody>
</table>

In this section, we alternatively run the following regression assuming a linear functional form:

$$Liquidity_{j,t} = \alpha + \beta_1 \lambda_{j,t} + \beta_2 \lambda_{-j,t} + \beta_3 \psi_{j,t} + \beta_4 \text{ANT}_{j,t} + \beta_5 \sigma_{\eta,t} + \beta_6 r_t + \beta_7 \sigma_t + \epsilon_{j,t},$$

where “$Liquidity_{j,t}$” of bond $j$ in week $t$ denotes the liquidity measure; trade volume ($V_j$), price dispersion ($\sigma_{P_j}$), or price impact ($\delta_j$). We run this regression separately for each measure.

An important limitation of this naïve specification is that it does not account for the proportionality between the liquidity measures and the predictors as uncovered by our theoretical formulas. In other words, the specified linear functional form does not capture the multiplicative relation between the predictors in jointly determining the liquidity measures, and so, may be inconsistent with the actual data generating processes. Being mindful of this possibility, we present the results of this estimation in Table C.2. The adjusted $R^2$s are 0.135, 0.031, and 0.011 in Table C.2, very low in comparison to to 0.486, 0.213, and 0.208 of Table 3, respectively. Although the coefficients have similar signs in both tables, we base our main conclusions on Table 3, since it more accurately corresponds to the functional form of the theoretical model.