Abstract

Liquidity can evaporate quickly during market turmoil. This paper proposes a nonlinear rational expectations equilibrium model of high-frequency endogenous liquidity provision to understand better such fragile liquidity. With fast trading speed and private information, high-frequency traders can either compete with designated market makers (DMMs) by providing liquidity or attempt to profit from speculative trades that consume liquidity. Absent an obligation to provide liquidity, high-frequency traders can switch between supplying and consuming liquidity, depending on their beliefs. The risk from this endogenous liquidity provision, coupled with limits to participation by DMMs, intensifies the adverse selection faced by DMMs. This can generate a gap between liquidity supply from DMMs and liquidity demand by informed traders. As a result, endogenous liquidity provision produces fragile liquidity, with the possibility of market crashes when high-frequency traders switch from liquidity provision to liquidity consumption in extreme times.

JEL classification: G10, G14

Keywords: Endogenous liquidity provision, fragile liquidity, machine learning
1. Introduction

The May 2010 Flash Crash continues to reverberate through financial markets. It triggered heated debate about the relation between market crashes and high-frequency trading, in particular through endogenous liquidity provision. Anand and Venkataraman (2016) provide evidence that, without explicit obligations, investors who supply liquidity simultaneously scale back this activity when conditions are unfavorable. This contributes to the covariation in liquidity supply within and across stocks. Hence, a growing reliance on high frequency traders (HFTs) acting as liquidity providers has raised concerns with market regulators and investors. Will a possible withdrawal from liquidity provision by HFTs result in illiquidity that destabilizes markets? Will this liquidity withdrawal introduce and intensify market fragility during turbulent times?

We argue that HFTs’ endogenous liquidity provision and designated market maker’s (DMMs) limited market participation (e.g., due to frictions induced by trading automation) play important roles in liquidity attenuation, which may yield flash-crashes. Unlike DMMs, HFTs can either provide liquidity and compete with DMMs, or they may take liquidity and attempt to profit from speculative trades based on their private beliefs. With no obligation to supply liquidity, HFTs will switch between supplying and consuming liquidity. Apart from uncertain asset values, this additional risk of HFTs’ switching focus, coupled with limited participation of DMMs, intensifies the adverse selection faced by DMMs. Hence, DMMs may provide less liquidity, which generates a gap between DMMs’ liquidity supply and informed traders’ liquidity demand.

1 Key events include the May 6, 2010 U.S. “flash-crash” where U.S. equity indices dropped by 5-6% and recovered within half an hour; followed by the October 15, 2014 Treasury Bond crash, where the yield on the benchmark 10-year U.S. government bond, dipped 33 basis points to 1.86% and reversed to 2.13% by the end of the trading day; as well as the August 25, 2015 ETF market freeze, during which more than a fifth of all U.S.-listed exchange traded funds and products were forced to stop trading. Brogaard et al. (2017) use three approaches to identify extreme price movement. The first is straightforward and simply labels all intervals that belong to the 99.9th percentile of 10-second absolute midpoint returns for each stock as EPMs. The second method is more sophisticated and accounts for predictable return correlations through time and across firms. The third is the Lee and Mykland’s (2012) method that accounts for contemporaneous (local) volatility. Lee and Mykland (2012) identify 45,200 EMP during 2008 and 2009.

With highly profitable market making and no obligation to provide liquidity, HFTs have an incentive to either supply or demand liquidity, depending on their beliefs. While the literature generally finds that HFTs act as liquidity providers, there is evidence that HFTs pull back when market conditions become unfavourable (Raman, Robe and Yadav, 2014; Bessembinder et al, 2016; Anand and Venkataraman, 2016; and Korajczyk and Murphy, 2019).

2 “The issue is whether the firms that effectively act as market makers during normal times should have obligations to support the market in reasonable way in tough times”, Mary L. Shapiro, SEC chairman, the Economic Club of New York on September 7, 2010.
To explore these issues more formally, we use a two-dimensional structure of liquidity provision: based on orders from DMMs and orders from a high-frequency endogenous liquidity provider (ELP). In our setting the ELP is also an HFT – they have a technological advantage that allows them to decide whether to provide or consume liquidity based on their private information. In addition to the ELP and DMM there is also an informed trader. To characterize information asymmetry and learning, we consider multiple sources of randomness in the payoff to the risky asset. Specifically, the informed trader and ELP have both distinct and common information about the risky asset payoff.

Using this information structure, we build a nonlinear rational expectations equilibrium model of fragile liquidity with a high-frequency ELP. Due to the nonlinearity and complexity in the DMM’s posterior beliefs, we introduce a reinforcement learning approach to obtain nonlinear demand and price functions. This brings new evidence and insights of reinforcement learning techniques to theoretical microstructure research by synthesizing these two streams of literature. The price function when ELP consumes liquidity (acts as a liquidity taker) is concave with respect to aggregate order flows and has much higher slope than under the regime that ELP provides liquidity (acts as a liquidity maker). This implies that liquidity under the taker regime is more fragile (can change dramatically) than under the maker regime. In addition, liquidity under the taker regime deteriorates with more limited DMM’s market participation.

These results show that ELP’s endogenous liquidity provision, coupled with limited DMM participation (resulting from trading technology and trading automation) can shrink significantly the liquidity supply from the DMM, but expand aggregate liquidity demand during market turmoil. This induces strategic complementarities in liquidity mismatches: traders demand more liquidity when the aggregate liquidity supply falls. An ELP that normally provides liquidity now consumes liquidity, which can intensify the adverse selection faced by DMMs. This then serves as a conduit for a loss of liquidity and possible market crashes during market turmoil. That is, ELP propagates initial shocks, induces a liquidity mismatch and leads to an intensified loss of liquidity.4

This mechanism is illustrated in Figure 1. Without ELP trading, the DMM provides liquidity to informed and liquidity traders and the adverse selection faced by the DMM is measured by

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4 Our contribution is to consider an ELP that uses an endogenous cut-off strategy. That is, we model an ELP that provides liquidity when the private information innovation is relatively small (below a threshold value) but takes liquidity as a speculator when the information shock is relatively large (above the threshold value). The resulting equilibrium is characterized by two regimes: making or taking liquidity.
randomness of the asset payoff (the top panel). With ELP trading (the bottom panels), during
typical times (when only the informed trader consumes liquidity) both the DMM and ELP act as
liquidity providers (see the bottom left panel). However, if the ELP ceases to supply liquidity and
instead acts as a speculator, this will amplify the initial liquidity mismatch between DMM’s
liquidity provision and informed trader’s liquidity demand (the bottom right panel).

We further explore two particular channels for an ELP to affect the adverse selection faced by
the DMM. First, it is possible that developments in trading technology and trading automation
(such as supercomputers and co-location with the exchange) have intensified competition among
HFTs and led to a concentration of liquidity provision by a small number of HFT firms. This
concentration seems to limit DMM liquidity supply during typical times but increases order
toxicity during market turmoil. Therefore, it increases DMM’s adverse selection. To explore this
possibility, we amend the model structure removing endogenous liquidity provision (HFT liquidity
provision becomes exogenous) and focus on the influence of limited DMM participation. This
simplification results in a closed-form equilibrium where more limited DMM participation results
in higher adverse selection for the DMM and deteriorating liquidity.

Our second channel for an ELP to influence adverse selection is through ELP’s endogenous
liquidity provision decision. While exogenous selective liquidity does not provide additional
information to the DMM’s efforts to infer the fundamental value; endogenous liquidity provision
means there is increasing toxicity of the aggregate market order flow when the ELP consumes
liquidity. To understand how the ELP’s endogenous order choice affects the DMM’s adverse
selection, we consider a linear approximation to the equilibrium. Characterized by the making
intensity, this approximation shows that the DMM’s adverse selection increases with the ELP’s
optimal threshold value of endogenous liquidity provision. Comparing this finding with results
from the full model using reinforcement learning, it is apparent that the linear approximation fails
to capture important nonlinearities and overestimates liquidity available under the taker regime.
Most empirical framework focuses on this linear situation structure. However, our results show
that this approximation only provides partial insights. However, when market participants are
diversified across numerous risky assets and where market structure becomes more complicated,
being able to precisely characterize how innovative and comprehensive trading behaviours affect
price and liquidity dynamics becomes very important for investors and regulators. Hence, linear
framework can be helpful in capturing some general mechanisms but may fail to deliver results precisely.

Our economy with endogenous liquidity provision generates several implications for market regulators. First, market crashes can occur regardless of the value of the underlying asset. Without changing the fundamental value, shocks to ELP’s signals can trigger dramatic price changes. That is, the equilibrium price function is discontinuous and has a jump, either increasing or decreasing, with respect to the ELP’s signal when the ELP switches from being a liquidity maker to a liquidity taker. Second, as the information for the informed trader and the ELP becomes more similar, liquidity under the taker regime decreases while liquidity under the maker regime increases. This more severe liquidity break should raise the concern of market regulators and investors, especially as markets become more integrated. Lastly, as the ELP’s signal becomes nosier, liquidity under both taker and maker regimes increase. This suggests that there may be benefits to slower ELPs, echoing recent regulator concern about the superior trading capabilities of high-frequency traders.

1.1 Background

Empirical research on high-frequency trading is typically designed to capture key market characteristics and features models that embed speed or information-based frictions in more traditional settings (see Menkveld (2016) for an excellent survey on both theoretical and empirical research about HFT). For example, most of the theoretical literature assumes that liquidity providers are deterministic and have no superior information (as is the case in Kyle, 1985 and Glosten and Milgrom, 1985). More recent evidence, however, shows that certain informed traders, namely endogenous liquidity providers, use limit orders extensively. Our goal is to fill these gaps — e.g. endogenous liquidity provision expands liquidity but also builds up liquidity fragility. Our theoretical results provide a better foundation for understanding the effects of endogenous liquidity provision on overall market quality (see Brogaard et al., 2018; Kirilenko et al., 2017 and Christensen et al., 2014 for related evidence).

Our work is also related to models of financial crises (see Barlevy and Veronesi, 2003; Yuan, 2005; Huang and Wang, 2010; Cespa and Foucault, 2014) and financial contagion (see King and

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5 This growing literature focuses on three aspects: a) what information high-frequency traders trade on (Hu, Pan and Wang, 2017; Van Kerveland and Menkveld, 2017), b) how high-frequency trading incorporates information (O’Hara, Yao and Ye, 2014; Brogaard, Hendershott and Riordan, 2017), and c) what is the impact of high-frequency trading on market quality (Budish et al., 2015; Yang and Zhu, 2016 and Weller, 2016).
Our analysis shares some features with the models outlined in these papers. This includes features such as cross-market arbitrageurs who transmit liquidity shocks which results in order imbalances, market frictions (such as borrowing constraints or trading speed) that limit some traders’ participation. What is unique to our model is the feature whereby the ELP has no obligation for market making and submits different types of orders based on market conditions. I.e., show how liquidity can fracture when an ELP switches from being a liquidity maker to a liquidity taker.

Our work also contributes to market microstructure papers that consider multi-dimensional sources of randomness, using risk-neutral agents in static (Romer, 1993; Gervais, 1997; Avery and Zemsky, 1998)⁶ or dynamic models (Li, 2012; Back et al., 2013).⁷ Some recent models also consider risk averse investors (Gao et al., 2013; Banerjee and Green, 2015).⁸ Our work differs from this literature in that we make the ELP’s trading role endogenous. Instead of emphasizing the possibility of multiple equilibria, our model provides a unique equilibrium with two regimes, where the outcome depends on the ELP’s information signal.

Lastly, we demonstrate how reinforcement learning techniques can play an important role in more intricate model structures by providing solutions for nonlinear equilibrium. Varian (2014), Abadie and Kasy (2017), and Mullainathan and Spiess (2017) provide excellent discussions of how machine learning can be applied to analyze economic problems involving big data. Recent studies further apply this into finance literature.⁹ Nevertheless, this research has focus on empirical applications, while our intent is to apply machine learning in solving theoretical models. When models have increasingly diverse agents and researchers consider more complex market structures

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⁶ Gervais (1997) considers a static Glosten and Milgrom (1985) model in which the market maker is uncertain about the precision of the informed trader's signal. Romer (1993) and Avery and Zemsky (1998) consider models in which the proportion of informed traders is uncertain (and is not learned through time).

⁷ Li (2012) and Back, Crotty, and Li, (2013) consider generalizations of the continuous-time, Kyle-model of Back (1992) that allow for uncertainty about whether the strategic trader is informed.

⁸ Banerjee and Green (2015) develop a fully revealing equilibrium to consider the feature whereby an uninformed trader is not certain whether his counterparty is informed or is a liquidity trader. In their setting, uncertainty about the status of others leads to a nonlinear price schedules that react asymmetrically to news. Gao et al. (2013) introduce uncertainty regarding the proportion of informed traders in a rational expectation equilibrium model with asymmetric information. They show that there are multiple nonlinear equilibria and price informativeness is non-monotonic in proportion of informed traders.

equilibria become nonlinear and hence difficult to solve. While being able to characterize precisely how more innovative forms of trading affects price and liquidity dynamics is important for investors and regulators, this need for richer models poses significant technical challenges to researchers.

We show how machine learning techniques, combined with model structures based on microstructure theory, can provide analytical insights to important market microstructure problems. We believe that machine learning’s ability to handle the complex problems, coupled with characteristics that are non-parametric and do not pre-specify a functional form, is well-suited to microstructure research. Unlike prior research, such as Bernardo and Judd (2000), we do not rely on a parameterization of price adjustment rules, asset demand, or the projection method. Our solution technique is flexible and can be generalized to study rational expectation models in the presence of other frictions. Our model contribution is related to Breugem and Buss (2019), but our setup is more complicated; it includes a continuous state space and more complex interactions between traders.

The paper is organized as follows. We present the economy with endogenous liquidity provision in Section 2. In Section 3, we illustrate the reinforcement learning approaches and present the main results about nonlinear rational expectation equilibrium. Section 4 explores two channels (DMM limited market participation and ELP endogenous liquidity provision) whereby ELP activity can intensify the DMM’s adverse selection cost. Section 5 discusses the model implications. We conclude in Section 6.

2. An Economy with Endogenous Liquidity Provision

This section describes a rational expectations equilibrium model of liquidity with a high-frequency endogenous liquidity provider (ELP). A distinct feature of this model is that high-frequency trader’s endogenous liquidity provision operates through a cut-off strategy where a threshold of liquidity provision is determined endogenously. We show that the withdrawal of the ELP’s liquidity can have significant effects on market clearing. This possible loss of liquidity from the ELP (replaced with a concomitant use of liquidity by the ELP) results in fragile liquidity.
2.1. Assets and Information

There are three, risk-neutral traders in the economy: the ELP who can choose between market orders (demanding or taking liquidity) and limit orders (providing or making liquidity); an informed trader who submits market orders and uses liquidity; and a designated market maker (DMM) who submits limit orders to provide liquidity. Unlike the ELP, the DMM is required to post limit orders, hence their designation. Additionally, there are liquidity traders who randomly submit market orders and thereby take liquidity. The existence of liquidity traders prevents both the ELP and informed trader’s private information from being revealed fully in equilibrium.

In the model time is discrete with two dates ($t = 1, 2$). At date 1, all traders submit orders based on their information and trades are cleared. At date 2, assets pay off and all traders consume. There is a risky asset, which is in zero net supply. Let $\bar{p}$ denote the price of the risky security. The risky asset has multiple random components, each combining to create a normally distributed payoff or value of $\bar{V}$, where:

$$
\bar{V} = \delta_f + \delta_d + \delta_0,
$$

where $\delta_i \sim N(0, \sigma_i^2)$ are mutually independent draws from normal distributions. The sub-innovation $(\delta_f, \delta_d, \delta_0)$ are observed by ELP only, informed only, and both traders, respectively. Specifically, before trading, both the informed trader and ELP are informed about one source of value, $\delta_0$; in addition, each is uniquely endowed with information about one other component of the risky asset payoff. The informed trader knows the realization of $\bar{s} = \delta_0 + \delta_d$ at time 1. The ELP observes private information described by the signal $\bar{\theta} = \delta_0 + \delta_f + \bar{\varepsilon}_\theta$ at time 1. In this setup the innovation $\bar{\varepsilon}_\theta \sim N(0, \sigma_{\varepsilon, \theta}^2)$ measures the ELP’s information speed, which can be influenced by degree of technology investment.

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10 Throughout the paper, a tilde signifies a random variable.
11 Although we do not explicitly provide micro-foundation for ELP’s private information, it fits the cross-learning setup, which likes Cespa and Foucault (2014) and Goldstein et al. (2014). When the same or similar assets trade in multi-venues, this setup allows high-frequency ELP to “back-run” (inferring fundamental information from order flows) at faster markets and to “front-run” (anticipating the order flows and fundamental value) at slower market. Therefore, the noise term $\bar{\varepsilon}_\theta \sim N(0, \sigma_{\varepsilon, \theta}^2)$ in ELP’s private signal measures how fast ELP process the order flow information from faster market (i.e., we call it information speed). The rise of technology investment on computer hardware, algorithms and connection to exchange servers has witnessed the development of HFT and speeded up information acquisition and dissemination. This just means that, with increasing information speed, the parameter $\sigma_{\varepsilon, \theta}$ decreases.
The payoff structure in expression (1) is introduced to characterize multi-dimensional information in the risky asset’s payoff. Past empirical research and market practice suggest high-frequency traders (such as an ELP) and active mutual fund investors (e.g. the informed trader in our model) focus on different types of information. High-frequency traders rely mainly on order flow information through back running, front running and cross-market inference. Active mutual fund investors rely on investment processes that consider fundamental and other research as well as due diligence. Hence, the information components $\delta_0$, $\delta_f$ and $\delta_d$ in equation (1) can be thought of as different dimensions of the information in the risky asset payoff and serve to characterize heterogeneous information between the informed trader and ELP.

Given this structure, the signal $\delta_0$ received by the informed trader and ELP represents a common information component; whereas $\delta_f$ and $\delta_d$ become trader-specific information. When the markets become more correlated, the information traded and disseminated on different markets might also become more similar. Therefore, the high-frequency ELP might observe a signal that is closely related to the signal seen by the informed trader. Our information structure allows us to explore the impact of increasing information commonality between high-frequency institutions and investors such as mutual funds.

### 2.2. Endogenous Liquidity Provision

Using the structure outlined above traders trade at date 1 to maximize their risk-neutral payoffs subject to quadratic trading costs, as in Banerjee et al. (2018),

$$
\max_{x_i} \mathcal{E}[x_i(\bar{V} - \bar{p})l_i] - \frac{1}{2\gamma} x_i^2
$$

(2)

where $\gamma$ measures the reciprocal of trading cost, $l_i$ is the information set for trader $i$ and $x_i$ is the demand of trader $i \in \{E, I, D\}$, representing ELP, informed trader and DMM, respectively. There is also a liquidity (noise) trader with supply $\bar{z} \sim \mathcal{N}(0, \sigma_z^2)$ for the risky asset, which is independent of all other variables in the economy.

Based on any relevant signal, a liquidity maker submits a price-contingent demand schedule; whereas a liquidity taker submits a market order. Specifically, the informed trader submits market orders $x_i(\bar{s})$ after obtaining private signal $\bar{s}$ and DMM submits limit orders $x_o(\bar{u}, \bar{p})$ after observing the overall market order flow $\bar{u}$ conditional on price $\bar{p}$. 
The ELP can choose to make liquidity by submitting limit orders when their signal $\tilde{\theta}$ is small (given the signal has a zero expected value we refer to these outcomes as typical market conditions). However, the ELP will take liquidity by submitting market orders in turbulent market conditions, i.e. when the $\tilde{\theta}$ is large in magnitude. This cut-off strategy $x_E$, can be written as:

$$x_E = x_T(\tilde{\theta}) \mathbb{1}_{|\tilde{\theta}|>\theta^*} + x_M(\tilde{\theta}; \tilde{u}, \tilde{p}) \mathbb{1}_{|\tilde{\theta}|\leq\theta^*},$$

where $\mathbb{1}_{|\tilde{\theta}|>\theta^*}$ is the indicator function, $x_T$ and $x_M$ are the orders when the ELP chooses to be liquidity taker and maker, respectively. Parameter $\theta^* > 0$ represents the cut-off level of the information component $\tilde{\theta}$ for ELP to switch between liquidity taker and maker, which is to be determined endogenously in equilibrium.

Developments in trading technology and trading automation (such as supercomputers and co-location with the exchange) have intensified competition among high-frequency traders and led to a concentration of liquidity provision by a small number of high-frequency firms. This concentration seems to limit DMM liquidity supply when high-frequency firms also provide liquidity. In order to take this into consideration, we introduce DMM’s market participation $\tau$, introduced in (4b).

Current theoretical literature assumes that high-frequency traders can be either liquidity makers or liquidity takers, and that this role is determined exogenously. For example, Yang and Zhu (2019) model the strategic interaction between fundamental investors and high-frequency liquidity takers or 'back-runners' (whose only information is about the past order flow of fundamental investors; back-runners partly infer fundamental investors' information from their order flow and exploit it in subsequent trading). Menkveld and Zoican (2018) show that speeding up the processing of orders by the exchange does not necessarily improve liquidity, given their modelling of a high-frequency market maker. The innovative feature of the model in this paper is that the ELP applies a cut-off strategy and can either profit from making liquidity or from taking liquidity and speculating on the size of information shock based on their signal. Empirically, high-frequency traders typically act as liquidity makes; however, they can shift to taking liquidity as warranted by market conditions (Raman, Robe and Yadav, 2014; Anand and Venkataraman, 2015; and Korajczyk and Murphy, 2015).
2.3. Equilibrium

Based on the ELP’s switching behaviour as outlined above, there are two equilibrium regimes: a ‘taker regime’ where the ELP uses liquidity; and a ‘maker regime’ where the ELP provides liquidity. In the taker regime three traders (informed, ELP, and liquidity) take liquidity. In equilibrium, their orders are absorbed by DMM, who is the only liquidity provider in this setting. Therefore, the market clearing condition in a taker regime is:

\[ u_T + x_D(u_T, \bar{p}) = 0 \quad \text{with} \quad u_T = x_I(\bar{s}) + x_T(\bar{\theta}) + \bar{z}. \]  

(4a)

In the maker regime, the informed and liquidity traders take liquidity; while the ELP and DMM compete to provide liquidity. With the ELP’s speed advantage, ELP’s limit orders have a higher priority than the DMM’s limit order.\(^{12}\) This trading speed advantage is characterized by parameter \( \tau \) in the following market clearing condition,

\[ u_M + x_D(\bar{\theta}; u_M, \bar{p}_M) + \tau x_D(u_M, \bar{p}_M) = 0 \quad \text{with} \quad u_M = x_I(\bar{s}) + \bar{z}. \]  

(4b)

Here \( \tau \) measures the limited market participation of the DMM. For \( \tau = 0 \), the ELP completely crowds out the DMM and is the only liquidity provider. As \( \tau \) increases the DMM provides a greater share of the available liquidity. As can be seen from expressions (4a) and (4b), the ELP provides liquidity only in the maker regime. With no obligation to supply liquidity, the ELP switches between making and taking liquidity endogenously. We formalize a rational expectations cut-off equilibrium, which requires clearing of the market under both maker and taker regimes.

**DEFINITION 1 (Rational Expectations Cut-off Equilibrium):** The rational expectations equilibrium in the economy is characterized by asset price \( \bar{p} \), demand functions \( (x^*_i, x^*_E, x^*_D) \) and optimal cut-off threshold value \( \theta^* \) such that

a) demand function \( x_i \) maximizes each trader’s expected payoff as described in expression (2);

b) the risky asset price function \( \bar{p} \) clears the market as described in equations (4a) and (4b);

c) given the price and demand functions it is optimal for the ELP to provide liquidity according to equation (3), where the parameter \( \theta^* \) is a threshold value such that the ELP is indifferent between taking and making liquidity a signal \( \bar{\theta} \) where \( |\bar{\theta}| = \theta^* \);

\(^{12}\) This is consistent with empirical observations of the concentration of liquidity provision and reduced participations of retail investors in financial market. HFT takes large proportion of market-making trading volume, resulting in limited participation of DMM whose limit orders can only be executed partially when competing with ELP.
d) for every realization of the signals $\tilde{\theta}$ and $\tilde{s}$, the beliefs of all traders are consistent with a jointly conditional probability distribution.

Due to the switching strategy of ELP, the equilibrium becomes nonlinear. Specifically, with endogenous liquidity provision as outlined in expression (3), the ELP either submits market orders or limit orders by applying a cut-off strategy, depending on their private information. Hence, the ELP’s trading behaviour provides additional uncertainty and information to the DMM and informed traders about the risk asset’s value. This leads to a nonlinear inference problem for the DMM (conditional on a truncated normal distribution). As a result, the equilibrium price function is nonlinear, and its specific functional form is unknown. Hence, the liquidity taker’s trading strategies, which also determine the aggregate order flow, are also nonlinear.

3. Equilibrium Trading Strategies and Price

This section first characterizes the nonlinear equilibrium with respect to the equilibrium demand and price functions and the endogenous threshold value of the ELP. Due to the nonlinearity in the DMM’s posterior beliefs, we do not have a closed form equilibrium solution. Hence, we introduce a reinforcement learning approach without parameterizing (conjecturing) price and demand function forms and conduct a numerical analysis of the nonlinear rational expectation equilibrium. The results show fragile liquidity that is a consequence of endogenous liquidity provision and limited market participation. This fragility, or possibility for the loss of liquidity, raises the possibility of market crashes when ELP’s switch from liquidity making to liquidity taking in atypical times (when the ELP’s signal suggests that the risky asset value is much different than expected).

3.1. Nonlinear Equilibrium with Endogenous Liquidity Provision

In the model, a trader’s strategy depends on the trading strategies of the others and the market maker’s pricing rule, which in turn determines traders’ trading strategies. This endogenous feedback where one agent’s strategy affects another’s, which in turn affects their own strategy (see Foster and Viswanathan, 1996) can be characterized as a fixed-point problem, which can complicate the analysis in general.
Related to ELP’s endogenous liquidity provision decision and DMM’s limited market participation, the DMM faces uncertainty about realized equilibrium regimes (either taker or maker regime) when submitting orders. Therefore, apart from the payoff uncertainty, the DMM encounters two additional uncertainties about the structure of the aggregate market order flow when inferring the fundamental value. This is through two channels. The first channel is the ELP’s endogenous switching, characterized with the probability $\mu = P(1_{|\bar{\theta}| > \theta^*})$ that ELP makes a speculative trade and takes liquidity. The second channel is through limited market participation $\tau$, which describes the proportion of DMM’s orders that are executed under the maker regime. The consideration of both channels is characterized by the following optimization problem for DMM:

$$\max_{x_D} \mu \left[ E[x_D(\bar{V} - \bar{p})|\bar{\theta}| > \theta^*] - \frac{1}{2y} x_D^2 \right] + (1 - \mu) \left[ E[\tau x_D(\bar{V} - \bar{p})|\bar{\theta}| \leq \theta^*] - \frac{1}{2y} (\tau x_D)^2 \right].$$

This interaction between the ELP’s endogenous liquidity provision and the DMM’s limited market participation plays an important role in explaining liquidity fragility and events such as flash-crashes. Related analysis about the adverse selection faced by the DMM is carried out through these two channels.

To characterize the equilibrium, we first derive the equilibrium when the threshold value is exogenously given. Definition 1 and the optimization problem given in expression (5) lead to following lemma that describes the demand functions of the informed, ELP, and DMM.

**LEMMA 1:** Given the threshold value $\theta^*$, the demand function of the informed trader $x_I(\bar{s}) = \gamma \bar{s} - \gamma f_{p,I}(\bar{s})$;

the demand functions of the ELP when consuming or supplying liquidity are:

$$x_T(\bar{\theta}) = \gamma \frac{\sigma_0^2 + \sigma_0^2}{\sigma_T^2 + \sigma_0^2 + \sigma_{\varepsilon,0}^2} \bar{\theta} - \gamma f_{p,E}(\bar{\theta}), \quad x_M(\bar{\theta}; \bar{u}, \bar{p}) = \gamma f_{V,E}(\bar{\theta}, \bar{u}) - \gamma \bar{p};$$

and the demand function of the DMM is:

$$x_D(\bar{u}, \bar{p}) = \gamma \frac{f_{V,D}(\bar{u})}{\mu + \tau^2(1 - \mu)} - \gamma \frac{\mu + \tau(1 - \mu)}{\mu + \tau^2(1 - \mu)} \bar{p};$$

where

$$f_{p,E}(\bar{\theta}) = E(\bar{p}_T|\bar{\theta} = \delta_0 + \delta_T + \bar{\varepsilon}_0); \quad f_{V,E}(\bar{\theta}, \bar{u}) = E(\bar{V}|\bar{\theta} = \delta_0 + \delta_T + \bar{\varepsilon}, \bar{u});$$

$$f_{p,I}(\bar{s}) = \mu E(\bar{p}_T|\bar{s} = \delta_0 + \delta_T, |\bar{\theta}| > \theta^*) + (1 - \mu) E(\bar{p}_M|\bar{s} = \delta_0 + \delta_T, |\bar{\theta}| \leq \theta^*);$$

$$f_{V,D}(\bar{u}) = \mu E[\bar{V}|\bar{u}_T, |\bar{\theta}| > \theta^*] + \tau(1 - \mu) E[\bar{V}|\bar{u}_M, |\bar{\theta}| \leq \theta^*]$$
are, respectively, the rational expectation functions of the ELP about the equilibrium price and fundamental value, the informed about the equilibrium price, and the DMM about the fundamental value conditional on their information.

Applying Lemma 1 to market clearance conditions (4a) and (4b), we obtain the following exogenous equilibrium price functions for given threshold value.

**PROPOSITION 1:** Given the threshold value \( \theta^* \), the equilibrium price functions under taker and maker regimes can be represented, respectively,

\[
\hat{p}_T(\bar{u}) = \frac{\theta \bar{u} + \gamma f_{v, \theta}(\bar{u})}{\gamma \Phi}, \quad \hat{p}_M(\tilde{\theta}, \bar{u}) = \frac{\theta \bar{u} + \gamma \theta f_{v, \theta}(\tilde{\theta}, \bar{u}) + \gamma r f_{v, \theta}(\bar{u})}{\gamma (\theta + r \Phi)},
\]

where \( \Phi = \mu + r(1 - \mu) \) and \( \theta = \mu + r^2(1 - \mu) \).

We next characterize the endogenous equilibrium threshold value. Unlike the DMM, the ELP has no obligation to supply liquidity. Thus, the ELP endogenously provides liquidity, depending on their information and market conditions. The equilibrium threshold signal value equates the ELP’s expected payoffs under both the taker and maker regimes. Proposition 2 gives the DMM threshold signal that fulfils this requirement.

**PROPOSITION 2:** The endogenous threshold value \( \theta^* \) is determined by,

\[
\theta^* = \left\{ \left[ \frac{\sigma^2 + \sigma_\theta^2}{\sigma^2 + \sigma_\theta^2 + \sigma_{\epsilon,\theta}^2} \theta - f_{p, \theta}(\theta) \right]^2 = E \left[ \frac{(r \Phi f_{v, \theta}(\tilde{\theta}, \bar{u}) - \gamma r f_{v, \theta}(\bar{u}) - \theta \bar{u})^2}{\gamma (\theta + r \Phi)} \right] \cdot \tilde{\theta} = \theta \right\}.
\]

Propositions 1 and 2 show that the equilibrium is determined by the conditional expectation functions of the informed, ELP and DMM. In the following discussion we focus on the filtering problem of the traders’ expectation functions (in Lemma 1), instead of dealing with the equilibrium demand and price functions directly.

Following the rational expectations equilibrium as given in Definition 1 and Lemma 1, the beliefs of all traders should be consistent with the jointly conditional probability distribution in equilibrium. This means that, in the equilibrium,

\[
f_{p, \theta}(\tilde{\theta}) = E \left[ \frac{\theta(\tilde{u}) + \gamma f_{v, \theta, \theta}(\tilde{u})}{\gamma \Phi} \right] \cdot \tilde{\theta}, \bar{u} = \gamma s - \gamma f_{p, \theta}(\hat{z}) + \gamma \frac{\sigma^2}{\sigma^2 + \sigma_\theta^2 + \sigma_{\epsilon,\theta}^2} \tilde{\theta} - \gamma f_{p, \theta}(\hat{\theta}) + \hat{z}.
\]

Similar arguments are also applied to other expectation functions in Lemma 1 (these are given in Appendix A2). Therefore, the equilibrium is characterised by a ‘fixed-function’ problem of the
expectation functions. In the traditional linear equilibrium framework, this is reduced to a ‘fixed-point problem’. There are two prominent requirements that allow this reduction to work for traditional methods: the specific equilibrium structure needs to be known a priori; and the filter problem can be solved. However, in our model, due to the ELP’s endogenous switching and the DMM’s inference problem, the equilibrium becomes nonlinear. The equilibrium structure is unknown, which makes the equilibrium inference problem difficult via traditional methods. In the following, we introduce a reinforcement learning method. The reinforcement learning approach is non-parametric and does not pre-specify functional forms. It simplifies the complexity created by an unspecified equilibrium structure and non-linear filter problem and is relatively flexible in calculating the nonlinear equilibrium.

3.2. Reinforcement Learning

In this subsection we introduce the reinforcement learning algorithm and provide intuition about how to compute equilibrium expectation functions, while leaving details to Appendix B. 13 Essentially, highly non-parametric reinforcement learning can be used to extract the conditional expectation functions without pre-specifying a functional form from experimental observations through bootstrapping; it captures characteristics that parametric models may not recognize. In brief, we start with zero initial expectation functions and a given threshold value $\theta^*$, and use reinforcement learning algorithms to gradually update the expectation functions based on simulation realizations. Specifically, we use $f_h^k$ to represent the expectation function with $h \in \{p,l\}; \{p,E\}; \{V,E\}; \{V,D\}$ after $k$ -th updating ($f_h^0 = 0$), which is used as prior expectation for $(k + 1) - th$ updating. When the learning converges, the expectation functions satisfy equation (6).

To generate experimental observations for $(k + 1) - th$ updating, we draw $n$ realization values of randomly input variables and denote $(\delta_{\theta}^{k,l}, \delta_f^{k,l}, \delta_D^{k,l}, \delta_g^{k,l}, \delta\theta^{k,l})$ as the $i-th$ observation in this round. Based on these values and the expectation functions $f_h^k$, we calculate the demand functions $f_h^k$, we calculate the demand functions of the informed, ELP and DMM based on Lemma 1. We then use the market clearing condition and Proposition 1 to obtain the equilibrium price and consequently the order flows and ELP’s value earned through taking and making liquidity as $(\bar{p}^{k,l}, \bar{u}^{k,l}, \bar{D}_{E,T}^{k,l}, \bar{D}_{E,M}^{k,l})$. With $n$ observations, the input-output pairs form a distribution for the four intermediate expectation functions, denoted $g_h^k$.

13 Under machine learning context, reinforcement learning is an algorithm on how software agents ought to take actions in an environment so as to maximize some notion of cumulative rewards.
The expectation functions are then updated according to $f_{h}^{k+1} = \beta_{h}^{k} g_{h}^{k} + (1 - \beta_{h}^{k}) f_{h}^{k}$, a weighted average of the prior expectation $f_{h}^{k}$ in round $k$ and the realized expectation $g_{h}^{k}$.\textsuperscript{14} Panel A in Figure 2 shows updating evolution process for DMM’s expectation function about the fundamental value $f_{V,D}$.

We jointly apply this learning to all the four expectation functions in Lemma 1. We then introduce a convergence test (see details in Appendix B3) to determine the number of experiments in order to make the reinforcement learning converge to the expectation functions in the equilibrium. We show the convergence of the learning to the expectation function, leading to the equilibrium fixed-function expectation functions. As an example, Panel B in Figure 2 illustrates the convergence to the solution of the ‘fixed-function’ problem. It shows snapshots of the 1\textsuperscript{st}, 2\textsuperscript{nd}, 4\textsuperscript{th}, 8\textsuperscript{th}, 32\textsuperscript{nd} and 120\textsuperscript{th} updating. At the earlier rounds, the difference between the prior expectation (blue solid line) and updated expectation (red solid line) is significant. As the number of experiment increases, the difference between the blue and red solid lines vanishes; so that the expectation functions converge.

[Figure 2]

In the procedure above, we calculate equilibrium expectation functions given the threshold value. Based on Proposition 2, the endogenous equilibrium $\theta^{\ast}$ is determined such that the ELP is indifferent between consuming and supplying liquidity. The details are in Appendix B4.\textsuperscript{15} In sum, we show that the nonlinear equilibrium characterized by Propositions 1 and 2 can be solved numerically using the reinforcement learning method.

\textsuperscript{14} We provide the details of how to calculate the realized expectation function $g_{h}^{k}$ and the weighting function in Appendix B. Intuitively, we put more weight to the realized expectation function in the earlier experiments and less weight in the later experiments when the experimental tends to be stable. This ensures a fast learning at the beginnings (depending more on the current outcomes due to the scarce historical data) but a constant learning speed in the latter experiments.

\textsuperscript{15} In the exogenous equilibrium, utility functions are based on the exogenously given threshold value $\overline{\theta}^{\ast}$. After the $N$-th experiment (when the belief converges), we calculate the value $\overline{\theta}^{\ast}$ that makes the ELP be indifferent in choosing to making and taking liquidity, $\overline{\theta}^{\ast}(\overline{\theta}^{\ast}) = \{\theta\mid U_{E,M}^{\ast}(\theta, \overline{\theta}^{\ast}) = U_{E,M}(\theta, \overline{\theta}^{\ast})\}$. This value $\overline{\theta}^{\ast}(\overline{\theta}^{\ast})$ depends on the exogenously given parameter $\overline{\theta}^{\ast}$. The endogenous equilibrium condition means a fixed-point problem of the exogenous cut-off level $\overline{\theta}^{\ast}$ such that the optimal threshold value $\overline{\theta}^{\ast}$ from comparing the utilities is the same under $\overline{\theta}^{\ast}$, that is $\overline{\theta}^{\ast}(\overline{\theta}^{\ast}) = \overline{\theta}^{\ast}$.
3.3. Results

We now report our main numerical results of the nonlinear endogenous liquidity provision equilibrium in Figure 3. Panel A of Figure 3 reports equilibrium price functions under maker and taker regimes (based on the equilibrium expectation functions given in the Appendix B6). It shows that the equilibrium price functions, especially under the taker regime, are nonlinear. The slope of the price function reflects the sensitivity of equilibrium prices to order flows, measuring liquidity (as in Kyle, 1985). Panel A shows that the price function under the taker regime is concave (in the absolute order flow) and has much higher slope than under the maker regime. This implies that liquidity under the taker regime is more fragile than under the maker regime.

The price under the taker regime is solely determined by the DMM. This means that the nonlinear price function under the taker regime is mainly due to the DMM’s nonlinear inference. Therefore, the DMM’s adverse selection is the source of fragile liquidity. More explicitly, the DMM’s expectation function is S-shaped, with a higher slope when the order flow is relatively balanced. Intuitively, a balanced aggregate order flow indicates that the orders from the informed trader and liquidity trader are opposite to the orders from the ELP. Therefore, the balanced aggregate order flow contains less information for the DMM, which increases the DMM’s sensitivity to aggregate order flow. We provide more analysis of the DMM inference as a channel for fragile liquidity in Section 4.

This result leads to several important implications. Firstly, in contrast to the existing market microstructure models with constant price impact, empirical studies (by examining the impact of individual transactions in limit order book market) find that price change is more sensitive to small trading volume but less so to larger trading volume. This is consistent with our concave price functions. Early studies by Hasbrouck (1991) and Hausman et al. (1992) find strongly concave functions without fitting the price function form. Keim and Madhavan (1996) also observe a concave impact function for block trades.

Secondly, the results generate fragile liquidity. A relatively small shock in one of the state variables (both fundamental and non-fundamental) can lead to a disproportionately large change in liquidity, as shown in Panel B in Figure 3. For example, liquidity can fall markedly if a small shock were to alter the ELP’s private signal relative to a threshold value. This could trigger the
ELP to switch from being a liquidity maker to a liquidity taker.\textsuperscript{16} We conduct comparative statics analyses by comparing equilibrium prices across small variations in both parameters and realized state variables.\textsuperscript{17} Given the parameters and realizations of information noisy shocks and noise trading shocks, the kinked contour lines of different pairs in Panel B represent equilibrium iso-price lines. The contour lines consist of two subspaces: taker regime and maker regime. Panel B shows that the contour lines have a steeper slope and are more densely distributed in the taker regime than in the maker regime, which indicates weaker liquidity under taker regime. Furthermore, the distances between iso-price lines under taker regime are varying, which is also consistent with a nonlinear price function under taker regime.

Panel C in Figure 3 shows the price impact of limited participation. The key observation is that the liquidity under the taker regime becomes worse with more limited DMM’s market participation. This is because more limited DMM’s market participation intensifies the DMM’s adverse selection cost. The ELP has no obligation to supply liquidity; and endogenously provides liquidity depending on their information and market conditions. When conditions become unfavourable, the ELP withdraws from making liquidity and takes liquidity instead. This means that DMM faces higher adverse selection under a taker regime. More limited DMM market participation is consistent with a higher probability that the DMM makes liquidity under a taker regime, which intensifies adverse the selection cost.

4. Fragile Liquidity

The results in Section 3 show that it is the DMM’s inference problem that leads to nonlinear price function and fragile liquidity. This section provides an analysis of the underlying mechanism through which an HFT’s endogenous liquidity provision and designated market maker’s (DMMs) limited market participation (e.g., due to frictions induced by trading automation) contribute to liquidity fragility.

\textsuperscript{16} For example, under high-frequency trading setup, increasing noise trading at other market might lead to more aggressive aggregate trading volume ELP observed from order flow information about other market, which pushes the realization of signal above the threshold value and triggers the high-frequency trader switching and liquidity crashes.

\textsuperscript{17} Other static models of market crashes (e.g., Gennotte and Leland 1990; Barlevy and Veronesi 2003; Yuan 2005; Breon-Drish 2012; or Cespa and Foucault 2014) also use this approach.
Specifically, we explore two channels for ELP to affect the DMM’s adverse selection. First, it is possible that developments in trading technology and trading automation (such as supercomputers and co-location with the exchange) have intensified competition among HFTs and led to a concentration of liquidity provision by a small number of HFT firms. HFT’s speed and information advantages limit the DMM’s market participation. Second, the ELP endogenously switches between providing and consuming liquidity based on an endogenous cut-off threshold value on the ELP’s information. Without the liquidity supply obligation, the ELP withdraws liquidity supply when market conditions become unfavourable. Both channels are integrated and affect jointly the adverse selection faced by the DMM.

To understand better the underlying mechanism of fragile liquidity in the full model (FM) in Section 3, we disentangle the two channels by switching off several model elements. Specifically, we first examine a limited market participation (LMP) through exogenous liquidity provision, instead of the endogenous liquidity provision in the FM. We then consider the ELP’s endogenous liquidity provision (ELP) with no-market-participation from the DMM when the ELP provides liquidity. To better disentangle these effects, we further remove the possibility of cross learning between the informed trader and the ELP by assuming that there is no common information component (i.e., $\sigma_0^2 = 0$) for both channels. The connections of the two channels to the full model (FM) are summarized in the following table.

<table>
<thead>
<tr>
<th>Cross learning</th>
<th>ELP liquidity provision</th>
<th>DMM market participation</th>
</tr>
</thead>
<tbody>
<tr>
<td>FM $\sigma_0^2 \neq 0$</td>
<td>$x_E = x_{E,T}(\hat{\theta}) 1_{[\hat{\theta} &gt; \theta]} + x_{E,M}(\hat{\theta}; \tilde{u}, \tilde{p}) 1_{[\hat{\theta}</td>
<td>\tilde{\theta}]^*}$</td>
</tr>
<tr>
<td>LMP $\sigma_0^2 = 0$</td>
<td>$x_E = x_{E,T}(\hat{\theta}) 1_{[\omega = 1]} + x_{E,M}(\hat{\theta}; \tilde{u}, \tilde{p}) 1_{[\omega = 0]}$</td>
<td>$\tau \geq 0$</td>
</tr>
<tr>
<td>ELP $\sigma_0^2 = 0$</td>
<td>$x_E = x_{E,T}(\hat{\theta}) 1_{[\hat{\theta} &gt; \theta]} + x_{E,M}(\hat{\theta}; \tilde{u}, \tilde{p}) 1_{[\hat{\theta}</td>
<td>\tilde{\theta}]^*}$</td>
</tr>
</tbody>
</table>

### 4.1. Limited Market Participation

To focus on the DMM’s limited market participation, we assume that the ELP’s liquidity provision is determined exogenously. This allows for closed-form solutions that helps to provide the underlying economic intuition for the LMP channel. We show that the limited market participation intensifies DMM’s adverse selection and lowers the liquidity provision from the DMM, reducing market liquidity.
When the ELP’s liquidity provision is exogenously given, the ELP randomly chooses to either consume liquidity with probability $\mu$ or supply liquidity with probability $(1-\mu)$, that is,

$$x_E = x_T(\tilde{\theta}) \cdot 1_{\sigma=1} + x_M(\tilde{\theta}; \tilde{u}, \tilde{p}) \cdot 1_{\sigma=0},$$

(7)

where $\sigma$ is a binomial random variable, independent of others, satisfying $P(\sigma = 1) = \mu$ and $P(\sigma = 0) = 1 - \mu$. Hence, the ELP’s liquidity provision decision contains no information about the fundamental payoff. To simplify the analysis, we assume that the ELP’s switching probability and speed advantage are related by

$$\mu(\tau, c) = \frac{\tau}{c + \tau},$$

(8)

in which parameter $c \geq 0$ measures the ELP’s incentive of providing liquidity; the ELP is more willing to provide the liquidity for high value of $c$. Expression (8) means that as $\tau$ increases (exogenously), the ELP’s speed advantage dissipates and the DMM’s market participation increases (having more market power). When this occurs, the ELP faces more competition when providing liquidity and hence reduces providing liquidity. That is, an increase in the DMM’s participation increases the likelihood that the ELP takes liquidity (more like an informed trader). However, this change is exogenous, partially reflected by the parameter $c$. Therefore, the DMM realizes that the order flow from the ELP contains no information about asset values.

To see the impact of LMP on the adverse selection, we now examine how aggressively different types of traders’ trade based on their trading intensities. We consider the making intensity for liquidity providers and the taking intensity for liquidity consumers. Note that making intensity reflects how liquidity maker reacts to aggregate order flow; higher making intensity represents higher adverse selection for the DMM. Formally, we introduce taking and making intensities as follows.

**DEFINITION 3:** The taking intensity, $\beta_i$ and $\beta_E$, of the informed trader and ELP and the making intensity $\phi_D$ of the DMM are defined by, respectively,

$$\beta_i = \frac{\partial x_l(\tilde{s})}{\partial \tilde{s}}, \quad \beta_E = \frac{\partial x_M(\tilde{\theta})}{\partial \tilde{\theta}}, \quad \phi_D = \frac{\partial x_M(\tilde{u}, \tilde{p})}{\partial \tilde{u}}.$$  

(9)

Following this definition, we can show how the DMM’s making intensity depends on the taking intensities of the informed and ELP.

**LEMMA 2:** Conditional on the taking intensity, $\beta_i$ and $\beta_E$, of the informed and ELP, the DMM’s making intensity $\phi_D$ is given by
Lemma 2 shows that DMM’s making intensity has two components; the first component corresponds to the taker regime while second component corresponds to the maker regime of the ELP. Intuitively, the maker regime component, under which both ELP and DMM provide liquidity, is affected by the DMM’s limited participation (τ). However, the taking regime component, under which DMM solely supplies liquidity, is also affected by the limited participation. This is because, when submitting the order, DMM has uncertainty about whether the order is executed fully or partially due to actions of the ELP. Therefore, the DMM takes both states into consideration when providing liquidity, which makes the taker regime also be affected by the DMM’s limited participation (τ). Given the informed and ELP’s taking intensities, limited market participation (lower τ) increases the DMM’s making intensity and, hence, the adverse selection. In equilibrium, the DMM’s making intensity, \( \bar{\phi}_D \), affects the trading intensities, \( \beta_i \) and \( \beta_E \), of the informed and ELP, which in turn affects the DMM’s market making intensity. This feedback determines the equilibrium intensities, \( \bar{\beta}_i \) and \( \bar{\beta}_E \), of the informed trader and the ELP, respectively.

**PROPOSITION 3:** With LMP, there is a linear equilibrium in which the taking intensities, \( \bar{\beta}_i \) and \( \bar{\beta}_E \), of the informed and ELP are the unique solution of the equation system (A13) in Appendix A3.

The effect of limited market participation on the DMM’s making intensity characterized by Lemma 2 and Proposition 3 is illustrated in Figure 4. Panel A in Figure 4 depicts the DMM’s making intensity with respect to the limited participation for three different degrees of ELP’s incentive of providing liquidity, represented by parameter \( c \). It delivers two main results. Characterized by the making intensity, the DMM’s adverse selection increases in ELP’s incentive to provide liquidity, but declines with the DMM’s market participation.

The intuition behind these insights into the DMM’s adverse selection is as follows. First, when the ELP is more willing to provide liquidity, the DMM’s order has more chance being crowded out, thereby increasing the market making competition and hence the DMM’s making intensity and adverse selection. Second, and more importantly, greater participation means that the DMM has a higher chance receiving orders under a maker regime. For example, \( \tau = 0 \) means that the ELP
crowds out the DMM completely under a maker regime so that the DMM only receives orders under taker regime when the ELP’s signal is large in magnitude. This increases the DMM’s adverse selection. As \( \tau \) increases, the DMM has more market power in competing with the ELP when providing liquidity, reducing the DMM’s adverse selection. When providing liquidity, the DMM takes their limited participation and the uncertainty of ELP trading into consideration. As well, order toxicity is higher in the taker regime since both the informed trader and the ELP have private information and take liquidity, circumstances under which the DMM’s order is fully executed. Hence, more participants in the market means the DMM has a higher chance of receiving benign orders, thereby reducing adverse selection.

Based on the above analysis, the impact of market participation with endogenous liquidity provision on the DMM’s trading intensity in the full model can be illustrated in Panel B of Figure 4. This panel shows that the DMM’s expectation function becomes more concave with respect to the (absolute) aggregate order flow as the DMM’s participation becomes more limited (as \( \tau \) decreases), consistent with Panel A of Figure 4 under the linear model. As a consequence, the S-shaped price function becomes more significant, as is illustrated in Panel C of Figure 3.

Due to the nonlinear equilibrium, the impact of market participation with endogenous liquidity provision on the DMM’s adverse selection is more complicated. This is because more market participation (higher \( \tau \)) not only influences the DMM’s making intensity and hence the adverse selection, but also affects aggregate order flow and endogenous liquidity provision decision. In fact, we have the following decomposition of this relation between DMM trading intensity and adverse selection (see Appendix B7)

\[
\frac{df_{V,D}(\tilde{u}; \theta^*, \tau)}{d\tau} = \frac{\partial f_{V,D}(\tilde{u}; \theta^*, \tau)}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial \theta^*} + \left( \frac{\partial f_{V,D}(\tilde{u}; \theta^*, \tau)}{\partial \theta^*} + \frac{\partial f_{V,D}(\tilde{u}; \theta^*, \tau)}{\partial \theta^*} \frac{\partial \theta^*}{\partial \tau} \right) \frac{\partial \theta^*}{\partial \tau}
\]

Essentially, the direct effect in (10) describes the limited market participation channel. This direct effect is generated by fixing the trading strategies of the informed trader and ELP and changing only \( \tau \) to determine the DMM’s expectation function. We focus on how the DMM market participation influences the direct effect and hence adverse selection. Panel B shows that, as market participation increases, the DMM’s expectation function becomes less nonlinear and also less steep for moderate aggregate order flow, which means DMM has a decreasing making intensity and
hence lower adverse selection. This result is consistent with the channel of the linear equilibrium of LMP described in Panel A. The indirect effect related to the endogenous threshold value is the focus of the following analysis.

[Figure 4]

4.2. Endogenous Liquidity Provision

With the LMP (Lemma 2 and Proposition 3), exogenous liquidity provision does not provide additional information to the DMM’s efforts to infer the fundamental value. With endogenous liquidity provision as outlined in expression (3), the order flow from the ELP becomes informative. Learning by the DMM from ELP’s endogenous liquidity provision decision leads to a nonlinear inference problem. To focus on the effect of endogenous liquidity provision decision, we assume that the DMM is completely crowded-out when the ELP provides liquidity (i.e., $\tau = 0$). Consequently, the DMM’s demand function can be described as

$$ x_D = \gamma E[\tilde{V} | \tilde{u}_T = x_1(\tilde{s}) + x_{E,T}(\tilde{\theta}) + \tilde{z}, |\tilde{\theta}| > \theta^*] - \gamma \tilde{p}. $$

Equation (11) means that the DMM receives order only when the ELP consumes liquidity. However, the ELP only consumes liquidity and submits a speculative market order when his signal is much different from what was expected. Thus, the DMM takes the ELP’s selective liquidity into consideration, which means there is increasing toxicity from the aggregate market order flow (when $|\tilde{\theta}| > \theta^*$). In general, the DMM knows that they receive order only when market condition are unfavourable, which significantly intensifies their adverse selection. To understand how the ELP’s endogenous order choice affects the DMM’s adverse selection, we consider a linear approximation as outline below.

ASSUMPTION 1: In calculating the posterior expectation about the fundamental payoff when the ELP takes liquidity, the DMM applies the projection theorem by using the posterior covariance and variance,

$$ E[\tilde{V} | \tilde{u}_T = x_1(\tilde{s}) + x_{E,T}(\tilde{\theta}) + \tilde{z}, |\tilde{\theta}| > \theta^*] \approx \frac{COV(\tilde{V}, \tilde{u}_T, |\tilde{\theta}| > \theta^*)}{VAR(\tilde{u}_T, |\tilde{\theta}| > \theta^*)} \tilde{u}. $$

This assumption allows us to obtain a closed-form linear equilibrium and helps to identify the mechanism for liquidity fragility.
PROPOSITION 4: Given the optimal threshold value for ELP’s cut-off strategy $\theta^*$, under Assumption 1, there exists a linear equilibrium, where the price and demand functions are described as:

$$
\tilde{p} = (\lambda_T \tilde{u}_T) \mathbf{1}_{|\tilde{u}| > \theta^*} + (\lambda_{\theta, M} \tilde{\theta} + \lambda_M \tilde{u}_M) \mathbf{1}_{|\tilde{u}| < \theta^*};
$$

$$
x_i = \beta_i \delta_d; \quad x_T = \beta_e \theta; \quad x_M = \gamma (\phi_{E, \theta} \tilde{\theta} + \phi_{E} \tilde{u} - \tilde{p}); \quad x_D = \gamma (\phi_{D} \tilde{u} - \phi_{D, P} \tilde{p});
$$

where coefficients ($\beta_i, \beta_e, \phi_{E, \theta}, \phi_{E}, \phi_{D}, \lambda_T, \lambda_{\theta, M}, \lambda_M$) are the functions of parameters ($\sigma_f^2, \sigma_{e, \theta}, \sigma_d^2, \sigma_z^2$) given by (A14)-(A17) in Appendix A4. Furthermore, the probability that ELP consumes liquidity and the optimal threshold value for ELP’s cut-off strategy are determined by

$$
\mu = f_\mu (\theta^*; \sigma_f, \sigma_{e, \theta}) = 2N \left( - \frac{\theta^*}{\sqrt{\sigma_f^2 + \sigma_{e, \theta}^2}} \right);
$$

$$
\theta^* = f_\theta (\mu; \sigma_f, \sigma_d, \sigma_{e, \theta}, \sigma_z, \gamma)
$$

$$
= \frac{(\lambda_M - \phi_E) (\sigma_f^2 + \sigma_{e, \theta}^2)}{\sigma_f^2} \sqrt{\frac{(\beta_i^2 \sigma_d^2 + \sigma_z^2)}{1 - \frac{\tau \mu + \tau^2 (1 - \mu)}{(1 + \tau) \mu + 2 \tau^2 (1 - \mu)}}^2}.
$$

Function $N(\cdot)$ represents the cumulative normal function.

With the closed-form linear equilibrium of Proposition 4, we can examine the effect of the ELP’s endogenous liquidity choice on the DMM’s adverse selection, market liquidity, and the amplification of adverse selection from the ELP moving from making to taking liquidity.

A. Adverse Selection

Based on Proposition 4, we first examine the effect of endogenous liquidity provision on the DMM’s making intensity and hence adverse selection; this result is illustrated in Panel C of Figure 4. Characterized by the making intensity, it shows that the DMM’s adverse selection increases with the ELP’s optimal threshold signal value. We know that the ELP takes liquidity and submits a speculative order only when his signal is unexpected. Hence, a higher threshold value means that when the ELP consumes liquidity, the fundamental value is more extreme, which increases order flow toxicity for the DMM. Intuitively, the posterior volatilities about $\text{VAR} [\delta_f | \tilde{\theta} > \theta^*]$ and $\text{VAR} [\tilde{\epsilon}_d | \tilde{\theta} > \theta^*]$ increase in the threshold value, which then significantly increases DMM’s making intensity under taker regime (i.e., $\partial E [\mathcal{V} | \tilde{u}_T, |\tilde{\theta}| > \theta^*] / \partial \tilde{u}$).

For the nonlinear equilibrium, changing in threshold value has two effects,
\[
\frac{df_{V,D}(\bar{u}; \theta^*)}{d\theta^*} = \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \bar{u}} \frac{\partial \bar{u}}{\partial \theta^*} + \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \theta^*}
\]

Similar to expression (10), the direct effect in (15) describes the changing in adverse selection as the threshold value increases. We illustrate this effect with results from reinforcement learning in Panel D of Figure 4. It shows that, as the threshold value increases, the DMM’s expectation function about the fundamental payoff becomes more nonlinear and much steeper for moderate aggregate order flow. That is, higher threshold values mean the ELP’s signal can become more extreme, which increase the toxicity of DMM’s observed order flow. This intensifies the DMM’s adverse selection, generating significant S-shaped price impact when the ELP takes liquidity (as illustrated in Panel A of Figure 3). These results show that, channelled by ELP, the results and underlying mechanisms explored in the linear approximation remain for the nonlinear equilibrium.

**B. Liquidity**

With the endogenous feedback between traders’ trading strategies and the DMM’s pricing rule, we further examine the effect of ELP on market liquidity. Based on the linear approximation, Proposition 4 leads to the following result on the difference in market liquidity across maker and taker regimes.

**COROLLARY 1:** Equilibrium liquidity under a maker regime is higher than under a taker regime, i.e. \( \lambda_M < \lambda_T \).

Corollary 1 shows a liquidity mismatch between the taker and maker regimes. In particular, due to the actions of the ELP, liquidity falls when the ELP takes liquidity (relative to when the ELP makes liquidity). This provides an insight into the different price impact of the nonlinear equilibrium in Panel A of Figure 3 in the two regimes.

To understand better the role of asymmetry information of HFT and the effect of the ELP’s switching uncertainty to market participants, we compare the linear ELP equilibrium to two linear equilibrium scenarios. The first is the standard Kyle model without HFT to explore the role of HFT. The second considers HFT who only demands liquidity by submitting market orders; for convenience of discussion, we refer this to HFTm. Comparing Kyle model with HFTm shows the effect of asymmetry information, while comparing HFTm with ELP shows the effect of the ELP’s endogenous switch behaviour. The results are reported in Table 1. It shows that, comparing with
Kyle model, the information asymmetry resulting from the presence of informed high-frequency liquidity taker reduces liquidity (by about 10%) and increases the adverse selection (by about 25%). Comparing with HFTm, the ELP’s endogenous switching improves liquidity in the taker regime (by about 10%) but worsens the liquidity in the taking regime (by about 10%). Therefore ELP increases the adverse selection, consistent with Corollary 1.

Table 1

We further compares the linear approximation and the nonlinear equilibria. Table 1 shows that the nonlinearity further reduces the liquidity in the taker regime. Therefore, the linear approximation captures the same underlying mechanism, however, fails to capture the nonlinearity and overestimates the liquidity under taker regime. Furthermore, with an higher threshold signal value for the DMM in the nonlinear equilibrium (by about 50%), liquidity is more fragile than with the linear approximation. This implies that there is a less chance for liquidity to break down in the nonlinear equilibrium (comparing with the linear model), but when this occurs, the impact becomes more significant. This comparison has important implication for empirical studies. Most empirical framework focuses on the linear analysis. However, our results show that this analysis might be incomplete. When market participants are more diverse and market structures are more complex, being able to characterize how trading behaviour affects price and liquidity dynamics is important for investors and regulators. While linear frameworks can be helpful in capturing general mechanisms, they but may fail to deliver precise insights.

C. Amplification

Proposition 4 also shows that the probability $\mu$ for the ELP to take liquidity (called the ‘uncertainty’ in the following) and the threshold value $\theta^*$ for the ELP’s cut-off strategy (called the ‘liquidity provision decision’ in the following) are linked through two best response functions in equations (13) and (14). This link helps to explore how changes in market conditions influence the ELP’s endogenous liquidity provision decision. We consider an exogenous change in a parameter (e.g. to reflect a change in trading costs). This exogenous change influences the liquidity provision

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18 As for the nonlinear model, we run regression on the observations for the last experimental to estimate the liquidity indicated by the response of price to noise demand.

19 Given the ELP’s endogenous liquidity provision decision, the DMM forms beliefs about the ELP’s trading behaviour. However, the DMM’s pricing rule will affect the ELP’s order type (making and taking) based on their expected profits. Therefore the uncertainty and liquidity provision decisions are determined endogenously.
decision \( \theta^* \), which in turn affects the uncertainty \( \mu \) that feeds back to the threshold value \( \theta^* \), as follows.

**COROLLARY 2.** Let \( Q \) be one of the exogenous parameters \( (\sigma_f, \sigma_d, \sigma_e, \theta, \gamma) \) that determines the probability \( \mu \) that ELP takes liquidity and the threshold \( \theta^* \) of ELP’s cut-off strategy. The effect of \( Q \) on the uncertainty or liquidity provision decision is given by

\[
\frac{d \mu}{dQ} = \kappa \times \left( \frac{\partial f_i}{\partial Q} + \frac{\partial f_i}{\partial \frac{\partial f_j}{\partial Q}} \right)
\]

for \( i, j = \theta^*, \mu \), in which \( \kappa \) is an instability multiplier defined by

\[
\kappa = \left( 1 - \frac{\partial f_\mu(\theta^*)}{\partial \theta^*} \frac{\partial f_\mu(\mu)}{\partial \mu} \right)^{-1}
\]

which is larger than one, \( \kappa > 1 \).

Corollary 2 illustrates two consequences of the feedback mechanism. First, an exogenous change in the liquidity provision decision spills over to the uncertainty when \( \frac{\partial f_\mu(\theta^*)}{\partial \theta^*} \neq 0 \), while the change of the uncertainty also feeds back to the liquidity provision decision when \( \frac{\partial f_\theta(\mu)}{\partial \mu} \neq 0 \). We can show that both \( \frac{\partial f_\mu(\theta^*)}{\partial \theta^*} \) and \( \frac{\partial f_\theta(\mu)}{\partial \mu} \) are negative. Therefore, both the uncertainty and liquidity provision decision are substitutes; an increase in \( \mu(\theta^*) \) leads to a decline in \( \theta^*(\mu) \) (so that \( \kappa > 1 \)).

Second, due to the substitution effect, the direct effect of a small shock to a fundamental parameter is amplified through feedback. Consider, for instance, the effects of a small reduction in the trading cost, denoted by \( \Delta \gamma > 0 \). The direct effect of this reduction reduces the ELP’s liquidity provision (by \( (\partial f_\theta / \partial \gamma) \Delta \gamma \)), increases the uncertainty (due to the negative spill-over effect \( \frac{\partial f_\mu(\theta^*)}{\partial \theta^*} \neq 0 \) ) by \( (\partial f_\mu(\theta^*)/\partial \theta^*)(\partial f_\theta / \partial \gamma) \Delta \gamma \), although all the fundamental parameters (fundamental value volatility \( \sigma_f^2 \) and signal noise volatility \( \sigma_e^2 \)) have not yet changed. Due to the amplifying effect, the ELP’s liquidity provision reduces even more, triggering another feedback loop: the decrease in \( \theta^* \) leads to an increase in \( \mu \), which further decreases \( \theta^* \), etc. Therefore, the total effect of an initial decrease in the trading cost is an order of magnitude larger than its direct effect.

In summary, the interdependence of the uncertainty \( \mu \) and liquidity provision decision \( \theta^* \) forms a feedback loop and creates a multiplier (\( \kappa \)) that amplifies the initial effect of an exogenous parameter change in equilibrium. Panel A in Figure 5 illustrates this feedback mechanism by
changing the trading cost which then triggers feedback loops. We further examine the effect of changes to various exogenous parameters on the endogenous threshold signal value of cut-off strategy in the nonlinear equilibrium and report the results in Panel C in Figure 5, which helps to obtain some implications in Section 5.

Specially, optimal threshold value increases with both common information and ELP’s information noise. With higher common information, ELP can better use his information to infer informed traders’ information. Thus, payoff under maker regime increases. Instead, when consuming the liquidity, competition between informed and HFT is more intensified, which reduces ELP’s payoff under taker regime. In summary, ELP prefers to provide liquidity most of the time, and consequently the optimal threshold cut-off value increases. With increasing noise component in ELP’s information signal, ELP has less information advantage to predict the fundamental variable. Less precise information impedes ELP’s ability to make directional bet. This implies that ELP prefers to be market maker whose profits depends on the aggregate trading volume. Consequently, the optimal threshold cut-off value increases in the noise-to-signal ratio.

[Figure 5]

5. Implications

5.1. Stock Market Crashes

Following Gennotte and Leland (1990), we define a market crash as a scenario in which the equilibrium price function is discontinuous in any of its continuous arguments. This definition captures the essential element of a crash that a small change in the underlying fundamentals is associated with a disproportionately large change in asset prices. Different from the literature, we offer the endogenous liquidity provision as an explanation for stock market crashes. To better illustrate the mechanism, we conduct the following analysis based on Proposition 4 of linear approximation.

COROLLARY 3: The price jump size from \( \hat{p}_M(\hat{\epsilon}, \hat{z}) \) to \( \hat{p}_T(\hat{\epsilon}, \hat{z}) \) due to ELP’s switch is given by
\[ J(\tilde{\theta}, \tilde{z}) = \left[ \frac{\hat{\lambda}_T}{1 + (\mu \lambda_T + (1 - \mu)\lambda_M)} - \frac{\sigma_T^2}{\sigma_T^2 + \sigma_{\tilde{e}, \theta}^2} \right] \tilde{\theta} + (\hat{\lambda}_T - \hat{\lambda}_M)\tilde{z}. \]  

Corollary 3 implies that, depending on different state variable values, price can either increase, decrease or remain the same as ELP switches trading roles when information shocks are incorporated into the threshold signal value of the cut-off strategy. Note that price reacts more to the ELP’s signal noise in maker regime than taker regime. This means the term in the square brackets in (18) is negative. Also, liquidity is better under the maker regime relative to the taker regime (i.e., \( \hat{\lambda}_M < \hat{\lambda}_T \)), which means the term multiplied by \( \tilde{z} \) in (18) is positive. This creates a trade-off for the effects of price jumps from liquidity trading. When the noise trading is relatively high, liquidity breaks down; the information noise shock hits the threshold signal value of the cut-off strategy, which leads to a positive price jump. When noise trading is relatively low, so breaks in liquidity leads to a negative price jump.

We illustrate in Panel B of Figure 5 that, even without changing the fundamental value component, the shocks about the ELP’s information noise themselves can trigger price jumps. Market crashes can occur regardless of the value of the underlying asset. Graphically, we can confirm that there is a price jump, either increasing or decreasing when the ELP switches from liquidity maker to liquidity taker. Therefore, there exists state variable values, under which the equilibrium price function is discontinuous with ELP’s information signal.

We are not the first one to model market crashes. However, the existing literature of market crashes focuses on very different mechanisms. For example, through uninformed traders’ inference problem, Barlevy and Veronesi (2003) and Yuan (2005) both derive an upward-sloping demand function that leads to market crashes. We show how market crashes can happen through the ELP’s switching from liquidity maker to liquidity taker; this can be triggered by small shocks to either fundamental or non-fundamental information. Hence, stock market crashes are closely related to liquidity fragility. Panel B of Figure 3 shows that market demand elasticity can drop precipitously due to the ELP switching roles (more densely distributed price lines under the maker regime). As a result, the equilibrium price sensitivity to shocks can change dramatically with different prices.
5.2. Information Differentiation

In Section 4, the informed and ELP have different signals with a common component. In this section we investigate how the similarity of informed and ELP signals affect trading and liquidity fragility. The similarity between informed and ELP signals is measured by the ratio of common part ($\sigma_0^2$) to the total volatility of fundamentals for informed trader’s ($\sigma_0^2 + \sigma_f^2$) and ELP’s ($\sigma_0^2 + \sigma_d^2$) and the results are reported in Table 2. It shows that, as the information for informed trader and ELP becomes more similar, the liquidity under the taker regime decreases while the liquidity under the maker regime increases. This implies that breaks in liquidity are more severe when signals are similar. This is a consequence of an increase in the threshold value for the ELP’s signal, the liquidity and DMM’s making intensity in the maker regime, decreasing liquidity in the taker regime, and the trading intensity of the both the informed trader and the ELP.

Information similarity influences both optimal threshold and aggregate order flow. Thus, it affects the DMM’s expectation functions,

$$
\frac{d f_{V,D}(\bar{u}; \theta^*)}{d \sigma_0} = \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \bar{u}} \frac{\partial \bar{u}}{\partial \sigma_0} + \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \theta^*} \frac{\partial \theta^*}{\partial \sigma_0} + \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \sigma_0} \frac{\partial \sigma_0}{\partial \sigma_0} \tag{19}
$$

The latter two components of (19) reflect the adverse selection faced by the DMM. The indirect effect captures the change in adverse selection cost through inference and selective liquidity. From Figure 5, we can see that information similarity increases the ELP’s optimal threshold value. From Section 4 we know that higher optimal threshold signal value intensifies adverse selection. Thus, information similarity significantly increases the adverse selection through the indirect effect. Higher information similarity also reduces any information advantage, hence both the informed trader and ELP trade less aggressively. This directly eases the DMM’s adverse selection. Our numerical results show significant increases in the adverse selection, suggesting that the indirect effects are dominant in our analysis.

The results in Table 2 also show that market liquidity under taker regime, which is solely determined by the DMM, declines. Reduced liquidity in the taker regime makes informed trader trade less aggressively, due to a high expected price impact. This in turn benefits the ELP in the maker regime because the ELP trades with a less aggressive informed trader. Hence, as the information for informed trader and ELP becomes more similar, the liquidity under taker regime decreases while the liquidity under maker regime increases.
When markets become more correlated, the information traded and disseminated on different markets might also become more similar. Therefore, the high-frequency ELP, who usually trades on order flow information through back running, front running and cross-market inference, might observe information that highly related to signals known to informed traders. Our analysis suggests that information similarity will make liquidity more fragile, potentially resulting in more liquidity breaks.

[Table 2]

5.3. ELP’s Noise-to-signal Ratio

Another important question is how the ELP’s information quality affects trading and liquidity. The ELP’s signal comes from order flow information such as front-running, back running and cross-market learning; while the noise in the signal can be a consequence of the ELP’s speed advantage. Both information quality and speed advantage of the ELP can be measured by the reciprocal of the noise-to-signal ratio. This insight is used to compute the results in Table 2. From the table, as the ELP’s signal becomes noisier, the liquidity under both taker and maker regimes improves. This implies that market benefits from low information quality or slow trade from the ELP, which supports recent regulator’s concern about superior trading speed of high-frequency trading.

With increasing noise-to-signal ratio, the ELP tends to trade less aggressively. Hence, the information advantage of the informed trader increases and they trade more aggressively (as noted in Table 2). The effect on the ELP is much stronger than the effect on the informed trader. Therefore, overall traders trade less aggressively in the noise-to-signal ratio.

The implications for the DMM’s adverse selection are given in the following

\[
\frac{df_{V,D}(\bar{u}; \theta^*)}{d\sigma_{e,\theta}} = \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \bar{u}} \frac{\partial \bar{u}}{\partial \sigma_{e,\theta}} + \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \theta^*} \frac{\partial \theta^*}{\partial \sigma_{e,\theta}} + \frac{\partial f_{V,D}(\bar{u}; \theta^*)}{\partial \sigma_{e,\theta}}
\]

(20)

We know that increasing the noise-to-signal ratio increases adverse selection indirectly through the inference selective liquidity; but lowers adverse selection through the direct effect, as noted in expression (20). Our numerical analysis shows that the direct effect dominates, with improved liquidity in the taker regime, which is solely determined by the DMM.

Although liquidity in the taker regime is still lower than in the maker regime and the threshold value increases with the noise-to-signal ratio, the overall liquidity improvement and less aggressive trading reduce the adverse selection seen by the DMM (demonstrated by the last row in Table 2.
Thus, an increase in either information transparency to reduce the ELP’s noise-to-signal ratio or trading speed does not necessarily improve market quality and reduce liquidity fragility.

6. Concluding remarks

We study a rational expectation equilibrium model of fragile liquidity with high-frequency endogenous liquidity provision. Both designated market maker (DMM) and endogenous liquidity provider (ELP) can provide liquidity for the position takers in a context where markets are fragmented due to both an informational and a participation friction. We show that the ELP’s endogenous liquidity provision, coupled with DMM’s limited participation resulting from developments of trading technology and trading automation, can shrink significantly the liquidity supply from the DMM, but expand aggregate liquidity demand during market turmoil.

We use two different approaches to analyze market equilibrium. With a linear approximation, we are able to obtain a closed-form linear solution and show that endogenous liquidity provision is the source of fragile liquidity. With machine learning methods we solve the nonlinear equilibrium model. Contrasting results between the nonlinear equilibrium and linear approximation, shows the linear approximation captures the same underlying mechanism for liquidity fragility, but fails to deliver the precise results of the nonlinear equilibrium. We demonstrate how machine learning techniques can be very useful in exploring nonlinear equilibria in complex models of price formation.
REFERENCE


Table 1: Model Comparison

This table compares the market dynamics of the equilibrium result of (i) the Kyle model (no HFT), (ii) HFTm (HFT submits market orders to consume liquidity only), (iii) linear (approximation) equilibrium, and (iv) nonlinear equilibrium. The making intensity is measured by respond of liquidity maker’s demand to the aggregate market order flows; while the liquidity is measured by the respond of price to the noise demand. We report the market condition, including liquidity when ELP submits market order (in the taker regime) and limit order (in the maker regime), together with the threshold value and the probability of taking liquidity. We also report the taking intensity of informed trader and ELP trader and the making intensity of ELP and DMM. The result of linear equilibrium is based on the linear approximation assumption, while the nonlinear equilibrium results are based on the reinforcement learning (described in the Appendix B) with ten-million times of simulated data (after the convergence). The liquidity is the slope of regressing the price on the noise demand. The trading intensity is the slope of regressing the expected price or fundamental value on observed signal for taking or making intensity respectively. The parameter values are $\sigma_f = 10$, $\sigma_d = 10$, $\sigma_{e,0} = 8$, $\sigma_Z = 5$ and $\gamma = 2$.

<table>
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<tr>
<th></th>
<th>Threshold value</th>
<th>Probability</th>
<th>Liquidity under taker regime</th>
<th>Liquidity under maker regime</th>
<th>Informed taking intensity</th>
<th>ELP making intensity</th>
<th>ELP taking intensity</th>
<th>DMM making intensity</th>
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<td>Kyle</td>
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<td>1.26</td>
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<tr>
<td>Linear</td>
<td>26.77</td>
<td>3.66%</td>
<td>2.79</td>
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<td>0.50</td>
<td>1.00</td>
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<td>Nonlinear</td>
<td>42.10</td>
<td>0.10%</td>
<td></td>
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</table>


Table 2: Market Condition and Trading Intensity

Market dynamics of the nonlinear equilibrium (based on machine learning) with respect to different information similarity (in Table A) and noise level (in Table B). In Table A, the information similarity of informed trader’s and ELP’s information is measured by the ratio of common part ($\sigma_0^2$) to the total volatility of fundamentals for informed trader’s ($\sigma_0^2 + \sigma_f^2$) and ELP’s ($\sigma_0^2 + \sigma_d^2$), respectively. The standard deviation of the total volatility is selected as 10. For example, $\sigma_0^2$ and $\sigma_d^2$ should be 10 and 90 when the information similarity is 10%. Other selected values of parameters are $\sigma_{\epsilon,\theta} = 8$, $\sigma_z = 5$ and $\gamma = 5$. In Table B, the noise level of ELP’s signal is measured by $\sigma_{\epsilon,\theta}$ which is the first role in the table. Other selected values of parameters are $\sigma_0 = 0$, $\sigma_f = 10$, $\sigma_d = 10$, $\sigma_z = 5$ and $\gamma = 5$. The value in the parenthesis is the ratio of threshold value to the standard deviation of theta. In both A and B, the market condition and trading intensity are defined and estimated as in Table 1.

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<th>Information similarity</th>
<th>0%</th>
<th>2%</th>
<th>4%</th>
<th>6%</th>
<th>8%</th>
<th>10%</th>
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<td>1.679</td>
<td>1.685</td>
<td>1.666</td>
<td>1.718</td>
<td>1.723</td>
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<tr>
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<td>1.111</td>
<td>1.102</td>
<td>1.093</td>
<td>1.084</td>
<td>1.075</td>
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<td>Informed taking intensity</td>
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<td>0.749</td>
<td>0.747</td>
<td>0.745</td>
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<td>0.739</td>
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<td>0.902</td>
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<td>1.991</td>
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<table>
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<td>30.4</td>
<td>36.5</td>
<td>44.1</td>
<td>51.9</td>
</tr>
<tr>
<td>(2.08)</td>
<td>(2.20)</td>
<td>(2.37)</td>
<td>(2.58)</td>
<td>(2.82)</td>
<td>(3.02)</td>
<td></td>
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<tr>
<td>Liquidity under taker regime</td>
<td>1.732</td>
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<td>1.659</td>
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<td>Liquidity under maker regime</td>
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<td>1.120</td>
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<td>Informed taking intensity</td>
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<td>ELP taking intensity</td>
<td>0.409</td>
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<td>0.300</td>
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<td>ELP making intensity</td>
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<tr>
<td>DMM making intensity</td>
<td>1.824</td>
<td>1.771</td>
<td>1.733</td>
<td>1.689</td>
<td>1.659</td>
<td>1.619</td>
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Figure 1 The mechanism of how endogenous liquidity provision builds up fragility. The yellow shaded areas in the probability densities represent possible realizations of noisy information about the asset payoff for HFT. The shaded area in the plot on the left (right) corresponds to small (large) information shocks in normal (turbulent) times. We depict instances of liquidity demand (from informed trader), liquidity supply (from the DMM) and both liquidity demand and supply (from the ELP). The top panel illustrates the setting without high-frequency ELP, the lower left panels are for normal times, and the lower right panels are for turbulent times. Without HFT, the liquidity demand from the informed trading matches the liquidity supply from the DMM in both normal and turbulent times. The two lower panels illustrate the effect of high-frequency ELP on liquidity. During normal times (the lower left panel), ELP liquidity reduces the liquidity supply from DMM but covers the liquidity mismatch between the DMM’s liquidity supply and informed trader’s liquidity demand. During turbulent times (the lower right panel), the ELP takes liquidity, thereby exacerbating the liquidity mismatch.
Figure 2 Reinforcement learning process. Panel A describes the evolution of DMM’s expectation function with reinforcement learning. The curves represent the evolution process of DMM’s expectation functions $f_{v,m}^k$ about the fundamental. At the end of $k$th run, we plot the updated $f_{v,m}^k$ as dash line in the figure. The finally converged $f_{v,m}^k$ is also plotted as the solid line. The sequence numbers of runs are shown in the legend. Panel B describes updating process for the expectation functions with reinforcement learning. This figure plots the updating for DMM’s expectation on the fundamental during different stages of experiment. At each panel, the blue line and red line represent the expectation functions before and after the updating of given run. The red dots are the average of observed fundamental value which is the input for the updating during the given run. The six panels for the snapshots of updating are, from left to right and top to bottom: 1st, 2nd, 32nd and 120th. Parameters are $\sigma_f = 10$, $\sigma_d = 10$, $\sigma_{e,0} = 8$, $\sigma_Z = 5$, $\gamma = 2$, $\tau = 0$. 
Panel A describes price function under taker (blue) and maker (red) regime. For maker regime, the equilibrium price depends on both $\theta$ and $\delta$. The solid and dashed lines represent for value, respectively. Panel B describes the iso-price lines in the information noise shock $\varepsilon$ and noise trading shock $\zeta$. The red and blue solid lines represent the maker and taker regimes, respectively; the dashed line represents the threshold boundary between ELP’s liquidity taker and maker regions, with the realization values of the systematic and idiosyncratic risk components $\delta_j = 0$ and $\delta_d = 0$. Panel C describes the influence of $\tau$. Left and right in Panel C show price function under taker and maker regime, respectively. The Blue lines show $\tau = 0.02$ while the red lines show $\tau = 0$. Other parameters remain the same as in Figure 2.

**Figure 3 Equilibrium functions.** Panel A describes price function under taker (blue) and maker (red) regime. For maker regime, the equilibrium price depends on both and. The solid and dashed lines represent for value, respectively. Panel B describes the iso-price lines in the information noise shock $\varepsilon$ and noise trading shock $\zeta$. The red and blue solid lines represent the maker and taker regimes, respectively; the dashed line represents the threshold boundary between ELP’s liquidity taker and maker regions, with the realization values of the systematic and idiosyncratic risk components $\delta_j = 0$ and $\delta_d = 0$. Panel C describes the influence of $\tau$. Left and right in Panel C show price function under taker and maker regime, respectively. The Blue lines show $\tau = 0.02$ while the red lines show $\tau = 0$. Other parameters remain the same as in Figure 2.
Figure 4 DMM’s inference Problem. Panel A describes DMM’s optimal making intensity as a function of the DMM’s market participation for different degrees of ELP’s incentive for supplying liquidity. Panel B shows DMM’s expectation function with different $\tau$. We run experiments for each given $\tau$ to learn the rational expectations for all type agents. The expectations of DMM $f^{\text{ml}}_{v,M} = (\mu f^{\text{ml}}_{v,M} + (1-\mu)\tau f^{\text{ml}}_{v,M})/(\mu + (1-\mu)\tau)$, in which $f^{\text{ml}}_{v,M}/f^{\text{ml}}_{v,M}$ is the DMM’s value expectation in the regime ELP submits market/limit order, respectively, and $\mu$ is the probability of ELP submitting market order. Panel C describes the effect of the optimal threshold value on the making intensity under linear approximation. Panel D shows DMM’s expectation functions with different $\theta$ based on the rational learning for all types of traders. Parameter values are $\sigma_f = 10$, $\sigma_d = 10$, $\sigma_{v,\theta} = 8$, $\sigma_z = 5$, $\gamma = 2$ in Panel C and $\tau = 0$ in Panel D.
Figure 5 Endogenous liquidity provision and Implications. Panel A describes the best response function of the endogenous liquidity provision for trading costs of \( \gamma = 2, 2.2 \), representing by the blue dashed and solid lines respectively; the red dashed line represents the best response function for the uncertainty, while the pink dotted line represents the initial direct effect of the reduction in trading cost from \( \gamma = 2 \) to \( \gamma = 2.2 \). Panel B describes the equilibrium price with respect to information noise shocks for three realizations of noise trading \( Z = 21.19, 11.19, 1.19 \), representing by the dotted, dashed and solid lines, respectively; here the realization values of systematic and idiosyncratic risk components are \( \delta_f = 0 \) and \( \delta_d = 0 \). Panel C describes the effect of various model parameters on the optimal threshold value.
This appendix provides the proofs of lemmas and propositions in Appendix A and details about reinforcement learning in Appendix B.

Appendix A: Proofs

A1. Proof of Lemma 1:

We firstly derive the equilibrium trading strategies and price functions given the ELP’s optimal threshold value of cut-off endogenous liquidity provision decision. For convenience, we first introduce the following notations on the conditional expectation functions of the ELP on the price and payoff, the informed on the price and the DMM on the payoff, respectively,

\[ f_p(E) = E_p(E_{\theta|\bar{\theta}} = \delta_0 + \delta_r + \epsilon_0); \quad f_v(E) = E_v(E_{\theta|\bar{\theta}} = \delta_0 + \delta_d + \epsilon_0, \bar{u}); \]
\[ f_p(\bar{\theta}) = \mu E_{\bar{p}|\bar{\theta}|\bar{\theta}| > \theta^*} + (1 - \mu) E_{\bar{p}|\bar{\theta}|\bar{\theta}| \leq \theta^*}; \]
\[ f_v(\bar{u}) = \mu E_{|\bar{u}|\bar{\theta}|, |\bar{\theta}| > \theta^*} + \tau (1 - \mu) E_{|\bar{u}|\bar{\theta}|, |\bar{\theta}| \leq \theta^*}; \]

where \( \mu = P(1|\bar{\theta}| > \theta^*) \) represents the probability when the ELP makes directional bet and consumes the liquidity.

**The ELP’s trading strategy:** When receiving an extreme signal on private information \( \bar{\theta} \) (i.e., \( |\bar{\theta}| > \theta^* \)), the ELP submits a market order to consume liquidity, conditional on that the ELP knows that the equilibrium price takes the form \( \bar{p}_r \). Applying the Bayesian rule and calculating the conditional expectation, we have:

\[ x_r(\bar{\theta}) = \gamma E_{|\bar{\theta}|}\bar{\theta}| = \gamma \frac{\sigma_r^2}{\sigma_r^2 + \sigma_{\bar{\theta}}^2} \bar{\theta} - \gamma f_{p,E}(\bar{\theta}). \]  
\[ (A1) \]

In contrast, when receiving moderate signal (i.e., \(|\bar{\theta}| \leq \theta^* \), the ELP submits a limit order to supply liquidity, conditional on his private information \( \bar{\theta} \) and the aggregate market order \( \bar{u}_r \). This inference problem of the ELP can be summarized by the expectation function \( f_{v,E}(\bar{\theta}, \bar{u}) \).

Accordingly, the ELP’s trading strategy under maker regime is given by

\[ x_m(\bar{\theta}; \bar{u}, \bar{p}) = \gamma f_{v,E}(\bar{\theta}, \bar{u}) - \gamma \bar{p}. \]  
\[ (A2) \]

**Informed trader’s trading strategy:** Different from the ELP, when submitting market order, informed trader is uncertain about equilibrium price regime. Informed trader anticipates that the equilibrium price is under maker and taker regimes with probability \( 1 - \mu \) and \( \mu \), respectively; while \( \mu = P(1|\bar{\theta}| > \theta^*) \). Applying the Bayesian rule and calculating the conditional expectation, we have
\[ x_i(\bar{s}) = \gamma E[\bar{V} - \bar{p}|l_i] = \gamma [\bar{s} - \mu E(\bar{p}_i|\bar{s} = \bar{s}_0 + \bar{d}_0, |\bar{\theta}| > \theta^*)\]
\[ -(1 - \mu)E(\bar{p}_M|\bar{s} = \bar{s}_0 + \bar{d}_0, |\bar{\theta}| \leq \theta^*)] = \gamma \bar{s} - \gamma f_{\mu,i}(\bar{s}). \]  

DMM’s trading strategy: Except for the additional uncertainty about equilibrium regime, the DMM is now facing uncertainty about order execution. Considering the ELP’s speed advantage, the DMM’s optimization problem can be described as:

\[ \max_{x_D} \mu \left\{ E[x_0(\bar{V} - \bar{p})] - x_D^2 \right\} + (1 - \mu) \left\{ E[\tau x_D(\bar{V} - \bar{p})] - x_D^2 \right\}. \]  

Under different equilibrium regime, the information sets for the DMM are different. Calculating the first order condition, we obtain the demand function for the DMM as follow,

\[ x_D = \gamma \left[ \frac{\mu E(\bar{V} | \bar{u} = x_i(\bar{s}) + \bar{z}, |\bar{\theta}| > \theta^*) + \tau (1 - \mu) E(\bar{V} | \bar{u} = x_i(\bar{s}) + \bar{z}, |\bar{\theta}| \leq \theta^*)}{\mu + \tau^2(1 - \mu)} \right] - \frac{\mu + \tau(1 - \mu)}{\mu + \tau^2(1 - \mu)} \bar{p}. \]  

A2. Proof of Propositions 1 and 2

To solve the market equilibrium, we substitute the demand functions into the market clearance condition under taker and maker regimes, respectively. Under the taker regime, the DMM is the only liquidity provider, therefore,

\[ \bar{u}_T = \gamma \left[ \frac{\mu f_{T,D}(\bar{u}) + \tau (1 - \mu) f_{T,D}(\bar{u})}{\mu + \tau^2(1 - \mu)} - \frac{\mu + \tau(1 - \mu)}{\mu + \tau^2(1 - \mu)} \bar{p} \right] = 0 \]  

Simplifying (A6), we obtain the equilibrium price under taker regime. Instead, under the maker regime, both the DMM and ELP provide liquidity,

\[ \bar{u}_M + \gamma f_{V,E}(\bar{\theta}, \bar{u}) - \gamma \bar{p} + \tau \left( \frac{\mu f_{T,D}(\bar{u}) + \tau (1 - \mu) f_{T,D}(\bar{u})}{\mu + \tau^2(1 - \mu)} - \frac{\mu + \tau(1 - \mu)}{\mu + \tau^2(1 - \mu)} \bar{p} \right) = 0 \]  

Simplifying (A7), we obtain the equilibrium price under taker regime.

We now characterize the endogenous equilibrium by determining the endogenous threshold value. Unlike the DMM, the ELP has no obligation to supply liquidity, depending on his information and market conditions. Note that both the demand functions and expected payoffs can be expressed as functions of the exogenous threshold value. We then endogenously determine the optimal threshold value by making the expected payoffs for the ELP be equal under taker and maker regimes. Calculating the payoffs for taker and maker regimes, respectively,
$U_{E,M} = E \left[ \frac{1}{\gamma} \left( \frac{\gamma \tau \Phi f_{V,E}(\tilde{\theta}, \tilde{u}) - \gamma \tau f_{V,M}(\tilde{u}) - \theta \tilde{u}}{\gamma (\theta + \tau \Phi)} \right) \right]^2 \tilde{\theta} = \theta$; \hfill (A8)

$U_{E,T} = \frac{1}{\gamma} \left( \frac{\sigma_1^2 + \sigma_0^2}{\sigma_1^2 + \sigma_0^2 + \sigma_{e,\theta}^2} \theta - f_{p,E}(\theta) \right)^2$. \hfill (A9)

Comparing (A8) and (A9) leads to the optimal threshold value for cut-off endogenous liquidity provision decision in Proposition 2.

In equilibrium, the beliefs of all traders are consistent with the jointly conditional probability distribution, implying that

$f_{p,E}(\theta) = E \left( \frac{\theta(\tilde{u}) + \gamma f_{V,D}(\tilde{u})}{\gamma \Phi} \right) | \tilde{\theta}, \tilde{u} = \gamma \tilde{s} - \gamma f_{p,I}(\tilde{s}) + \gamma \frac{\sigma_1^2 + \sigma_0^2}{\sigma_1^2 + \sigma_0^2 + \sigma_{e,\theta}^2} \tilde{\theta} - \gamma f_{p,E}(\tilde{\theta}) + \tilde{z} \right)$.

$f_{V,E}(\tilde{\theta}, \tilde{u}) = E \left( \tilde{V} | \tilde{\theta}, \tilde{u} = \gamma \tilde{s} - \gamma f_{p,I}(\tilde{s}) + \tilde{z} \right)$.

$f_{p,I}(\tilde{s}) = \mu E \left( \frac{\theta(\tilde{u}) + \gamma f_{V,D}(\tilde{u})}{\gamma \Phi} \right) \left| \tilde{s}, \tilde{u} = \gamma \tilde{s} - \gamma f_{p,I}(\tilde{s}) + \gamma \frac{\sigma_1^2 + \sigma_0^2}{\sigma_1^2 + \sigma_0^2 + \sigma_{e,\theta}^2} \tilde{\theta} - \gamma f_{p,E}(\tilde{\theta}) + \tilde{z}, |\tilde{\theta}| > \theta^* \right)$

$+ (1 - \mu) E \left( \frac{\theta(\tilde{u}) + \gamma f_{V,D}(\tilde{u})}{\gamma (\theta + \tau \Phi)} \right) \left| \tilde{s}, \tilde{u} = \gamma \tilde{s} - \gamma f_{p,I}(\tilde{s}) + \tilde{z}, |\tilde{\theta}| \leq \theta^* \right)$

$f_{V,D}(\tilde{u}) = \mu E \left( \tilde{V} | \tilde{u} = \gamma \tilde{s} - \gamma f_{p,I}(\tilde{s}) + \gamma \frac{\sigma_1^2 + \sigma_0^2}{\sigma_1^2 + \sigma_0^2 + \sigma_{e,\theta}^2} \tilde{\theta} - \gamma f_{p,E}(\tilde{\theta}), |\tilde{\theta}| > \theta^* \right)$

$+ \tau (1 - \mu) E \left( \tilde{V} | \tilde{u} = \gamma \tilde{s} - \gamma f_{p,I}(\tilde{s}) + \tilde{z}, |\tilde{\theta}| \leq \theta^* \right)$.

The above four expressions describe the ‘fixed-function’ problem that characterizes the threshold value of the endogenous equilibrium.

**A3. Proof of Lemma 2 and Proposition 3**

Lemma 2 and Proposition 3 describe the equilibrium for limited market participation (LMP). The proof is similar to Appendices A1 and A2. To focus on the LMP, we switch off the possibility of cross learning between the informed trader and the ELP by assuming that there is no common information component (i.e., $\sigma_0^2 = 0$); and further examine limited market participation through exogenous liquidity provision. In this case, the equilibrium has a closed form linear solution. Under these assumptions, we calculate traders' expectation functions based on a linear equilibrium.

Specifically, we assume that the informed and ELP market orders are given by $x_{r}(\tilde{\theta}) = \tilde{\mu}_{e}\tilde{\theta}$ and $x_{i}(\tilde{s}) = \tilde{\mu}_{i}\tilde{s}$. We can then derive DMM’s demand function. Different from (A5), there is no information about the ELP’s selective liquidity provision due to it is exogenous; hence, we have
\[ x_D = y \left[ \frac{\mu E[\tilde{V} | \tilde{u}] = x_I(\tilde{s}) + x_{E,T}(\tilde{\theta}) + \tilde{z}_1 + \tau(1 - \mu)E[\tilde{V} | \tilde{u}] = x_I(\tilde{s}) + \tilde{z}_1 - \mu + \tau(1 - \mu)\tilde{p}}{\mu + \tau^2(1 - \mu)} \right] \]  
\[ = y \left( \frac{1}{\beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \sigma_z^2} \right) \beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \sigma_z^2 + \frac{c}{\beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \sigma_z^2} \right) - y \frac{1 + c}{\beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \sigma_z^2} \tilde{p}. \]

Also, based on linear price functions \( \tilde{p}_T = \lambda_T \tilde{u} \), and \( \tilde{p}_M = \lambda_{\theta,M} \theta + \lambda_M \tilde{u} \), we obtain the market orders of the informed and ELP,

\[ x_T(\tilde{\theta}) = yE[\tilde{V} - \tilde{p}|I_E] = y \frac{\sigma_f^2}{\sigma_f^2 + \sigma_{\tilde{e}\theta}^2} \tilde{\theta} - y \lambda_T \beta_{\tilde{e}} \tilde{\theta}; \]
\[ x_I(\tilde{\delta}_a) = yE[\tilde{V} - \tilde{p}|I_I] = y[1 - \mu \lambda_T \beta_I - (1 - \mu) \lambda_M \beta_I] \tilde{\delta}_a. \]

Therefore, equations (A-10) to (A-12), coupled with the other equilibrium conditions (similar to Appendices A1 and A2), lead to

\[ E[V|I_E] = \frac{\sigma_I^2}{\sigma_f^2 + \sigma_{\tilde{e}\theta}^2} \tilde{\theta} + E[\tilde{\delta}_a | \tilde{u} = \beta_I \tilde{\delta}_a + \tilde{z}] = \frac{\sigma_I^2}{\sigma_f^2 + \sigma_{\tilde{e}\theta}^2} \tilde{\theta} + \frac{\beta_I \sigma_a^2}{\beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \sigma_z^2} \tilde{u}; \]

\[ y(\phi_D \tilde{u} - \phi_{D,P} \tilde{p}) + u = 0; \]
\[ \tilde{u} + \tau y(\phi_D \tilde{u} - \phi_{D,P} \tilde{p}) + y(\phi_{E,\theta} \tilde{\theta} + \phi_E \tilde{u} - \tilde{p}) = 0; \]

which yield the following system of equations determining market equilibrium,

\[ \frac{(c + 2\tau)(1 + (1 + 2c)\tau)}{(c + \tau)(1 + (1 + 2c)\tau)} \beta_i \]
\[ + y \frac{(1 + (1 + 2c)\tau + (1 + ct) \beta_i (\sigma_f^2 + \sigma_{\tilde{e}\theta}^2)) \beta_i}{\beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \beta_i \sigma_a^2 + \sigma_z^2} \]
\[ + y \frac{(1 + (1 + 2c)\tau)(1 + ct) \beta_i (\sigma_f^2 + \sigma_{\tilde{e}\theta}^2)) \beta_i}{\beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \beta_i \sigma_a^2 + \sigma_z^2} \]
\[ = 0; \]
\[ 2(\sigma_f^2 + \sigma_{\tilde{e}\theta}^2) \beta_E + \frac{cy}{\beta_{\tilde{e}}\sigma_f^2 + \beta_{\tilde{e}}\sigma_a^2 + \beta_i \sigma_a^2 + \sigma_z^2} \]
\[ = 0. \]

**A4. Proof of Proposition 4**

**Trading strategies and price functions:** The linear approximation helps to obtain a closed-form linear solution. Based on Assumption 1, we conjecture an equilibrium with linear price and demand functions. We first specify the price function as

\[ \tilde{p} = \tilde{p}_T 1_{[\tilde{p}_T > 0]} + \tilde{p}_M 1_{[\tilde{p}_M > 0]}, \]

with \( \tilde{p}_T = \lambda_T \tilde{u}_T \) and \( \tilde{p}_M = \lambda_{\theta,M} \theta + \lambda_M \tilde{u}_M \),

where \( \tilde{u}_T \) and \( \tilde{u}_M \) are defined in (4a) and (4b) and coefficients \( \lambda_T \) and \( \lambda_M \) are determined in the equilibrium.
Trader $i$ chooses the demand function $x_i$ to maximize the expected utility conditional on his belief. The procedure to derive the equilibrium is similar to Appendices A1 and A2. As in Appendix A3, the linear character helps to solve the inference problem. Therefore traders’ trading strategies can be linearly represented as:

$$x_i = \hat{\beta}_i \delta_a; \quad x_{E,T} = \hat{\beta}_E \delta; \quad x_{E,M} = \gamma \left( \hat{\phi}_{E,T} \delta + \hat{\phi}_{E} \bar{u} - \bar{p} \right); \quad x_D = \gamma \left( \hat{\phi}_D \bar{u} - \hat{\phi}_{D,P} \right);$$

where coefficients $(\hat{\beta}_i, \hat{\beta}_E, \hat{\phi}_{E,T}, \hat{\phi}_E, \hat{\phi}_D)$ are the functions of parameters $(\sigma^2_f, \sigma^2_{\xi}, \alpha^2_a, \alpha^2_a, \sigma, \tau)$ given by

$$\hat{\beta}_i = \frac{1}{\frac{1}{y} + (\mu \lambda_T + (1 - \mu) \lambda_M)}; \quad \hat{\beta}_E = \frac{\sigma^2_f}{\sigma^2_f + \sigma^2_{\epsilon}}; \quad \hat{\phi}_{E,T} = \frac{\beta_i \alpha^2_a}{\beta_i^2 \alpha^2_a + \sigma^2_f}; \quad \hat{\phi}_E = \frac{\mu + \tau(1 - \mu)}{\mu + \tau^2(1 - \mu)}; \quad \hat{\phi}_D = \frac{\beta_i \alpha^2_a}{\beta_i^2 \alpha^2_a + \sigma^2_f};$$

$$\mu = 2N \left( -\frac{\theta^*}{\sigma^2_f + \sigma^2_{\xi}} \right);$$

here, parameters $\sigma^2_f$ and $\sigma^2_{\epsilon}$ represent conditional variance $VAR(\delta_f | \bar{\delta} > \theta^*)$ and $VAR(\bar{\epsilon}_\theta | | \bar{\delta} > \theta^*)$, respectively, to be introduced next.

**Updating posterior variance:** We show how to update the posterior distribution for fundamental component $\delta_f$ and information noise $\bar{\epsilon}_\theta$ through the definition of variance and expectation. Here, we focus on the fundamental component $\delta_f$. The noise part $\bar{\epsilon}_\theta$ follows a similar procedure and is thus omitted. We first focus on the maker regime updating,

$$E \left[ \delta_f | \bar{\delta} \leq \theta^* \right] = E \left[ E \left[ \delta_f | -\theta^* + \bar{\epsilon}_\theta < \delta_f < \theta^* + \bar{\epsilon}_\theta \right] \right].$$

In (A7), the outer layer expectation on the right-hand side is to calculate the expectation over noise term $\bar{\epsilon}_\theta$,

$$E \left[ \delta_f | \bar{\delta} \leq \theta^* \right] = \int_{-\infty}^{\theta^*} \left\{ E \left[ \delta_f | -\theta^* - t < \delta_f < \theta^* - t \right] \right\} dF_t(t).$$

For the variance, it is more complicated. Based on the definition, we have

$$VAR \left[ \delta_f | \bar{\delta} \leq \theta^* \right] = E \left[ E \left[ \delta_f | -\theta^* + \bar{\epsilon}_\theta < \delta_f < \theta^* + \bar{\epsilon}_\theta \right] \right]$$

$$VAR \left[ E \left[ \delta_f | -\theta^* + \bar{\epsilon}_\theta < \delta_f < \theta^* + \bar{\epsilon}_\theta \right] \right].$$

For the first part (the expected conditional variance), we have
\[ E \left[ \text{VAR} \left[ \bar{\delta}_t | - \theta^* - \varepsilon_\theta < \bar{\delta}_t < \theta^* - \varepsilon_\theta \right] \right] = \int_{-\infty}^{\infty} \{ \text{VAR} \left[ \bar{\delta}_t | - \theta^* - t < \bar{\delta}_t < \theta^* - t \right] \} \, dF_t(t). \]  

(A21) For the second part (the variance of conditional expectation), we have
\[ \text{VAR} \left[ E[\bar{\delta}_t | - \theta^* - \varepsilon_\theta < \bar{\delta}_t < \theta^* - \varepsilon_\theta] \right] = \int_{-\infty}^{\infty} \{ E[\bar{\delta}_t | - \theta^* - t < \bar{\delta}_t < \theta^* - t] - E \left[ \bar{\delta}_t \left| \| \varepsilon^\theta \| \leq \theta^* \right. \right] \}^2 \, dF_t(t). \]

Therefore the posterior distribution is pinned down by the truncated normal distribution,
\[ E[\bar{\delta}_t | - \theta^* - t < \bar{\delta}_t < \theta^* - t] \]
and
\[ \text{VAR}[\bar{\delta}_t | - \theta^* - t < \bar{\delta}_t < \theta^* - t]. \]

Both the expectations have “closed form” solutions. Consequently, we obtain the posterior distributions about the fundamental component \( \bar{\delta}_f \) and information noise \( \varepsilon^\theta \) for maker regime.

We then examine the posterior distribution for the fundamental component \( \bar{\delta}_f \) and information noise \( \varepsilon^\theta \) under taker regime. On the updating of the posterior mean, similar to the maker regime, we have
\[ E \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| \leq \theta^* \right. \right] > \theta^* = \int_{-\infty}^{\infty} \{ E[\bar{\delta}_f | \theta < -\theta^* - t \text{ or } \theta > \theta^* - t] \} \, dF_t(t). \]  

(A23) To update the posterior variance, based on the basic properties of truncated normal distribution, we have
\[ \text{VAR}[\bar{\delta}_f] = \mu \text{VAR}[\bar{\delta}_f | \| \varepsilon^\theta \| \leq \theta^*] + (1 - \mu) \text{VAR} \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| > \theta^* \right. \right] + \mu \left( E \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| > \theta^* \right. \right] - E[\bar{\delta}_f] \right)^2 \]  

(A24) \[ + (1 - \mu) \left( E \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| \leq \theta^* \right. \right] - E[\bar{\delta}_f] \right)^2. \]

Solving the above equation and simplifying the expression, we have
\[ \text{VAR} \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| > \theta^* \right. \right] = \frac{\text{VAR}[\bar{\delta}_f]}{\mu} - \left( \frac{1}{\mu} - 1 \right) \text{VAR} \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| \leq \theta^* \right. \right] \]  

\[ - \left( E \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| > \theta^* \right. \right] - E[\bar{\delta}_f] \right)^2 - (1/\mu - 1) \left( E \left[ \bar{\delta}_f \left| \| \varepsilon^\theta \| \leq \theta^* \right. \right] - E[\bar{\delta}_f] \right)^2. \]  

(A25) This procedure of updating the posterior variance is used in the DMM’s inference problem.

**Market clearance and optimal threshold value**: To solve the equilibrium, we substitute the demand functions into the market clearance condition under taker and maker regimes, respectively. Under the taker regime, the DMM is the only liquidity provider, therefore,
\[ \gamma (\phi_D u - \phi_{M\tau} \bar{\theta}) + u = 0. \]  

(A26) Comparing to the equilibrium price under taker regime \( \bar{p}_\tau = \lambda_{\tau} \bar{u} \), we have
\[ \hat{\lambda}_r = \frac{1}{\phi_{M,p}} \left( \hat{\phi}_D + \frac{1}{\gamma} \right) ; \]  

(A27)

Under the maker regime, both the DMM and ELP provide liquidity,
\[ \bar{u} + \tau y (\phi_{M,p} \bar{u} - \phi_{M,p} \bar{p}) + y (\phi_{E,\theta} \bar{\theta} + \phi_e \bar{u} - \bar{p}) = 0. \]  

(A28)

Comparing to the equilibrium price under maker regime \( \bar{p}_M = \lambda_{\theta,M} \bar{\theta} + \lambda_M \bar{u} \), we have
\[ \hat{\lambda}_M = \frac{\mu + \tau^2 (1 - \mu)}{(1 + \tau \mu + 2 \tau^2 (1 - \mu))(1 + \tau \phi_M + \phi_E)} ; \]  

(A29)

\[ \hat{\lambda}_{\theta,M} = \frac{\mu + \tau^2 (1 - \mu)}{(1 + \tau \mu + 2 \tau^2 (1 - \mu)) \sigma_f^2 + \sigma_{e,\theta}^2} ; \]  

(A30)

Lastly, we determine the equilibrium cut-off level. Calculating the utilities for taker and maker regimes, respectively:
\[ U_{E,M} = E \left[ \frac{y}{2} (\phi_{E,\theta} \bar{\theta} + \phi_e \bar{u} - \bar{p})^2 \right] = E \left[ \frac{y}{2} ((\phi_{E,\theta} - \lambda_{\theta,M}) \bar{\theta} + (\phi_e - \lambda_M) \bar{u})^2 \right] \]
\[ = \frac{y}{2} (\phi_{E,\theta} - \lambda_{\theta,M})^2 \bar{\theta}^2 + \frac{y}{2} (\phi_e - \lambda_M)^2 (\beta_f^2 \sigma_d^2 + \sigma_e^2) ; \]
\[ U_{E,M} = \frac{\beta_f^2 \bar{\theta}^2}{2 \gamma} . \]

Therefore, comparing the two utility functions, the equilibrium condition implies that the threshold value can be represented as
\[ \theta^* = \frac{(\hat{\lambda}_M - \phi_e)(\sigma_d^2 + \sigma_e^2)}{\sigma_f^2} \sqrt{\frac{1}{(1 + \gamma \hat{\lambda}_r)^2} \left( \frac{\beta_f^2 \sigma_d^2 + \sigma_e^2}{\tau \mu + \tau^2 (1 - \mu)} \right)^2} \]  

(A31)

A5. Equilibrium in Kyle and HFTm Models

**Kyle Model:** In this situation, there is no high-frequency trader. Without ELP, the position takers, informed trader and liquidity traders, trade with the DMM, who is the only liquidity supplier on the market. Informed trader’s trading strategy is similar to Appendix A4 (without additional uncertainty) with taking intensity
\[ \beta_I = \frac{\gamma}{1 + \gamma \lambda} \]  

(A32)

For the DMM, similar to Appendix A4 without additional uncertainty of the ELP’s endogenous switching, the making intensity is given by
\[ \phi = \frac{\beta_I \sigma_d^2}{\beta_f^2 \sigma_d^2 + \sigma_e^2} . \]  

(A33)
Substituting the demand functions of informed trader and DMM into the market clearance condition \((\lambda = \phi + 1/\gamma)\), we derive the following condition to be solved for the equilibrium:

\[
\frac{\gamma - \beta_I}{\gamma \beta_I} = \frac{\beta_I \sigma_d^2}{\beta_I^2 \sigma_d^2 + \sigma_\theta^2} + \frac{1}{\gamma \lambda}
\]  

\[(A34)\]

**HFTm Model:** In this situation, high-frequency trader demands liquidity by submitting market orders only. Therefore, the position taker, the informed trader, ELP and liquidity traders, trade with the DMM. Similar to Appendix A1, the taking intensities for informed trader and ELP are given, respectively, by

\[
\beta_I = \frac{\gamma}{1 + \gamma \lambda}; \quad \beta_E = \frac{\sigma_f^2}{\sigma_f^2 + \sigma_{\theta,\beta}^2} \frac{\gamma}{1 + \gamma \lambda}
\]  

\[(A35)\]

The DMM infers the fundamental information from the aggregate order flow, with the making intensity decided by,

\[
E[\bar{V}|\bar{u}] = E[\bar{V}| \bar{u} = x_I + x_E + \bar{z}] = E[\bar{V}| \bar{u} = \beta_I \delta_d + \beta_E \theta + \bar{z}]
\]  

\[
= \frac{\beta_I \sigma_d^2 + \beta_E \sigma_f^2}{\beta_I^2 \sigma_d^2 + \beta_E^2 (\sigma_f^2 + \sigma_{\theta,\beta}^2) + \sigma_\theta^2} \bar{u}.
\]  

Therefore, the equilibrium is determined by the above taking and making intensities,

\[
\lambda = \frac{\beta_I \sigma_d^2 + \beta_E \sigma_f^2}{\beta_I^2 \sigma_d^2 + \beta_E^2 (\sigma_f^2 + \sigma_{\theta,\beta}^2) + \sigma_\theta^2} + \frac{1}{\gamma \lambda}
\]  

\[(A37)\]

**A6. Proof of Corollaries 2 and 3**

**Proof of Corollary 2:** Taking total differentiation to the best response functions with respect to \(Q\), we have

\[
\frac{d\mu}{dQ} = \frac{\partial f_\mu}{\partial Q} + \frac{\partial f_\mu}{\partial \theta} \frac{d\theta}{dQ}; \quad \frac{d\theta}{dQ} = \frac{\partial f_\theta}{\partial Q} + \frac{\partial f_\theta}{\partial \mu} \frac{d\mu}{dQ}.
\]  

\[(A38)\]

Combing the above linear equations, we have

\[
\frac{d\mu}{dQ} = \frac{\partial f_\mu}{\partial Q} + \frac{\partial f_\mu}{\partial \theta} \left( \frac{d\theta}{dQ} + \frac{\partial f_\theta}{\partial \mu} \frac{d\mu}{dQ} \right);
\]  

\[(A39)\]

from which, we obtain the total effect for the uncertainty. The procedure to obtain the total effect for liquidity provision decision is the same.

**Proof of Corollary 3:** We assume the realizations of the fundamental components \(\delta_f\) and \(\delta_d\) to be zero. In equilibrium, the price function is given by

\[
\bar{p}(\bar{\varepsilon}_\theta, \bar{z}) = \bar{p}_T(\bar{\varepsilon}_\theta, \bar{z})1_{|\theta| > \theta^*} + \bar{p}_M(\bar{\varepsilon}_\theta, \bar{z})1_{|\theta| \leq \theta^*}.
\]
Substituting the state variable value into Proposition 4, we obtain

\[
\tilde{p}_T(\tilde{\epsilon}_\theta, \tilde{z}) = \frac{\tilde{\lambda}_T}{\tilde{\gamma} + (\mu \tilde{\lambda}_T + (1 - \mu) \tilde{\lambda}_M) \tilde{\epsilon}_\theta + \tilde{\lambda}_T \tilde{z}}
\]

(A40)

\[
\tilde{p}_M(\tilde{\epsilon}_\theta, \tilde{z}) = \frac{\tilde{\sigma}_f^2}{\tilde{\sigma}_f^2 + \tilde{\sigma}_{\epsilon, \theta}^2} \tilde{\epsilon}_\theta + \tilde{\lambda}_M \tilde{z}.
\]

(A41)

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**Appendix B: Reinforcement Learning for Nonlinear Equilibrium**

We demonstrate how machine learning techniques can be used to solve nonlinear equilibrium in more complicated financial market microstructure. Because of strategic trading behaviours of different group of investors with asymmetry information, most of the microstructure research relies on linear equilibrium to predict price efficiency and market liquidity. Facing computational constrain, it becomes difficult in dealing with nonlinear equilibrium, such as the one we encountered in this paper. Applying the reinforcement learning algorithm to the baseline model in Section 2, we first investigate how well the linear approximation equilibrium captures the same endogenous liquidity provision mechanism for the nonlinear equilibrium, together with predicting the variables of interest to market participants. We focus on both the similarity and difference among the results generated by machine learning techniques and linear approximation methodology. The analysis provides the robustness of the mechanism under the linear approximation methodology. It also brings some new evidence and insights of machine learning techniques to theoretical microstructure research by synthesizing the rational expectations equilibrium literature with machine learning literature. It is worth emphasizing that machine learning algorithms are often highly non-parametric and do not pre-specify a functional form. The non-parametricity should not be viewed as a drawback, as the algorithms are designed to be adaptive so that they can extract patterns in experimental observations through bootstrap that parametric models may not recognize. As a result, machine learning algorithms often provide more precise results and therefore are better candidates for our investigation of more complicated models.

To illustrate the nature of the equilibrium fixed point problem, we first review the problem in a Kyle type of model. Usually, in equilibrium traders and market maker are strategic in their trading and pricing, characterizing “forecast the forecasts of others” (Foster and Viswanathan, 1996). A
trader’s trading strategy depends on the trading strategies of the others and market maker’s pricing rules, which in turn also affects trader’s trading strategies. This endogenous feedback of “one agent’s strategy affects other agents’ strategies that affect himself own strategy” (Foster and Viswanathan, 1996) can be characterized by a fixed-point problem that considerably complicates the inference problem in solving the equilibrium. Under some specific model setups, the equilibrium structure can be linear, which makes the inference problem easier. The linear equilibrium is then obtained through a guess-and-verify procedure by solving a fixed-point problem. However, there are two prominent requirements that allow this to work: the specific equilibrium structure needs to be known in advance and the filter problem can be solved. In our model, the equilibrium structure is unknown, making the equilibrium inference problem almost impossible via traditional theoretical methodology. In the following, we apply machine learning technique in four steps to determine the equilibrium. We first calculate equilibrium expectation functions given the threshold value and then determine the endogenous threshold value.

**B1. Generating Observations:**

Before describing the belief updating after each experimental in detail, we first provide a short review of how we generate the observations in each experimental given the current belief. The observation $i$ in the $k$–th experimental we drew can be described as the input-output pair $\Omega^{k,i} = \{(\delta_0, \delta_f, \delta_d, \delta_d, \varepsilon), (\beta, \tilde{u}, \tilde{U}_{E,T}, \tilde{U}_{E,M})\}$. Specifically, given the current expectation functions, $f_{\beta,p,l,k}^i, f_\beta^k, f_{\beta,E}^k, f_{\beta,M}^k$, we drew the realization values of random input variables $(\delta_0, \tilde{\delta}_f, \tilde{\delta}_d, \tilde{\delta}_d, \varepsilon)$. Based on these values, we calculate the demand functions for different types of traders. With market clearance condition, we solve the market equilibrium and obtain the output variables $(\tilde{p}, \tilde{u}, \tilde{U}_{E,T}, \tilde{U}_{E,M})$, which form an observation about the input-output pair. Following, we describe the details about generating the observations:

20 More specific, in Kyle (1985), informed trader’s demand function is linear $x = \beta v$. Due to strategic trading and pricing, market maker sets a linear price $p = \lambda u$, with $\lambda = \frac{\beta \sigma^2}{\beta^2 \sigma^2 + \sigma_z^2}$. Informed trader’s trading strategy depends on the trading strategies of market maker’s pricing rules, which in turn also depend on the informed trader’s trading strategies. When informed trader solves the optimization problem, his trading strategy follows $x = \frac{v}{\lambda} = \frac{(\beta^2 \sigma^2 + \sigma_z^2)}{2 \beta \sigma^2} v$, creating a fixed point problem, $\frac{(\beta^2 \sigma^2 + \sigma_z^2)}{2 \beta \sigma^2} = \beta$, with a solution of $\beta = \sigma / \sigma_z$. 

9
**Step 1:** Draw the realizations of input random variables based on their distributions and calculate the fundamental value and signals that the informed trader and ELP receive: 
\[ z_{k,i} \sim N(0, \sigma_k^2); \epsilon_{0,k}^i \sim N(0, \sigma_{\epsilon_0}^2); \delta_{k,i}^z \sim N(0, \sigma_z^2); \delta_{k}^s \sim N(0, \sigma_s^2); \delta_{0}^k \sim N(0, \sigma_0^2). \] 
Based on the realization values, we calculate the realization values of fundamental value and signal \( v_{k,i} = \delta_{0}^k + \delta_{k}^z + \delta_{s}^k; \theta_{k,i} = \delta_{0}^k + \delta_{k}^z + \epsilon_{0,k}^i; s_{k,i} = \delta_{0}^k + \delta_{s}^k. \)

**Step 2:** Calculate the market order demand of informed trader as follows: 
\[ x_{m,i} = \gamma \left( \left( \delta_{0}^k + \delta_{s}^k \right) - f_{p,k}^{i} (\delta_{0}^k + \delta_{s}^k) \right). \]

**Step 3:** Calculate the order choice of the ELP. Different from informed trader, the ELP needs to make two decisions: order type and order size. If the realization value \( \theta_{k,i} \) satisfies \( |\theta_{k,i}| < \bar{\theta} \), the ELP chooses limited orders. By observing market order flow \( x_{T,i}^{k,i} + z_{k,i} \) and signal \( \theta_{k,i} \); the ELP’s limit order follows: 
\[ x_{M,i}^{k,i} = \gamma \left( f_{v,k}^{i} \left( \theta_{k,i} + z_{k,i} - p_{k,i} \right) \right). \]

If the realization value satisfies \( |\theta_{k,i}| > \bar{\theta} \), the ELP chooses market orders as following: 
\[ x_{T,i}^{k,i} = \gamma \left( \frac{\sigma_0^2 + \sigma_f^2}{\sigma_0^2 + \sigma_d^2 + \sigma_{\epsilon}^2} \theta_{k,i} - f_{p,k}^{i} (\theta_{k,i}) \right). \]

**Step 4:** Calculate the DMM’s limit order who observes the market order flow \( x_{T,i}^{k,i} + x_{E,i}^{k,i} + z_{k,i} \) and supplies limited order 
\[ x_{D,i}^{k,i} = \gamma \left( f_{D,k}^{i} \left( x_{T,i}^{k,i} + x_{E,i}^{k,i} + z_{k,i} \right) \right) - \gamma \left( \frac{\mu + \tau(1-\mu)}{\mu + \tau^2(1-\mu)} p_{k,i} \right). \]

**Step 5:** Determine the equilibrium financial market price. Based on the realization value \( \theta_{k,i} \), we have two different regimes: taker and maker. In the maker regime, 
\[ x_{T,i}^{k,i} + x_{T,i}^{k,i} + z_{k,i} + \gamma f_{D,k}^{i} \left( x_{T,i}^{k,i} + x_{T,i}^{k,i} + z_{k,i} \right) - \gamma \left( \frac{\mu + \tau(1-\mu)}{\mu + \tau^2(1-\mu)} p_{k,i} \right) = 0. \]
In the taker regime, 
\[ x_{T,i}^{k,i} + z_{k,i} + \gamma f_{E,k}^{i} \left( \tilde{\theta}_{k,i} + x_{T,i}^{k,i} + z_{k,i} \right) - \gamma p_{k,i} + \tau \left( \frac{\mu + \tau(1-\mu)}{\mu + \tau^2(1-\mu)} p_{k,i} \right) = 0. \]

**Step 6:** Calculating the ELP’s utilities under two different regimes, 
\[ U_{E,T}^{k,i} = x_{T,i}^{k,i} \left( v_{k,i} - p_{k,i} \right) - \frac{1}{2\gamma} \left( x_{T,i}^{k,i} \right)^2; \quad U_{E,M}^{k,i} = x_{M,i}^{k,i} \left( v_{k,i} - p_{k,i} \right) - \frac{1}{2\gamma} \left( x_{M,i}^{k,i} \right)^2. \]

By doing steps 1 to 5, we complete one simulation and obtain one observation to update the belief. Step 6 helps to determine the converge test and update the threshold value to be discussed.
B2. Updating the Expectations and Utilities:

We now explain how the beliefs are updated after each experimental. After obtaining \( n \) observations, the input-output pairs form a new distribution for the four expectation functions, denoted by, \( g_{p,t}^k, g_{p,k}, g_{v,k}^k, g_{v,0}^k \). The beliefs are then updated according to

\[
f_{p,t}^{k+1} = F\left(f_{p,t}^{k}, g_{p,t}^k\right);
\]

More specifically, in order to solve the model, we discrete the expectation function. The reason to choose discrete function is that we want to capture the non-linear properties.

**Discrete:** We first describe the way to discrete the expectation functions. Instead of using the continuous function for expectation, we choose a piece-wise function; and within each interval the value of dependent variable is fixed. We use \( f_{p,t} \) as an example.

The independent variables of the expectations of traders are a set of sections ranging from \(-H\) to \(H\) with \(L\) intervals and two additional interval \((−∞,−H)\) and \((H,∞)\) to cover all possible inputs. Each section is estimated independently to obtain possible nonlinear relationship between input and expectation. By doing this, we have \(2H_{L}+2\) sections in total.

The expectation of traders are given by

\[
f_{p,t}^{k}(s) = \begin{cases} c_{p,t}^{k,1}, & \text{if } s < -H \\ c_{p,t}^{k,l}, & \text{if } -H + (l-2)L \leq s < -H + (l-1)L, l = 2, \ldots, \frac{2H}{L} + 1 \\ c_{p,t}^{k,2}\left(\frac{H}{L}+1\right), & \text{if } s \geq H \end{cases}
\]

We draw the input from the normal distribution. The observations that locate in \((−∞,−H)\) and \((H,∞)\) are fewer, which might cause the imprecision estimation for these intervals.

**Updating:** We first describe how we discrete the observation functions, which have similar form as the expectation functions

\[
g_{p,t}^{k}(s) = \begin{cases} \alpha_{p,t}^{k,1}, & \text{if } s < -H \\ \alpha_{p,t}^{k,l}, & \text{if } -H + (l-2)L \leq s < -H + (l-1)L, l = 2, \ldots, \frac{2H}{L} + 1 \\ \alpha_{p,t}^{k,2}\left(\frac{H}{L}+1\right), & \text{if } s \geq H \end{cases}
\]

where the constants are determined by

\[
\alpha_{p,t}^{k,l} = \frac{\sum_{i=1}^{n} 1_{-H+(l-2)L \leq s < -H+(l-1)L} \times p_{j,i}^{k,l}}{\sum_{i=1}^{n} 1_{-H+(l-2)L \leq s < -H+(l-1)L}}
\]
This means to calculate the weight-average for each intervals as the value for observation functions.

The beliefs are then updated by updating the value for expectation function,

\[ c_{p,l}^{k+1,i} = \begin{cases} 
(1 - \beta_{p,l}^{k,i})c_{p,l}^{k,i} + \beta_{p,l}^{k,i}d_{p,l}^{k,i}, & i_{p,l}^{k,i} > 0; \\
0, & i_{p,l}^{k,i} = 0;
\end{cases} \]

where the weight function is defined by

\[ \beta_{p,l}^{k,i} = \max \left( \frac{1}{k - 1}, \beta_{MW} \sum_{j=1}^{k} t_{p,l}^{j,i} \right), \]

and

\[ t_{p,l}^{k,i} = \sum_{l=1}^{n} 1_{-H+(l-2)L \leq s_{l} < -H+(l-1)L}. \]

The intuition behind the chosen weighting function is as follows. The first term means that, as the number of experimental increases, more weight is given to historical data in early periods when the experimental results are becoming more reliable. The second term means that the lowest value for \( \beta_{p,l}^{k,i} \) is \( \beta_{\min} \), which provides the highest weight on the historical data. This prevents to rely on the historical results too much. The last term means the reliability of the newest test based on the number of random drawing. For \( (-\infty, -H) \) and \( (H, \infty) \), the frequency of random drawing that locates in these intervals is lower; hence, we rely more on new experimental.

On the utilities, we use the similar methodology. After each experimental, we can obtain a new distribution about the utility function. Similar procedure applies to incorporate this information into the utility function, except that we pay attention to the ex-ante utility. This means that both the ELP’s utility functions for maker and taker only depend on the ELP’s private signal. Also, the utility functions depend on the exogenously given threshold value.

\textbf{B3. Convergence Test:}

The convergence test helps to determine the number of experimental. Basically, when the belief converges in equilibrium, a new experimental provides insignificant additional information. With enough times of updating, the change in the belief becomes insignificant. We now introduce two convergence tests.

Firstly, in order to obtain the overall change of the belief, we firstly calculate the update of belief in each interval as

\[ \text{Err}_{p,l}^{k,i} = |c_{p,l}^{k-1,i} - c_{p,l}^{k,i}|. \]
This measurement reflects the amount of new information in each interval. We then calculate the equal-weighted and frequency-weighted for each belief by

\[
E_{\text{Err}}^{p,\text{ew},i} = \frac{1}{\sum_{l=1}^{2^{(l+1)}}} \sum_{l=1}^{2^{(l+1)}} 1_{t_{p,l}>0} E_{\text{Err}}^{p,i}; \quad E_{\text{Err}}^{p,\text{fw},i} = \frac{1}{\sum_{l=1}^{2^{(l+1)}}} \sum_{l=1}^{2^{(l+1)}} E_{\text{Err}}^{p,i} t_{p,l}.
\]

We then introduce the convergence condition for experimental. The beliefs converge when, for all errors, the errors are higher than the minimum value from the history,

\[
E_{\text{Err}}^{p,\text{ew},i} > \min\{E_{\text{Err}}^{p,\text{ew},i}, k = 1,2,...,N-1\}; \quad E_{\text{Err}}^{p,\text{fw},i} > \min\{E_{\text{Err}}^{p,\text{ew},i}, k = 1,2,...,N-1\}.
\]

The above condition means that under the current experimental we cannot incorporate more precision information into our experimental function.

Secondly, after each updating, we calculate the correlation between the prior expectation and updated expectation. In equilibrium, this correlation is above than certain threshold value (close to 1). In our experiments, both convergence requirements are satisfied in equilibrium.

### B4. Determine the Endogenous Threshold Value

We have presented the way to calculate the equilibrium expectation functions, which further determine the optimal demand functions. This procedure can be viewed as finding the exogenous equilibrium. We now explore the way to determine the endogenous optimal threshold value in the nonlinear equilibrium. The idea behind is straightforward. When the equilibrium converges, we obtain the equilibrium utility functions. In the exogenous equilibrium, the utility functions are based on the exogenously given threshold value $\theta^\star$. After the $N$-th experiment (when the belief converges), we calculate the value $\theta^\star$ that makes the ELB indifferent in between making and taking liquidity, $\theta^\star(\theta^\star) = \{\theta^2_{E,I}(\theta, \theta^\star) = U_{E,M}^N(\theta, \theta^\star)\}$. This value $\theta^\star(\theta^\star)$ depends on the exogenously given parameter $\theta^\star$. The endogenous equilibrium condition means a fixed-point problem of the exogenous cut-off level $\theta^\star$ such that the optimal threshold value $\theta^\star$ from comparing the utilities is the same under $\theta^\star$, that is $\theta^\star(\theta^\star) = \theta^\star$.

To calculate this value, we apply two different methodologies. The first way is ergodic. We calculate the equilibrium for all the possible threshold values and then determine the equilibrium. This method is accurate and provides the uniqueness and convergence; however, the calculation is complicated and time-consuming. Instead, in the comparative analysis, we adopt another method. Given the cut-off level used to calculate the equilibrium demand $\theta^\star$, we obtain the optimal threshold
value $\tilde{\theta}^*$ that comes from utilities comparison. In the next round, we use this optimal threshold value $\tilde{\theta}^*$ that comes from utilities comparison as the new threshold value,

$$\tilde{\theta}^{t+1} = \{ \theta \left| U_{E,T}^N(\theta, \tilde{\theta}^t) = U_{E,M}^N(\theta, \tilde{\theta}^t) \right. \}.$$ 

In equilibrium,

$$\tilde{\theta}^{t+1} = \tilde{\theta}^t.$$

**B5. Comparison for HFTm**

To demonstrate the efficiency of the machine learning approach, we first examine HFTm in which HFT demands liquidity by submitting market orders only. We show that the machine learning methodology provides the same results. When HFT demands liquidity only, there is no additional uncertainty for market participants and hence the equilibrium (in both the price and demand functions) are linear. With the reinforcement learning, we randomly select 200 different sets of model parameters, then calculate the equilibrium illiquidity measures and compare with the illiquidity measures under the theoretical linear equilibrium. The results are reported in Figure A1, showing the illiquidity measures in terms of the price to noise ratios for 200 sets of parameters based on two approaches. The correlation among two liquidity measures among the 200 sets is 0.999, showing that the results from the theoretical and machine learning approaches are highly correlated.

**B6. Expectation Functions**

We now apply the machine learning methodology to the endogenous liquidity provision model of nonlinear equilibrium. To highlight the nonlinearity in equilibrium, we compare the expectation functions to their linear approximation in Figure A1. It shows that, apart from DMM’s expectation on the payoff $f_{V,D}$, other three expectation functions are almost linear, with minor difference from the linear approximation equilibrium. We now provide some intuitions on these observations.

On informed trader’s expectation function $f_{p,I}$, the large information shocks are limited and rare enough to make the ELP’s liquidity provision relatively stable; thus, the informed trader’s expectation is mainly influenced by the price under the maker regime. In fact, a trader’s trading strategy depends on DMM’s pricing rules, which in turn affect the traders’ trading strategies. In the ELP model, the ELP completely crowds out the DMM. Different from the DMM, when making
the market, the ELP has no additional uncertainty about the structure of order flow information. Hence the ELP’s inference from price is nearly linear once the informed demand is nearly linear. Therefore, when the probability of taking regime is low, the equilibrium is similar to the case without speed’s influence, which is nearly linear as is illustrated in Figure A1. The same argument also explains the nearly linear expectation of the fundamental value $f_{V,E}$ for the ELP. Due to the almost linear expectation function, the informed trader’s demand schedule is also nearly linear, which further leads to the almost linear expectation on the fundamental value for the ELP.

Regarding DMM’s expectation functions, Figure A1 shows that the making intensity is S-shaped in the nonlinear equilibrium, which is higher when the order flow received by the DMM is essentially balanced comparing to the extreme order imbalance. Intuitively, when receiving market orders, the DMM knows that the ELP consumes liquidity due to her extreme private signals with large market orders. Therefore, a balanced aggregate order flow indicates that the orders from the informed trader and liquidity trader are opposite to the orders from the ELP. Therefore, the balanced aggregate order flow contains less information for the DMM, which increases DMM’s uncertainty to make the market. This is characterized by the DMM’s S-shaped expectation function, with high slope for balanced orders.

Finally, Figure A1 shows a nearly linear expectation function $f_{p,E}$ for the ELP on the price in the taker regime. This seems counter-intuitive at first sight, considering that the DMM’s pricing rule is non-linear due to his non-linear expectation function $f_{V,D}$. Note that the ELP only takes liquidity and submits market order when her private signals are in extreme. Therefore, the ELP anticipates that the aggregate order flow would be imbalanced when she submits market orders. The panel about the DMM’s expectation function in Figure A1 shows that, when the aggregate order flow is imbalanced, the DMM’s making intensity is lower. Hence, the expected price function for the ELP is nearly linear. In summary, the above analysis demonstrates how machine learning techniques can be used to explore the evolution of more intricacy market microstructure with nonlinear equilibrium.

**B7. Decomposition**

**DMM’s limited participation**: DMM’s limited participant influences DMM’s adverse selection directly and indirectly. Directly, DMM faces more adverse selection under taker regime than maker regime due to more volatile fundamental under taker regime. Higher value of $\tau$ means ELP’s
trading speed advantage decreases and DMM can participate market more actively. This means that DMM actually faces less adverse selection. Indirectly, higher value of $\tau$ also influences informed trader’s and ELP’s market order and ELP’s endogenous liquidity provision decision. Figure A2 numerically shows the decomposition to the direct and indirect effects. By changing the speed parameter and endogenously determine the expectation functions for all market participants, we generate the total effect. The direct effect is then generated by fixing the trading strategies of informed trader and ELP and changing only the speed parameter to determine DMM’s expectation function, while the indirect effect is the distance between total and indirect effects. Figure A2 shows that both direct and indirect effects are negative. This implies that increasing (decreasing) the speed parameter shrinks (intensifies) the direct and indirect adverse selection effects and therefore decreases (increases) the total adverse selection effect.

**Uncertainty of ELP’s endogenous liquidity provision:** The total effect of ELP’s uncertainty can be decomposed into a direct effect and an indirect effect. Changing the threshold value of ELP’s cut-off strategy affects DMM’s adverse selection through changing of both informed and ELP’s trading strategies and DMM’s inference about ELP’s cut-off strategy. On the direct effect, a high cut-off value increases DMM’s adverse selection cost since DMM knows that the fundamental becoming more volatile. On the indirect effect generating from the forecasts ELP’s endogenous liquidity provision strategy of others, with higher cut-off value, it also affects other trader’s strategy, which further affects information in the aggregate flow.

Figure A2 shows numerically such decomposition. We first generate the total effect by changing the cut-off level and endogenously determine the expectation functions for all market participants. We then generate the direct effect by fixing the trading strategies of informed trader and ELP and change only the cut-off level to determine the DMM’s expectation function. The indirect effect is then the distance between the total and indirect effects. Specifically, we use the reinforcement learning method to solve DMM’s inference problem under different situations. Figure A2 shows that both direct and indirect effects are positive. This means that increasing (decreasing) ELP’s cut-off value intensifies (shrinks) both direct and indirect adverse selection effects, which obviously increases (decreases) the total adverse selection.
Panel A

![Image of a scatter plot showing the similarity of liquidity measures from two approaches. Each blue square represents a single experiment and its sequential number is marked nearby.](image)

Panel B

![Image showing graphs of expectation functions for different scenarios.](image)

**Figure A1 Reinforcement learning.** Panel A describes equilibriums of model HFTm (high-frequency trader only consumes liquidity) with two hundred randomly selected combinations of $\sigma_f$, $\sigma_d$, $\sigma_e$, and $\sigma_z$ obtained by machine learning and linear solution respectively. The scatter plot shows the similarity of the liquidity measures from these two approaches. Each blue square represents a single experiment and its sequential number is marked nearby. In Panel B, the upper left/right and bottom left/right panels represent the expectation functions $f_{p,E}, f_{d,E}, f_{v,E}, f_{v,M}$, respectively. The red dash line represents the results under machine learning and the blue dotted line represents the results under linear approximation. The yellow line represents the difference which is marked on the right Y-axis.
Panel A

Figure A2 Decomposition for expectation function. Panel A represents the reaction of pricing-power-weighted expectation functions $f_{v_M}^m$ to the change of DMM’s speed $\tau$. The left/right upper and left/right bottom panels show the results with $\tau = 0.01$, $0.02$, $0.03$ and $0.04$, respectively. We first run the benchmark experiment with $\tau = 0$ and obtain the equilibrium $f_{v_M}^{m\ell}$. The equilibrium $f_{v_M}^{m\ell}$ is plotted in blue solid line in each panel. Secondly, we fix the expectations of informed trader and ELP while changing the $\tau$, and run the experiments to learn the expectation of DMM. This altered $f_{v_M}^{m\ell}$ is plotted in red dash line for each given $\tau = 0.01$, $0.02$, $0.03$ and $0.04$. At last, we change the $\tau$ and run the experiments to learn the expectations of all type agents. The altered $f_{v_M}^{m\ell}$ under such environment is plotted in green dash line. The $f_{v_M}^{m\ell} = (\mu f_{v_M}^m + (1 - \mu) \tau f_{v_M}^l)/(\mu + (1 - \mu) \tau), \text{ in which the } f_{v_M}^m/f_{v_M}^l \text{ is the DMM’s value expectation in the regime ELP submits market/limit order respectively and } \mu \text{ is the probability of ELP submitting market order.}$

Panel B

Panel B represents the reaction of expectation functions $f_{v_M}$ to the change of ELP’s endogenous liquidity provision cut-off strategy level $\theta$. The left/right upper and left/right bottom panels show the results with changing from $\theta = 30.4$ to $\theta = 10, 20, 40$ and $50$, respectively. The procedure is similar to panel A.