A Model of Capital Structure under Labor Market Search

Ping Liu†

January 2019

Abstract

This paper develops a competitive search equilibrium model of capital structure and labor outcomes. In the model, employers design capital structures and compete for workers subject to idiosyncratic productivity shocks and labor search frictions. The capital structure policy reflects the trade-off between the “strategic benefit” of debt in wage bargaining and the cost of debt in labor hiring. The model generates rich comparative static implications regarding the impacts of productivity, credit and labor market factors on leverage and labor outcomes. A calibration yields a large wage dispersion and a highly elastic labor market tightness with respect to productivity.

Keywords: capital structure, labor market search, competitive search equilibrium

*I am grateful to Heitor Almeida, Dirk Hackbarth, Zhiguo He, Timothy Johnson, George Pennacchi, Joshua Pollet, Uday Rajan, Stijn Van Nieuwerburgh, Alexei Tchistyj, Jiaxu Jessie Wang, Neng Wang, Toni Whited, Yufeng Wu, Yuhai Xuan, Jun Yang, Mao Ye for helpful comments. I also thank participants at 2018 SFS Cavalcade North America, 2018 Wabash River Finance Conference, Rutgers University, SUNY Buffalo, University of Georgia, and University of Illinois at Urbana Champaign.

†Ping Liu is from Purdue University. Email: liu2554@purdue.edu
1. Introduction

A growing theme of finance literature is the interaction between the labor market outcomes and corporate finance decisions. These studies document that the wage bargaining between employers and workers and the search friction in the labor market have a nontrivial effect on an employer’s capital structure policy. Meanwhile, an employer’s financial condition influences both its hiring policy and the volume of applications for its job vacancies. However, with a few exceptions, most theoretical research examines labor market and capital structure dynamics in isolated models. This paper bridges the gap between the empirical and theoretical research by providing a competitive search equilibrium model of capital structure policies in a frictional labor market featuring Mortensen-Pissarides search and matching framework.

I develop a new trade-off theory of capital structure decisions arising from labor market frictions that does not rely on the tax benefit of debt. In particular, a higher leverage policy grants the potential employer an edge in the wage bargaining process with workers once it matches with a searching worker. This so-called strategic role of debt provokes the potential employer to use higher leverage. However, higher leverage, besides increasing the potential employer’s expected post-match bankruptcy cost, also imposes an additional cost on its hiring probability: it discourages the job seekers from applying for the potential employer’s job vacancy, thus reduces the potential employer’s matching probability in the labor market. This will discourage the potential employer from using higher leverage. More importantly, I develop a tractable equilibrium framework that nests this new trade-off theory of individual employers’ capital structure decisions to a competitive search and matching model of labor

---

1 See, for example, Bronars and Deere (1991); Matsa (2010); Agrawal and Matsa (2013).
2 For example, Brown and Matsa (2016).
3 A few scholarly works put labor market and capital structure under the same umbrella(e.g., Wasmer and Weil 2004; Monacelli et al. 2011; Chugh 2013).
market. An appealing feature of this model is that the individual employers’ capital structure decisions and labor market aggregates are simultaneously determined by the same exogenous factors from the productivity, credit market and labor market. Another convenient feature is that I can derive a closed-form solution for the distribution of matching cash flows under the stationary equilibrium. This allows me to analyze the aggregate labor market quantities in a highly tractable manner.

The model is able to generate a Pareto-form wage distribution with a long and “fat” right tail, which is consistent with empirically observed wage distributions (e.g., Gabaix 2009 and references therein). Moreover, the model is able to replicate some well-known comparative static results in corporate finance literature and labor economics literature. Regarding the capital structure policies, a high-growth firm does not necessarily employ higher leverage, a well-known empirical fact that the classic trade-off theory of capital structure struggles to explain (e.g., Rajan and Zingales 1995; Miao 2005; Barclay et al. 2006). An increase in the worker’s bargaining power induces the employer to use higher leverage, so does a more generous unemployment benefit program. Both predictions are consistent with the empirical findings in, for example, Bronars and Deere 1991, Matsa 2010 and Agrawal and Matsa 2013. Regarding the labor market outcomes, a more generous unemployment benefit program is able to reduce the wage inequality in the economy (Koeniger et al. 2007).

Another interesting and testable prediction of this model is that a higher bargaining power on the worker’s side does not necessarily lead to a slacker labor market, i.e., lower vacancy-to-unemployment ratio. Unique in my model, although an employer obtains a smaller share of matching surplus when the worker has a higher bargaining power, it can counteract worker’s higher bargaining power by issuing more debt, thereby diverting more cash flow from matching surplus to interest payment. Consequently, the impact of workers’ bargaining power on employers’ labor market entry decisions thus labor market tightness is ambiguous.
Lastly, a calibration of the base case model reveals a much higher frictional wage dispersion, measured by the ratio between average and minimum wage, as well as a much more elastic vacancy-to-unemployment ratio with respect to average productivity, compared with most frictional search models of the labor market. In this sense, my model provides promising directions toward the reconciliation between the search models of labor market and two well-known puzzles in the labor economics literature: the excessively low model-implied mean-min wage ratio (Hornstein et al., 2011) and the excessively low model-implied elasticity of vacancy-to-unemployment ratio with respect to productivity (Hall, 2005; Shimer, 2005). The new trade-off theory is able to generate an average leverage ratio very similar to its empirical counterpart.

The structure of the model is as follows. I nest a standard dynamic capital structure model (Leland, 1994) to a frictional search and matching framework in the labor market, in the spirit of Mortensen-Pissarides (Mortensen and Pissarides, 1994). In my model, the labor market consists of a continuum one of workers and large continuum of potential employers ensuring free-entry. The workers are either employed or unemployed and searching for potential employers. Likewise, the employers are either occupied or idle and searching for unemployed workers. A “firm” is established once a potential employer matches with a searching worker. Search frictions prevent instantaneous matches between potential employers and unemployed workers. Motivated by empirical facts (Brown and Matsa, 2016), I assume that job seekers have information regarding each potential employer’s financial condition. Under this assumption, the leverage is a double-edged sword in determining the expected payoff to the potential employer from searching and matching in the labor market. On one hand, a potential employer’s higher leverage policy will reduce the matching cash

\[\text{Although I make several modeling assumptions to maintain tractability, the calibration results are robust across a wide range of plausible model parameters.}\]
flow that the potential employer has to share with the worker once they are matched. On the other hand, besides increasing the expected post-match bankruptcy cost, a potential employer’s higher leverage policy will reduce the expected payoff to a searching worker from matching with the potential employer. Thus the higher leverage policy will “discourage” the searching workers from applying for the potential employer’s job vacancy. This in turn will reduce the potential employer’s hiring intensity. Each potential employer, competing for workers, weighs the benefit against the cost of a higher leverage policy and designs its capital structure that maximizes the expected value of its labor search, subject to the constraint that it must provide searching workers with the expected value comparable to other potential employers. Otherwise, searching workers will apply for job vacancies of other potential employers. Consistent with Leland [1994], a potential employer implements its capital structure policy by issuing a perpetual debt after meeting with an unemployed worker. The matching production starts after the debt is issued, and subject to idiosyncratic shocks specific to each match. The employer and the worker will voluntarily part their ways and return to the labor market search when the matching performance is sufficiently poor.

A stationary cross-sectional distribution of matching cash flows emerges in the steady-state equilibrium. Firms exhibit variations in terms of leverage ratios. This is because the cash flows are matching-specific and are different across matches (“firms”). I am able to obtain the stationary cross-sectional distribution of matching cash flows in a closed form. In the model, debt, equity, wage and expected job tenure are matching-specific and can be explicitly expressed as functions of the matching cash flow. Therefore, in the equilibrium, the economy-wide average leverage ratio, wage dispersion, and the average length of job tenure are tractable and invariant over time. However, the actual composition of matches in the economy continually changes: some matches dissolve and others establish at each instant.

The equilibrium concept of the model—competitive search equilibrium—can be demon-
strated by the following example. Suppose the growth rate of all matching cash flows increases. In a single-firm analysis, such an increase has a cash flow effect in the sense that it in expectation increases the matching cash flow. Hence an employer has a higher incentive to issue more debt, to defend against workers’ wage bargaining in high cash flow scenarios. Moreover, it also increases the expected appreciation in the option value to default. This provides an extra incentive for the employer to issue more debt. However, numerous empirical literature has refuted the prediction that higher-growth firms have higher leverage (e.g., Rajan and Zingales, 1995, Barclay et al, 2006). This contradiction between empirical regularities and theoretical predictions can be reconciled by the current model. The mechanism is as follows: the potential employer entrants will observe the increase in expected matching value as a result of an increase in the growth rate of matching cash flow. Therefore, more potential employers will enter the labor market. Intensified competition for limited amount of workers will drive up the expected value of job searching and will force the employers to use lower leverage. Therefore the overall impact of cash flow growth rate on the leverage ratio in the equilibrium depends on the relative strength of the two forces in the single-firm analysis and the competition effect in the labor market.

The model also delivers other implications regarding the employers’ capital structure choices and aggregate labor market outcomes across different economic, labor market and credit market parameter values. Most of these implications are consistent with empirical evidence. For example, an increase in the growth rate of matching cash flow reduces the unemployment rate and leads to a tighter labor market. These are consistent with the procyclical nature of the employment and labor market tightness. An increase in the volatility of matching cash flow increases the debt usage but decreases the average leverage ratio in the economy, consistent with the inconclusive empirical evidence regarding the relationship between economic volatility and leverage (e.g., Titman and Wessels, 1988, Johnson, 2018).
negative shock in credit market results in a higher unemployment rate and a slacker labor market. These predictions are consistent with the most recent experience of the financial crisis, in which the employers disrupted by negative shocks in their credit supplies reduced their hiring (e.g., Chodorow-Reich, 2014).

This paper contributes to several research strands. First, recent research has established that macroeconomic conditions has important implications of firms’ financial policies. For example, Hackbarth et al. (2006) have shown that firms adapt their capital structure and default decisions according to the current position of macro-economy over the business cycle. This, in turn, has profound impact over observed debt levels, the cyclical behavior of leverage ratios and credit spreads. My paper has shown that labor market conditions also influence the employers’ capital structure policies. Unique in my paper, the individual employers’ capital structure decisions and labor market outcomes are jointly determined by the same set of exogenous parameters from the macro-economy, labor market and credit market. This allows me to examine the joint dynamics of labor market aggregates and micro-level corporate finance decisions across various conditions in the macro-economy, labor market and credit market. Another strand of literature has shown that debt market liquidity affects gains/losses of debt rollover and borrowers’ credit risk, thereby influencing borrowers’ default policies (He and Xiong, 2012). My paper extends this literature by considering how search friction and liquidity in the labor market affects employers’ capital structure choices and default policies.

Second, this paper complements the burgeoning macroeconomic literature that studies the relationship between financial market conditions and labor market conditions (e.g, Wasmer and Weil, 2004; Monacelli et al., 2011; Chugh, 2013). However, the underlying mechanisms through which the labor and financial markets are interrelated separate this paper from previous studies. Extant research focuses on the a financial accelerator mechanism similar to models proposed by Bernanke and Gertler (1989) and Kiyotaki and Moore (1997).
In those models, the amplification of productivity shocks through costly external financing of labor hiring and countercyclical finance premium influences employers’ hiring activities and leads to large labor market fluctuations. My paper develops a novel competitive search equilibrium that incorporates a new trade-off theory of capital structure arising from the role of debt in the wage bargaining and the potential employer’s hiring intensity. From a methodological point of view, the continuous time approach enables me to characterize the various aspects of labor market outcomes in closed-forms and facilitates the comparative static analyses.

Lastly, several papers provide microeconomic-level analyses of human capital and capital structure choices, taking labor market conditions as exogenous (e.g., Berk et al., 2010). My paper nests a micro-level trade-off theory of capital structure from wage bargaining and labor search frictions into a competitive search and matching equilibrium model in the labor market. Compared to these microeconomic-level studies, the current model also allows me to analyze the equilibrium properties of the labor market under employers’ endogenous capital structure policies across various conditions in the macro-economy, labor market and credit market.

The paper is organized as follows. Section 2 lays out model environment. Section 3 characterizes various asset value equations and defines the equilibrium concept. Section 4 solves the model analytically. Section 5 provides a calibration for the base case model and presents comparative static results. Section 6 concludes the paper and points toward possible directions for future research.

Monacelli et al. (2011) also consider the strategic role of debt in the wage bargaining. However, I also consider the implications of employers’ capital structure policies on their hiring intensities. The equilibrium concept in the current paper is also different from theirs.
2. Model environment

Time is continuous. Suppose that information is perfect and agents discount future cash flow at a constant risk-free rate \( r > 0 \). The risk neutrality assumption is standard in dynamic corporate finance literature and search models of labor market. The same analyses still go through under risk-neutral measure even if the agents are risk-averse (e.g., Harrison and Kreps [1979]).

2.1 Labor market search

I model the labor market search and match in line with standard frictional search models of labor market (e.g., Mortensen and Pissarides [1994]). The labor market consists of a continuum of ex-ante identical potential workers and potential employers. The measure of workers is normalized to one. The measure of potential employers is endogenously determined to ensure free entry.\(^6\) The workers are either employed or unemployed and searching for potential employers. Likewise, the employers are either occupied or idle and searching for unemployed workers. A potential employer posts a vacancy to the job seekers and incurs a flow cost, \( \kappa \), to keep the vacancy open. On the worker side, I assume that the labor market is so large that a searching worker can only select a subset of job vacancies to apply for. While searching for jobs, a worker enjoys a flow value of unemployment benefit, \( b \). I assume that \( b \) is so small that an unemployed worker will not reject matching opportunities with a potential employer. In other words, all the newly formed matches are socially efficient. The following assumption is important.

\(^6\)I assume that each potential employer can only post one vacancy in the job market. However, this assumption only facilitates the expressions and has no material consequences.

\(^7\)The benefits include, but are not limited to, unemployment allowance, leisure, social welfare, and income from self-employment.
Assumption 1. For each job vacancy they plan to apply for, searching workers have information about the potential employer’s capital structure policy after a match.

Whether the workers’ knowledge of potential employers’ capital structure policies is perfect or not is not crucial, but perfect knowledge allows me to derive the model solutions tractably without losing any important model implications. Therefore I assume workers’ knowledge is perfect.

Consistent most search models of labor market, both the job-search and the labor-hiring processes are frictional. Specifically, the flow of new worker-employer matches is captured by the homogeneous-of-degree-one concave matching function \( m(u, v) \), where \( u \) and \( v \) denote the economy’s unemployment and vacancy rates, respectively. Define \( \varepsilon = \frac{v}{u} \) as the labor-market tightness. Then the job finding rate \( g \) is a function of \( \varepsilon \) and \( g = \frac{m(u, v)}{u} = m(1, \varepsilon) = \tilde{g}(\varepsilon) \), representing the rate at which an unemployed worker meets a potential employer. The hiring rate \( h \) is also a function of \( \varepsilon \) and \( h = \frac{m(u, v)}{v} = m \left( \frac{1}{\varepsilon}, 1 \right) = \tilde{h}(\varepsilon) \), representing the rate at which a potential employer meets a searching worker. I assume that \( \lim_{\varepsilon \to 0} g(\varepsilon) = \lim_{\varepsilon \to \infty} h(\varepsilon) = 0 \) and \( \lim_{\varepsilon \to \infty} g(\varepsilon) = \lim_{\varepsilon \to 0} h(\varepsilon) = \infty \). Since both the job finding rate \( g \) and hiring rate \( h \) are functions of labor market tightness \( \varepsilon \), sometimes it is useful to express the hiring rate as \( h = \tilde{h}(\varepsilon) = \tilde{h}(\tilde{g}^{-1}(g)) = h(g) \), where \( h'(g) < 0 \).

2.1.1 Discussion of Assumption 1

Assumption 1 may seem extreme at the first sight, but it has empirical supports. First, most public firms stick to particular capital structures over the course of many years (Lemmon et al., 2008). More recently, using a newly available survey data from an online job-search platform, Brown and Matsa (2016) find that the job seekers’ information on potential employers’ financial conditions is consistent with potential employers’ credit default swap prices. Moreover, job seekers act upon their information and are reluctant to apply for job vacancies
posted by employers with poor financial conditions and high leverages. These findings empirically support my assumption about workers’ knowledge regarding the potential employers’ capital structures. Moreover, \textbf{Assumption 1} is harmless even if one has strong prior that workers’ information collection takes time. Consider the following thought experiment: employers build up their reputations for capital structure policies in the labor market through repeated matching and separating with workers. Meanwhile, workers Bayesian-learn about each potential employer’s reputation for capital structure policy. Because my analysis focuses on the economy at the steady state, without loss of generality, I may still assume that workers have perfect knowledge about potential employers’ capital structure policies and potential employers have no incentives to deviate from their long-term leverage targets. Lastly, the insights from this paper will unlikely be affected as long as the workers can glean some information about the potential employers’ capital structure policies.

2.2 Match

2.2.1 Debt contracts

A “firm” is established when an employer and a worker matches. Consistent with Monacelli et al. (2011), I make the second assumption regarding the timing and the distribution of debt issuance.

\textbf{Assumption 2.} \textit{The employer has one (and only) opportunity to issue debt (i.e., to implement its capital structure policy) after it matches with a worker and before the production starts. The proceeds from debt issuance are immediately distributed to the employer.}

I assume that the “firm” does not readjust debt after initial debt issuance for tractability. I also assume that the proceeds from debt issuance goes to the employer. Both assumptions are consistent with those in standard contingent claim models of capital structure (e.g.,
An alternative way to motivate Assumption 2 is to assume that a fixed investment amount $I$ is required to start the production after a match is formed and that the employer finances the fixed investment by issuing an optimal mixture of debt and equity. This alternative motivation will not change the employer’s objective function and will not alter any results.

In order to stay in a time-homogeneous environment, I assume that a debt contract in this paper is represented by a consol bond with a constant coupon rate $c$. This assumption is also standard in contingent claim models of capital structure (e.g., Leland [1994]).

### 2.2.2 Production

The production starts immediately after a match is formed and debt is issued. The match-specific cash flow of a match $i$ at time $t$ is equal to $X_{it}$. The cash flow $X_{it}$ starts at $X_0$ and evolves according to the following geometric Brownian motion process:

$$\frac{dX_{it}}{X_{it}} = \mu dt + \sigma dZ_{it}$$

$(Z_{it})_{t \geq 0}$ is a standard Brownian motion that represents match-specific uncertainty. Since the cash flow processes of all matches are ex-ante represented by the same form of geometric Brownian motion processes starting at $X_0$, I omit the match indicator $i$ hereafter.

### 2.2.3 Matching separation

A matching relationship can be terminated at any instant for endogenous or exogenous reasons. First of all, consistent with most studies in dynamic capital structure, I assume that the employer may declare “bankruptcy”, leaves the match and return to labor market search at any time. I further assume that a worker does not have the commitment power to enter
into a long-term employment contract. She can also quit the employment relationship at any
time and return to labor market search. This reflects the fact that in the United States, most
employment relationships are “at will”. An endogenous matching separation occurs when
either the employer or the employee voluntarily leaves the match and return to labor market
search. If an endogenous matching separation occurs, a fraction $0 < \alpha \leq 1$ of net present
value will be lost to the “bankruptcy” costs, leaving creditors with abandonment value net of
bankruptcy costs while leaving both the employer and the employee with nothing. Moreover,
there exists a match-specific Poisson process that governs the exogenous destruction of the
match, with intensity $s$. Upon an exogenous match destruction, the salvage values for all
financial claims contingent on the matching cash flow are zero. Upon either an endogenous
or an exogenous separation, the match ends. Upon the end of the match, the employer and
the worker do not inherit any financial claims and obligations from the old match and are
restored to their initial states. As will be shown later, this assumption allows me to derive
a stationary cross-sectional distribution of matching cash flows in a closed-form. It is also
consistent with the existing literature (e.g., Berk et al., 2010).

2.2.4 Wage bargaining

A consequence of lack of commitment power in an employment relationship is that the
wage determination process of a particular matching relationship is governed by continuous
bilateral bargaining between the employer and the worker. Following the literature, I take an
axiomatic approach and use continuous generalized Nash bargaining solutions to characterize
the bargaining outcome, conditional on the matching cash flow at time $t$. The bargaining
power of the worker is $\beta$ and the bargaining power of the employer is $1 - \beta$.

---

8The exogenous separation of a match is standard in the literature (e.g., Rogerson et al., 2005 and references therein). This could reflect the risk of technological obsolescence, natural disasters, worker relocations, and so forth.
3. Asset values and equilibrium

In this section, I begin characterizing the asset values contingent on a match: Debt $D(X; c, U, V)$, the employer’s and the worker’s expected discounted values from a match, $E(X; c, w, V)$ and $W(X; w, U)$, respectively\(^9\). Then I present the expected asset value of an unemployed worker $U$ and that of a potential employer, $V$, from labor market search. The equation for $U$ plays a central role in that it defines a unique relationship between an potential employer’s coupon choice and its hiring intensity. This section culminates with the equilibrium concept: the competitive search equilibrium with endogenous leverage.

3.1 Match-specific asset values

It is convenient to introduce the following notations. Let the de facto discount rate, $\delta := r + s$ and the expected present value of a perpetual stream of values $(X)_t$ starting at $x$:

$$\Pi(x) := E\left[\int_0^{\infty} e^{-\delta t} X_t dt | X_0 = x \right] = \frac{x}{\delta - \mu}$$ (3.1.1)

3.1.1 Debt value of a match

For a given coupon rate $c$, the expected value of a searching worker $U$ and the expected value of the potential employer $V$, the debt value $D(X)$ of a matched worker-employer pair

\(^9\)The value of debt $D$ depends on the employer’s coupon choice $c$, the expected value of a searching worker $U$ and the expected value of the potential employer from labor searching $V$. An employer’s expected discounted payoff $E$ from a match depends on the employer’s coupon choice $c$, wage rate $w$ and the expected value of the potential employer from labor search $V$. The worker’s expected discounted value $W$ from the match depends on the wage rate $w$ and the expected value from job search $U$. By employers’ free-entry condition, $V$ is always zero in equilibrium. In the following sections, to conserve space, I omit the parameters after the semicolon.
at cash flow level $X$ solves the following boundary value problem:

$$rD(X) = c + \mu XD'(X) + \frac{1}{2} \sigma^2 X^2 D''(X) - sD(X)$$

(3.1.2)

where $X$ is the cash flow threshold for endogenous matching separation. The boundary conditions are standard value-matching conditions:

$$D(X) = D^B = (1 - \alpha) \left( \Pi(X) - \frac{rU + rV}{\delta} \right)$$

(3.1.3)

$$\lim_{X \to \infty} D(X) = \frac{c}{\delta}$$

By standard results from dynamic capital structure literature (e.g., Goldstein et al., 2001). The solution to the above boundary value problem is

$$D(X) = \frac{c}{\delta} - \left( \frac{c}{\delta} - (1 - \alpha) \left( \Pi(X) - \frac{rU + rV}{\delta} \right) \right) \left( \frac{X}{X} \right)^{\vartheta}$$

(3.1.4)

Here $\vartheta$ is the negative root of the equation $\nu (\nu - 1) + \frac{2\mu}{\sigma^2} \nu - \frac{2\delta}{\sigma^2} = 0$:

$$\vartheta = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) - \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2\delta}{\sigma^2}}$$

(3.1.5)

3.1.2 Employer’s expected value from a match

Similarly, for a given coupon rate $c$, wage rate $w$, and the expected value of the potential employer $V$ from labor search, the employer’s expected value from a match solves the following free-boundary problem:

$$rE(X) = X - c - w + \mu X E'(X) + \frac{1}{2} \sigma^2 X^2 E''(X) - s(E(X) - V)$$

(3.1.6)
The boundary conditions are:

\[ E (X^E) = V \quad (value \ matching) \]
\[ E' (X) \big|_{X=X^E} = 0 \quad (smooth \ pasting) \] (3.1.7)
\[ \lim_{X \to \infty} \left( \frac{E}{X} \right) < \infty \quad (no \ bubble) \]

\(X^E\) denotes the employer’s cash flow threshold for endogenous matching separation.

### 3.1.3 Employee’s expected value from a match

For a given wage rate \(w\) and unemployment value \(U\), an employee’s expected value from a match \(W (X)\) satisfies:

\[ rW (X) = w + \mu X W' (X) + \frac{1}{2} \sigma^2 X^2 W'' (X) - s (W (X) - U) \] (3.1.8)

The boundary conditions are

\[ W (X^W) = U \quad (value \ matching) \]
\[ W' (X) \big|_{X=X^W} = 0 \quad (smooth \ pasting) \] (3.1.9)
\[ \lim_{X \to \infty} \left( \frac{W}{X} \right) < \infty \quad (no \ bubble) \]

\(X^W\) denotes the employee’s cash flow threshold for endogenous matching separation.

### 3.1.4 Unemployed workers and Potential employers

For now assume that any potential employer does not have control over its capital structure policy (level of \(c\)) and the labor market is segmented into \(J\) exogenously defined “sub-markets” indexed by \(j \in \{1, 2, \ldots, J\}\). Each “sub-market” \(j\) is characterized by a coupon rate \(c_j\). The
measure $v_j$ of potential employers with that coupon policy and the measure $u_j$ of workers applying for jobs posted by those potential employers form the sub-market $j$, with a labor market tightness $\varepsilon_j$.

By assumptions in Section 2.1, the labor market is so large that any unemployed worker can only select a subset of job vacancies to apply for. Which “sub-market(s)” will she optimally choose to enter and apply for jobs then? Let $U_j$ denote the expected value of searching for jobs in “sub-market” $j$. Then $U_j$ satisfies:

$$rU_j = b + g(\varepsilon_j)[W_j(X_0) - U_j]$$  \hfill (3.1.10)

where $b$ is the flow value of unemployment benefit that a potential worker collects each instant while searching for jobs.

Workers will enter the sub-market that provides them with the highest expected value of job search. Since workers are ex-ante identical and have perfect information about $c_j$ in each sub-market, all the sub-markets with nonempty job applicants must grant the same level of expected value to the unemployed workers; this value I denote as $U$. Bringing $U$ into (3.1.10), the expected value of an unemployed worker from job search satisfies the following equation:

$$rU = b + g(\varepsilon_j)[W_j(X_0) - U]$$  \hfill (3.1.11)

or equivalently,

$$g(\varepsilon_j) = \frac{rU - b}{W_j(X_0) - U}$$  \hfill (3.1.12)

For a given $U$, (3.1.12) defines a unique relationship between the coupon rate $c_j$ and job finding rate $g(\varepsilon_j)$ thus labor market tightness $\varepsilon_j$ in each sub-market $j$. In other words, $g(\varepsilon_j)$

10This is similar to the sub-market concept in Moen (1997) on wage posting in labor market.
is a function, specified by (3.1.12), of \( U \) and \( c_j \).

I now allow any potential employer to optimally choose its capital structure policy, i.e., \( c \). Since all the potential employers face the same matching productivity and the same optimization problem for coupon rate, they choose the same coupon rate in equilibrium\(^{11}\).

We have

\[
\begin{align*}
    rU = b + g(\varepsilon) [W(X_0) - U] & \quad \text{i.e.} \\
    g(\varepsilon) &= \frac{rU - b}{W(X_0) - U}
\end{align*}
\]  

(3.1.13)

(3.1.14)

A potential employer chooses an optimal coupon rate \( c_{\text{max}} \) such that the its expected value from labor search given unemployment value \( U \), denoted by \( V(c;U) \), is maximized. \( V(c;U) \) obeys the following equation\(^{12}\):

\[
\begin{align*}
    rV(c;U) &= -\kappa + h(g(\varepsilon)) \left[ E(X_0) + D(X_0) - V(c;U) \right]
\end{align*}
\]  

(3.1.15)

\( \kappa \) is the flow cost of keeping the vacancy open and \( h(g(\varepsilon)) \) is the employer’s hiring rate, which, by Section 2.1, is a function of workers’ job-finding rate \( g(\varepsilon) \). \( g(\varepsilon) \) is given by (3.1.14).

3.2 Competitive search equilibrium with endogenous leverage

A competitive search equilibrium with endogenous leverage consists of an optimal coupon rate \( c_{\text{max}} \), a separation threshold \( X \), a vector of asset values \( (D,E,U,V,W) \), a labor market tightness \( \varepsilon \), a stationary distribution density function \( f(X) \) of matching cash flows, such

\(^{11}\)As a result, \( U \) only depends on the aggregate debt(coupon) level in the labor market.

\(^{12}\)Strictly speaking, in the potential employer’s optimization problem, \( h(g(c;U)) \) should be \( h^c(g(c;U)) \), the potential employer’s belief about the relationship between its coupon choice and hiring rate. However, I focus on a stable rational expectation equilibrium concept, meaning that in equilibrium, the potential employer’s expectation always coincides with the true relationship between capital structure and the hiring rate, which is \( h(g(c;U)) \) and \( g \) is determined by (3.1.14). See Moen (1997) for a similar argument.
that the following conditions hold:

I. The debt value $D$ satisfies (3.1.4), the employer’s and worker’s expected value of a match $E$ and $W$ satisfied the boundary value problem (3.1.6) subject to (3.1.7), and (3.1.8) subject to (3.1.9), respectively. The expected value of a searching worker $U$ and a potential employer $V$ satisfies (3.1.13), (3.1.15), respectively.

II. Given $U$, $c_{\text{max}}$ maximizes the expected value of the potential employer from labor searching $V$, subject to (3.1.14), and $X = \max\{X^E, X^W\}$, where $X^E$ and $X^W$ are chosen to maximize $E$ and $W$, respectively.

III. Free entry: In equilibrium $V(c_{\text{max}}; U) = 0$

IV. Labor market tightness: The equilibrium job finding rate $g$ and thus labor market tightness $\varepsilon$ satisfies (3.1.14)

V. Stationary labor market: There exists a stationary distribution density function $f(X)$ of matching cash flows. $f(X)$ is such that the outflow from the unemployment population is equal to the inflow to the unemployment population at every $dt$, which is equivalent to the requirement that the inflow to employment population is equal to the outflow from the employment population at every $dt$.

Condition (I)—(IV) are standard requirements for competitive search equilibrium of labor market. Condition (V) requires that the steady-state distribution of matching cash flows is time-invariant. This is possible because the matching cash flows are subject to idiosyncratic shocks and a law of large number is assumed. However, I should emphasize that the actual identities of matches behind this stationary aggregate distribution change continually at every instant: Some new matches enter into the economy while others dissolve and the
matching parties return to labor market search\(^{13}\).

4. Solve the equilibrium

4.1 Wage and separation threshold

According to Section 3.2, the cash flow threshold that entails an endogenous matching separation is the higher value of the worker’s and the employer’s optimal separation thresholds. This significantly complicates the optimal stopping problem. In this subsection, I will show that the two separation thresholds always coincide with each other, which greatly simplifies my analyses. Then I derive a linear wage function of cash flow \(X\).

After a match is created and the debt is issued, the worker and the employer split the remaining matching surplus through continuous bilateral bargaining according to a generalized Nash bargaining rule. This bargaining rule selects the wage such that:

\[
\w(X) \in \arg \max_{w'} [W(X) - U]^{\beta} [E(X) - V]^{1-\beta}
\]

As a standard result in the search models of labor market, this maximization yields as a necessary and sufficient first-order condition:

\[
\beta [E(X) - V] = (1 - \beta) [W(X) - U]
\]

In equilibrium, the employer’s outside option \(V = 0\). The worker’s outside option value is \(U\). Therefore,

\[
\beta E(X) = (1 - \beta) [W(X) - U] \quad (4.1.1)
\]

\(^{13}\)Miao (2005) analyzes an industry equilibrium model using similar concept. Also refer to Dixit and Pindyck (1994).
Taking derivatives of both sides of (4.1.1) with respect to $X$ results in:

$$\beta E' (X) = (1 - \beta) W' (X)$$  \hspace{1cm} (4.1.2)

and

$$\beta E'' (X) = (1 - \beta) W'' (X)$$  \hspace{1cm} (4.1.3)

One direct consequence of equation (4.1.2) is that the employer and the worker of a match always agree to separate the matching relationship and return to search when $X$ hits the same threshold, i.e., $X^E = X^W = X$. Intuitively, this is because under generalized Nash bargaining, the worker’s and the employer’s payoffs from a match are both equity-like. Therefore, the asset values have expressions similar to those in standard contingent claim models (e.g., Goldstein et al. 2001). This greatly simplifies my analyses. The following lemma shows that the wage rate is linear in current cash flow $X$.

**Lemma 1.** In equilibrium, under generalized Nash bargaining, the wage rate is linear in $X$:

$$w(X) = \beta (X - c) + (1 - \beta) b + (1 - \beta) g(\varepsilon) [W(X_0) - U]$$

$$= \beta (X - c) + (1 - \beta) b + \beta g(\varepsilon) E(X_0)$$  \hspace{1cm} (4.1.4)

**Proof.** [A.1]  \hspace{1cm} Q.E.D.

### 4.2 Matching surplus

By the virtue of generalized Nash bargaining, the employer’s and the worker’s expected value of the match are proportional to matching surplus: $S = E + W - V - U$. Specifically,

$$E - V = (1 - \beta) S$$

20
and

\[ W - U = \beta S \]

The value function of being unemployed (3.1.13) can be expressed in term of \( S \):

\[ rU = b + g\beta S (X_0) \quad (4.2.1) \]

Similarly, the value function of a potential employer is

\[ rV = -\kappa + h[(1 - \beta) S (X_0) + D (X_0)] \quad (4.2.2) \]

Here \( g \) is the job-finding rate for an unemployed worker and \( h \) is the hiring rate of a potential employer and is a function of \( g \). By the definition of \( S \), given \( c \), \( U \) and \( V \), \( S \) solves the following free-boundary problem:

\[ rS (X) = X - c - rU - rV + \mu XS' (X) + \frac{1}{2}\sigma^2 X^2 S'' (X) - sS (X) \quad (4.2.3) \]

(4.2.3) is equivalent to

\[ \delta S (X) = X - c - rU - rV + \mu XS' (X) + \frac{1}{2}\sigma^2 X^2 S'' (X) \quad (4.2.4) \]

where \( \delta = r + s \). The boundary conditions for \( S (X) \) are

\[ S (X) = 0 \quad (value \ matching) \]

\[ S' (X) |_{X=X_0} = 0 \quad (smooth \ pasting) \quad (4.2.5) \]

\[ \lim_{X \to \infty} \left( \frac{S}{X} \right) < \infty \quad (no \ bubble) \]

\(^{14}\)See the discussion at the end of Section 2.1.
The free-boundary problem (4.2.4) subject to (4.2.5) is common in the contingent models of corporate finance and admits a closed-form solution (e.g., Goldstein et al., 2001). Taking the derivative of \( S(X) \) with respect to \( X \) and by the “smooth-pasting” condition yields the optimal cash flow threshold for matching separation \( X \). Proposition 1 summarizes the results.

**Proposition 1.** Given \( c, U \) and \( V \), the matching surplus \( S(X) \) is given by

\[
S(X) = \Pi(X) - \frac{c + rU + rV}{\delta} - \left[ \Pi(\bar{X}) - \frac{c + rU + rV}{\delta} \right] \left( \frac{X}{\bar{X}} \right)^{\vartheta} \quad (4.2.6)
\]

where \( \Pi(X) \) is given by (3.1.1) and \( \vartheta \) is given by (3.1.5). The optimal separation threshold \( \bar{X} \) is

\[
\bar{X} = \frac{-\vartheta}{1 - \vartheta} (\delta - \mu) \frac{c + rU + rV}{\delta} \quad (4.2.7)
\]

With the free-entry condition in the equilibrium, i.e., \( V = 0 \), the matching surplus becomes

\[
S(X) = \Pi(X) - \frac{c + rU}{\delta} - \left[ \Pi(\bar{X}) - \frac{c + rU}{\delta} \right] \left( \frac{X}{\bar{X}} \right)^{\vartheta} \quad (4.2.8)
\]

and the equilibrium optimal separation threshold \( \bar{X} \) is

\[
\bar{X} = \frac{-\vartheta}{1 - \vartheta} (\delta - \mu) \frac{c + rU}{\delta} \quad (4.2.9)
\]

Notice that the optimal separation threshold is increasing in \( c \). This is consistent with contingent models of capital structure (e.g., Leland, 1994). The optimal separation threshold \( \bar{X} \) also increases with \( U \), which is new to the literature. Intuitively, since both the employer and the employee of the match can trigger the matching separation, the optimal separation threshold incorporates the employee’s outside option value \( U \).
4.3 Optimal coupon $c^{\text{max}}$

The potential employer chooses the optimal coupon rate $c^{\text{max}}$ that maximizes its expected value from labor market search, $V(c; U)$, taken the expected value of an unemployed worker’s job search, $U$, as given. $V(c; U)$ satisfies the asset equation (4.2.2) and the potential employer’s hiring rate $h$ is a function of the job-finding rate $g$. As discussed in Section 3.1.4 for a given $U$, the relationship between the job-finding rate $g$, thus the employer’s hiring rate $h$, and the coupon rate $c$, is determined by (4.2.1)\textsuperscript{15} The initial expected value of matching surplus $S(X_0)$ and debt $D(X_0)$ is given by (4.2.6) and (3.1.4) with the optimal separation threshold $X$ is given by (4.2.7).

Taking the derivatives of both sides of (4.2.2) with respect to $c$, the first-order condition for the optimal coupon rate is given by the following proposition\textsuperscript{16}.

**Proposition 2.** In equilibrium, the optimal coupon rate $c^{\text{max}}$ satisfies the following first-order condition:

$$h \left[ \frac{\beta}{\delta} + \frac{1}{\delta} \left( \frac{\alpha \nu}{c^{\text{max}} + rU + rV} - \beta \right) \left( \frac{X_0}{X} \right)^\theta \right] + h(c) \left\{ \frac{(1 - \beta)}{\beta} \left[ \Pi (X_0) - \frac{rU + rV}{\delta} \right] + \frac{1 - \beta}{\beta} \frac{c^{\text{max}}}{\delta - \delta} \left( \frac{X_0}{X} \right)^\theta \right\} = 0$$

(4.3.1)

$$h(c) = \frac{\partial h}{\partial g} \frac{\partial g}{\partial c} = h_1 \cdot h_2 < 0, \text{ where } h_1 = h'(g), \text{ the derivative of the hiring rate } h \text{ with respect to the job-finding rate } g, \text{ and } h_2 = \frac{\partial g}{\partial c} = \frac{\delta}{X} \left[ 1 - \left( \frac{X_0}{X} \right)^\theta \right].$$

In the Section IA.2.2, I also present the second-order condition for optimal coupon, $c^{\text{max}}$ that satisfies (4.3.1) is optimal if and only if (IA.2.17) is simultaneously satisfied.

\textsuperscript{15}As discussed in Section 3.1.4 strictly speaking, $h$ should be the potential employer’s belief about the relationship between its coupon choice $c$ and its hiring rate $h$, given $U$. However, in equilibrium, the potential employer’s expectation is always the true relationship between capital structure and the hiring rate, which is dictated by (4.2.1).

\textsuperscript{16}For details, refer to Section IA.2
Equation (4.3.1) gives me an intuitive result regarding a potential employer’s optimal capital structure policy. When choosing its optimal capital structure, the potential employer balances three effects of $c$ on its expected value from labor search. All three effects have empirical supports. First, by using a higher leverage policy, the potential employer expropriates a larger share of the matching cash flow in the form of proceeds from debt issuance once it matches with a worker. The remaining size of the cash flow available to the worker shrinks. This so-called “strategic role of debt” is empirically documented by, for example, Matsa (2010), who finds that employers reduce debt after their states adopt new legislations to reduce the bargaining powers of labor unions. The second effect is the classic cost of financial distress. Since ceteris paribus, a higher leverage ratio triggers bankruptcy earlier and since bankruptcy is costly, a higher $c$ reduces the potential employer’s expected value from a match, because it forces “premature” separation of the match. This effect is absent in the traditional search models of labor market, as they typically overlook employers’ capital structure decisions. Meanwhile, the cost of financial distress associated with high leverage is widely documented in the trade-off theory of capital structure with risky debt (e.g., Myers, 2001). The last effect, which is novel to the theoretical literature on labor market search, is that a higher coupon rate $c$ reduces the arrival rate of the workers to the potential employer’s job vacancy, thereby reducing the potential employer’s matching intensity in the labor market. This effect is empirically identified by Brown and Matsa (2016), who find that job seekers have precise information about potential employers’ financial conditions. Job vacancies posted by potential employers with poor financial conditions and higher leverage have fewer applicants.

In equilibrium, a potential employer optimally designs its capital structure, which balances the benefit of leverage—the strategic role of debt—and two costs of leverage (the cost of financial distress and the negative impact of debt on the potential employer’s hir-
ing rate). Mathematically, the potential employer chooses the optimal $c$ that equalizes the following two absolute values of partial elasticities with respect to $c$: the partial elasticity of expected post-match employer value, $(1 - \beta) S(X_0) + D(X_0)$ with respect to $c$, and the partial elasticity of the potential employer’s hiring rate, $h$, with respect to $c$, i.e.,

$$|\eta_c [(1 - \beta) S(X_0) + D(X_0)]| = |\eta_c (h)| \quad (4.3.2)$$

$|\eta_c (.)|$ stands for the absolute value of respective partial elasticity with respect to $c$.

### 4.4 Expected job tenure

Next I derive a closed-form representation of expected remaining job tenure given the cash flow $X$, $T(X)$. Standard results from stochastic process literature (e.g., [Karlin and Taylor, 1981]) show that $T(X)$ solves the boundary value problem

$$\mu X T''(X) + \frac{1}{2} \sigma^2 X^2 T''(X) - sT(X) = -1 \quad (4.4.1)$$

subject to the following boundary conditions

$$T(X) = 0 \text{ and } \lim_{X \to \infty} T(X) = \frac{1}{s} \quad (4.4.2)$$

Heuristically, the expected remaining tenure is zero if $X$ hits the separation threshold, $X$. Meanwhile, if $X$ is very large, the only event that could end the match is an exogenous match-destruction event with the arrival intensity $s$. Solving explicitly the boundary value problem (4.4.1) subject to (4.4.2) for $T(X)$, I have the following proposition:

**Proposition 3.** Given the current cash flow level $X$ and cash flow threshold for matching
separation $X$, the expected residual duration of a job (expected remaining job tenure) is

$$T(X) = \frac{1}{s} \left[ 1 - \left( \frac{X}{X} \right)^\rho \right]$$

(4.4.3)

where $\rho = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) - \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2s}{\sigma^2}}$.

**Proof.** Section A.2. Q.E.D.

$T(X)$ is decreasing in the coupon rate $c$ and increasing in the current cash flow $X$.

### 4.5 Stationary distribution of matching cash flows

I close the equilibrium by characterizing the stationary cross-sectional distribution of the matching cash flows $X$s. Note that conditional on optimal coupon rate $c^{\text{max}}$, searching workers’ expected value $U$ and job-finding rate $g$, the leverage ratio, wage rate and expected remaining job tenure of any match in the economy can be conveniently expressed as functions of cash flow $X$. As will be shown in Section 5 by deriving the stationary cross-sectional distribution of $X$, I am able to calculate the steady-state average leverage ratio, unemployment rate, mean-minimum wage ratio and average expected remaining job tenure in the economy.

My economy is a stochastic growth economy featuring matching pair deaths and births. The stochastic process governing the dynamic evolution of a matching cash flow $X$ is assumed to be a geometric Brownian process. This implies that $X$ belongs to a class of Kolmogorov-Feller diffusion process. Let $\tilde{f}(X; X_0)$ be the transition density function of $X$ starting at $X_0$. From the classic results of Karatzas and Shreve (2012), the dynamics of

\footnote{For the application of power laws to city and population growth, see Gabaix (2009).}
\footnote{For the definition of Kolmogorov-Feller diffusion process, see Chapter 5, Definition 1.1 in Karatzas and Shreve (2012).}
\footnote{In what follows, I omit $X_0$ in $\tilde{f}(X; X_0)$ for simplicity.}
\( \tilde{f} (X) \) follow a Fokker-Planck equation, also known as a Kolmogorov forward equation of \( X \), \forall X \in [X, \infty) \setminus \{X_0\},

\[
\frac{df(X)}{dt} = -\frac{d}{dX} [\mu X f(X)] + \frac{1}{2} \frac{d^2}{dX^2} [\sigma^2 X^2 f(X)] - s \tilde{f}(X) \quad (4.5.1)
\]

Letting \( f(X) \) denote the stationary \( \tilde{f}(X) \), I have the following boundary value problems governing \( f(X) \):

\[
-\frac{d}{dX} [\mu X f(X)] + \frac{1}{2} \frac{d^2}{dX^2} [\sigma^2 X^2 f(X)] - s f(X) = 0 \quad (4.5.2)
\]

subject to \(^{20}\)

\[
f(X^+) = 0 \quad (4.5.3a)
\]

\[
\frac{1}{2} \sigma^2 X_0^2 [f'(X_0-) - f'(X_0+)] = s \int_{X^-}^{X^+} f(X) \, dX + \frac{1}{2} \sigma^2 X^2 f'(X+) \quad (4.5.3b)
\]

\[
g \left[ 1 - \int_{X^-}^{X^+} f(X) \, dX \right] = s \int_{X^-}^{X^+} f(X) \, dX + \frac{1}{2} \sigma^2 X^2 f'(X+) \quad (4.5.3c)
\]

The boundary conditions, despite their complexities, are intuitive under scrutiny. First, once the matching performance is so poor that the cash flow \( X \) reaches the endogenous default threshold \( X \), separation occurs immediately. In other words, \( X \) spends no time at \( X \). Mathematically, this requires \( \frac{1}{2} \sigma^2 X^2 f(X+) = 0 \). Since, \( \frac{1}{2} \sigma^2 X^2 \neq 0 \), I have \( (4.5.3a) \). Second, \( (4.5.3b) \) has an economic meaning as follows: At steady state, the total flows into the employment must commensurate the total flows out of the employment. The left-hand side is the total flows into the employment. The density \( f(X) \) is not differentiable at \( X_0 \),

\[\text{\(X^+ = \lim_{X' \uparrow X} X_0 \) and \( X^- = \lim_{X' \downarrow X} X_0\).}\]

\[\text{\(^{21}\)Mathematically, \( X \) is an attainable boundary that can be hit by the process in finite time period with a positive probability. Moreover, attainable boundaries are either absorbing or reflecting. In my case, it is absorbing.}\]

27
corresponding to the inflow of workers to the employment and all new matches starting at \( X_0 \). The right-hand side is the total flows out of the employment. The first term is intuitive. According to the second term, the flow of matches exiting at \( X \) per unit of time is \( \frac{1}{2}\sigma^2 X^2 \phi' (X+) \). This quantity can be achieved by integrating the left-hand side of (4.5.2) and then evaluate at \( X = X_22 \). More heuristically, over a small enough interval of time \( \Delta \), the diffusion term in \( dX_t = \mu X_t dt + \sigma X_t dZ_t \) dominates, and half of the approximate measure \( \phi \left( X + \sigma \sqrt{\Delta} \right) \times \sigma \sqrt{\Delta} \) of matches near the boundary \( X \) will exit the production. Finally, (4.5.3c) is the standard restriction on search models in the labor market (e.g., Mortensen and Pissarides, 1994), and it yields the Beveridge curve. The left-hand side is the outflow from the unemployment population, and the right-hand side is the inflow to the unemployment population, which, by definition, is also the outflow from the employment population.

The solution technique of the boundary problem (4.5.2) subject to boundary conditions (4.5.3a), (4.5.3b) and (4.5.3c), is similar to that in the continuous-time power law literature (e.g., Gabaix, 2009; Achdou et al., 2014). \( \forall X \in [X, \infty] \setminus \{X_0\} \), the following proposition characterizes the stationary cross-sectional distribution density function of \( X \):

**Proposition 4.** In equilibrium, given \( c, U \) and \( g \), \( \forall X \in [X, \infty] \setminus \{X_0\} \), the stationary cross-sectional distribution density function of \( X \) is:

\[
\phi(X) = \begin{cases} 
\zeta X^{-m_1-1}, & X > X_0 \\
\tilde{\zeta} X^{-m_0-1} \left[ 1 - \left( \frac{X}{X_0} \right)^{m_1-m_0} \right], & X \leq X < X_0 
\end{cases}
\]

where \( m_0 = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) - \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2s}{\sigma^2}} \) and \( m_1 = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) + \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2s}{\sigma^2}} \), \( \zeta \) and \( \tilde{\zeta} \) are uniquely determined by boundary conditions (4.5.3b) and (4.5.3c).

**Proof.** Section A.3  Q.E.D.

\( ^{22} \)Notice that by boundary condition (4.5.3a), \( \phi(X) = 0 \)
The expression of the stationary cross-sectional distribution density function \( f(X) \) of cash flow \( X \) takes the form of double-Pareto distribution density, as repeatedly shown in the stochastic growth literature (e.g., Gabaix 2009; Achdou et al. 2014).

We thus complete the solution of a competitive search equilibrium with endogenous leverage, defined in Section 3.2. In the next section I will calibrate a baseline model and do comparative static analyses to explore the model’s implications regarding employers’ capital structure policies and labor market outcomes.

5. Results

To study the model implications, I first calibrate a base case model using a set of carefully chosen parameter values. Then I conduct comparative static analyses.

5.1 Base case

In this section, I calibrate a base case model. To solve the model, First, the optimal coupon rate \( c^{\text{max}} \), unemployed workers’ expected value \( U \) and job-finding rate \( g \) are jointly determined by the employers’ free-entry condition, \( V = 0 \), the unemployed worker’s and the potential employer’s value equations (4.2.1) and (4.2.2), respectively, and the first-order condition for the employer’s optimal coupon choice (4.3.1). After checking the second-order condition for the optimality of the coupon choice obtained from the first step, the expected tenure is derived according to (4.4.3) of Proposition 3. The stationary cross-sectional distribution density function of matching cash flows, \( f(X) \), is derived according to (4.5.4) of Proposition 4. The unemployment rate \( u \) is equal to \( 1 - \int_{X}^{\infty} f(X) \, dX \).

I first explain the model parameter choices in Section 5.1.1. Then I solve a base case model using the selected parameters in Section 5.1.2.
5.1.1 Parameters

The parameter values are either taken from the existing literature or calibrated such that the model’s equilibrium quantities match corresponding empirical quantities. The abundance of literature in both corporate finance and search theory in the labor market allows me to pin down most parameter values using the former approach. For the remaining parameters that the literature does not have clear guidance for their values, I adopt the latter approach to calibrate their values. The model is tightly parameterized.

The risk-free rate $r = 2\%$ and cash flow rate volatility a typical firm in my economy $\sigma = 25\%$ are standard in the literature (e.g., He and Milbradt [2014], Morelec et al., [2012]) estimate that the the risk-neutral drift of cash flow process $\mu = 0.0067$ using data from Compustat, CRSP and Institutional Brokers’ Estimation System(IBES). Therefore, I set $\mu$ in my model equal to 0.0067. The bankruptcy cost parameter, $\alpha$ is set at 0.25, which is consistent with recent literature (e.g., Hackbarth and Mauer, 2011) \footnote{As a further reference, Bhamra et al. (2010) calibrate an economy with state change according to a two-state Markov chain. They set the proportional bankruptcy cost parameter of 30% in low-state and 10% in high state.}

Turning to the parameters concerning with labor market search, following the literature \footnote{The existing literature finds that Cobb-Douglas-type matching functions accurately recounts the matching process in the labor market (e.g., Petrongolo and Pissarides (2001) and citations therein.)}, I assume the matching function is of Cobb-Douglas form and is a function of unemployment rate $u$ and vacancy rate, $v$. Specifically, the instantaneous matching rate $m(u, v) = Au^\iota v^{1-\iota}$. Following Hall and Milgrom (2008), I set the elasticity of matching function with respect to unemployment rate $u$ at 0.5. Hall and Milgrom (2008) calibrate the matching efficiency parameter of 0.024 per weekday to match the job-finding rate and vacancy-unemployment ratio. This translates to about 6 per year. Correspondingly, I choose the matching efficiency parameter in my model, $A$, equal to 6. Existing literature (e.g., Shimer, 2005) estimates that the flow value of unemployment benefit, $b$ is equal to 40% of the flow product of a labor
market match. Correspondingly, I calculate $b = 0.1693$. This value accounts for 40% of average value-added over total asset of a S&P 500 firm from 1990 to 2018. Following the existing literature (e.g., Hall and Milgrom, 2008; Moen and Rosén, 2011), the flow cost of keeping a job vacancy open, $\kappa = 0.4$. The worker’s bargaining power, $\beta$, is set at 0.72, consistent with the empirical estimate by Shimer (2005).

It remains to determine the match-specific starting cash flow, $X_0$, and match-specific Poisson separation intensity, $s$. Unlike most other dynamic corporate finance models with geometric Brownian motion cash flow process, the starting point of match-specific cash flow, $X_0$, matters in my model because the inflow to employment at $X_0$. Shimer (2005) uses Current Population Survey and estimates that the quarterly job separation rate is 0.1 per quarter, equivalent to 0.4 per year. However, in the Mortensen-Pissarides framework without on-the-job search, the only source of job termination is exogenous separation. Unique in my model, either one of the two types of matching separations could occur at every instant: the exogenous separation and the endogenous separation when matching cash flow process hits the endogenous separation threshold. Unfortunately, existing literature does not provide clear guidance regarding values of these two parameters. I calibrate these two parameters to match the average profitability of a S&P 500 firm from 1990 to 2018, and the average of median years of tenure with current employer for employed wage and salary workers in the U.S. from 2008 to 2018.\footnote{Source: Bureau of Labor Statistics https://www.bls.gov/news.release/tenure.t01.htm} Average expected job tenure in my model is $\int_X^\infty T(X) f(X) \, dX$, where $T(X)$ is defined in Proposition 3 and $f(X)$ is defined in Proposition 4. I present the parameter values used in the base case model in Table 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Parameter & Value & Source \\
\hline
$\kappa$ & 0.4 & Current Population Survey \\
$\beta$ & 0.72 & Empirical estimate by Shimer (2005) \\
$X_0$ & & \\
$s$ & & \\
\hline
\end{tabular}
\caption{Parameter values used in the base case model.}
\end{table}
5.1.2 The baseline model

The equilibrium quantities for the baseline model are presented in Table 2. Proposition 4 characterizes the stationary cross-sectional distribution of matching cash flows $X_s$ in the equilibrium. Figure 1 presents this distribution. In the equilibrium, the stationary cross-sectional distribution of matching cash flows $X_s$ exhibits a double-Pareto form. According to Lemma 1, the wage rate in a match is linear in the matching cash flow $X$. The linearity means that the stationary cross-sectional distribution of wages has exactly the same shape as the cash flow distribution. Therefore, my model is able to generate empirically observed wage distribution: a Pareto-functional form with a long and “fat” right tail (e.g., Reed, 2001, 2003; Gabaix, 2009). To further explore the wage dispersion in my model, I calculate the ratio between mean and minimum wage, the “mean-min wage ratio”, a measure of frictional wage dispersion. Hornstein et al. (2011) first propose this measure of frictional wage dispersion. They point out that most search models of labor economics literature cannot generate a large enough ratio that matches the empirical quantity, given the observed size of the transition rate of workers. The mean-min wage ratio in my model is 1.79, which is much higher than that from most search models of labor market, and is close to the empirically observed quantity. Overall, the baseline model generates both an empirically plausible functional form of wage distribution and creates a large enough frictional wage dispersion. The mechanism behind the wage results is as follows: two crucial and distinct features of my model are idiosyncratic risk in the match-specific cash flow process and the endogenous separation of matches. A direct consequence of these two features is the redistribution of

\begin{footnote} 
For example, according to Hornstein et al. (2011), the mean-min wage ratio in the baseline search model is 1.05, indicating that the average wage can be only 5% higher than the lowest wage. This number is way below the empirical counterpart. 
\end{footnote} 

\begin{footnote} 
A conservative estimates of mean-min wage ratio from empirical data is between 1.7 and 1.9. See, for example, Hornstein et al. (2011) and references therein. 
\end{footnote} 

\begin{footnote} 
Allowing an employer to choose its own capital structure leads to an endogenous matching separation at a higher cash flow threshold. 
\end{footnote}
workers from the lower to the upper part of the cross-sectional cash flow distribution, which, in turn, explains the power-law-type tail distribution of the matching cash flows, thus the wage distribution. Heuristically, the matches with sufficiently low cash flows will not survive in the steady-state equilibrium because the matching parties will find it optimal to part their ways and return to the search in the labor market. Meanwhile, matches with high cash flows will not be “abandoned” by the matching parties and are thus more likely to exist in the equilibrium.\footnote{Good matches could still dissolve because of the exogenous Poisson process that obliterates the match.} This results in a Paretian right tail in the wage distribution and a larger frictional wage dispersion.

\[\text{[Insert Figure 1 here.]}\]

Next I examine whether my model could potentially reconcile the high standard deviation of vacancy-to-unemployment ratio relative to the standard deviation of labor productivity. \cite{Shimer2005} first observes that the standard deviation of vacancy-to-unemployment ratio is almost 20 times as large as the standard deviation of labor productivity, but according to standard search models of labor market, the standard deviation of vacancy-to-unemployment ratio should be roughly the same as that of the labor productivity. Since I do not introduce aggregate cash flow shocks in my model, a full resolution of this puzzle is beyond the scope of this paper. Nevertheless, \cite{Shimer2005} calculates that the elasticity of vacancy-to-unemployment ratio with respect to matching productivity is 1.7, which is much lower that observed in the U.S. data \cite{Hall2005}. Correspondingly, I calculate the elasticity of vacancy-to-unemployment ratio ($\varepsilon$) with respect to the average matching productivity, defined as the average match-specific cash flow, $X^{31}$. The closed-form stationary cross sectional distribu-
tion of matching cash flows facilitates the calculation. From Proposition 4, the job-finding rate, thus vacancy-to-unemployment ratio, and the average matching productivity is linked through the following equation.

\[
X = \tilde{\zeta}X^{-m_0+1} \left[ \frac{1}{m_0-1} \left(1 - \left(\frac{X}{X_0}\right)^{m_0-1}\right) - \frac{1}{m_1-1} \left(1 - \left(\frac{X}{X_0}\right)^{m_1-1}\right) \right] + \frac{\zeta}{m_1-1}X_0^{-m_1+1}
\]

The elasticity in my model is 15.4778, much larger than that from standard Mortensen-Pissarides framework. The intuition for this much larger elasticity is as follows. Unlike most existing search models of labor market, in my model, the vacancy-to-unemployment ratio and the average matching productivity are jointly determined. Vacancy-to-unemployment ratio affects the average matching productivity in the economy through a “redistribution” channel. The higher the vacancy-to-unemployment ratio, the more new matches will spawn at initial productivity (cash flow) level, \(X_0\), to replace the matches that “die” at the lower productivity level \(X\). In other words, the vacancy-to-unemployment ratio, through the matching function, “redistributes” the matching cash flows from a lower level \(X\) to a higher initial level \(X_0\). Therefore, it boosts the average matching productivity in the economy. The efficiency of this “redistribution” channel depends on the distance between \(X_0\) and \(X\) versus the distance between \(X_0\) and the average matching productivity \(X\). The shorter the distance between \(X_0\) and \(X\) is, compared with the distance between \(X_0\) and \(X\), the less efficient the “redistribution” channel will be. It, in turn, requires a larger increase in vacancy-to-unemployment ratio to raise the average matching productivity by 1%. In the calibrated exercise, \(X_0\) is closer to the matching separation threshold, which is 0.3105 according to (4.2.9), compared with the average matching cash flow, \(X\). Therefore, a 1% increase in
average productivity will require a large increase in the vacancy-to-unemployment ratio.\footnote{In untabulated results, I find that the elasticity of vacancy-to-unemployment ratio with respect to average matching productivity increases with the coupon rate. This confirms the mechanism I present because a higher coupon rate increases the endogenous matching separation threshold, thereby bringing \( X \) closer to \( X_0 \).} \footnote{\textcite{Hall2005} also calculates industry-average leverage ratio using a similar definition. In my model, as standard in the search literature of labor market, the wage is determined by generalized Nash-bargaining after the employer makes the coupon payment. In this sense, both shareholders and workers hold equity-like claims. This is consistent with empirical observation that many firms offer employees and executives equity-type compensations, for example, through employee stock ownership plans and equity compensation plans. Therefore market leverage is defined as the ratio between the average market value of debt and the sum of the average market values of debt and matching surplus. Alternatively, one may think that the leverage ratio in this paper is the market value of debt over the firm’s fair enterprise value.}

Does the trade-off model from labor market search and wage bargaining generate a capital structure that is comparable to what we observe in the data? To answer this question, I calculate the equilibrium average market leverage ratio in the economy. The closed-form stationary cross-sectional distribution of matching cash flows allows me to characterize the market leverage ratio according to the following equation:

\[
\text{market leverage} = \frac{\int_X^{\infty} D(X; c, U) f(X) \, dX}{\int_X^{\infty} [D(X; c, U) + S(X; c, U)] f(X) \, dX}
\]

The numerator is the average market value of debt and the denominator is the sum of the average market values of debt and matching surplus.\footnote{\textcite{Miao2005} also calculates industry-average leverage ratio using a similar definition. In my model, as standard in the search literature of labor market, the wage is determined by generalized Nash-bargaining after the employer makes the coupon payment. In this sense, both shareholders and workers hold equity-like claims. This is consistent with empirical observation that many firms offer employees and executives equity-type compensations, for example, through employee stock ownership plans and equity compensation plans. Therefore market leverage is defined as the ratio between the average market value of debt and the sum of the average market values of debt and matching surplus. Alternatively, one may think that the leverage ratio in this paper is the market value of debt over the firm’s fair enterprise value.}

On average, the model-implied market leverage ratio is 0.1826, which is very close to the empirically documented market leverage ratio\footnote{\textcite{Hallingetal2016} document that the average market leverage ratio for U.S. COMPUS-TAT firms is 0.178 and for international firms is 0.198 from 1984 to 2009.}. Overall, the trade-off model in this paper—weighing the benefit of using debt as a strategic bargaining tool in the wage determination process, against the cost of

\[\text{Debt} = \frac{\text{Debt}}{\text{Debt} + (1 - \beta) \text{Surplus}},\]

If I assume that the labor market satisfies “Hosios rule” \footnote{If I assume that the labor market satisfies “Hosios rule” \cite{Hosios1990}, that the risk-free rate \( r = 0.05 \), also common in the literature, and define market leverage as \( \frac{\text{Debt}}{\text{Debt} + (1 - \beta) \text{Surplus}} \), then the model-implied market leverage ratio is 0.2155. This number is also close to its empirical counterpart \cite{Hallingetal2016}.} \cite{Hosios1990}, that the risk-free rate \( r = 0.05 \), also common in the literature, and define market leverage as \( \frac{\text{Debt}}{\text{Debt} + (1 - \beta) \text{Surplus}} \), then the model-implied market leverage ratio is 0.2155. This number is also close to its empirical counterpart \cite{Hallingetal2016}.
debt arising from competition for workers and bankruptcy cost—generates a leverage ratio consistent with empirically observed capital structure practice without relying on the tax benefit of debt. In the U.S., the 2017 tax law drastically reduced the maximum allowable tax deduction for corporate interest expense. So tax shield in the traditional trade-off model of capital structure may become a second order consideration going forward. Therefore, the development of a new trade-off theory and the demonstration of its empirical relevance are also practically important.

Overall, the baseline model generates a wage distribution of empirically accurate shape. It also points to promising directions toward solving two puzzles that vex the traditional Mortensen-Pissarides search model of labor market: the overly low model-implied frictional wage dispersion and the inelastic model-implied vacancy-to-unemployment ratio with respect to productivity, compared with their empirical counterparts. The model also generates a leverage ratio that is consistent with empirical observations. In the following subsection, I conduct a series of comparative static analyses to further explore the model implications.

[Insert Table 2 here.]

### 5.2 Comparative statics

In my model, employers’ capital structure choices and labor market outcomes jointly respond to changes in exogenous parameters. I focus on five parameters: the drift and volatility of match-specific cash flow, $\mu$ and $\sigma$ respectively, the flow value of unemployment benefit, $b$, the workers’ bargaining power, $\beta$, and the bankruptcy cost, $\alpha$. For each comparative static analysis, I allow for one model parameter to vary across a plausible range, holding other model parameters at the level dictated in the base case.
5.2.1 Drift of match-specific cash flow $\mu$

As mentioned in the Section 1, the change in optimal coupon rate after an increase in $\mu$, depends on the relative strength of increased benefit of debt in the single-firm analysis against the increased cost of debt from intensified labor market competition. Therefore, in the current calibration, coupon rate exhibits a non-monotonic relationship with different values of $\mu$, as in Figure 2a. Moreover, the average market leverage decreases with $\mu$ according Figure 2b. Thus the trade-off mechanism in my paper is able to explain the empirical fact that high-growth firms do not necessarily have higher leverage\textsuperscript{37}. A higher growth rate of matching cash flow also results in a lower unemployment rate and a tighter labor market, i.e., a higher vacancy-to-unemployment ratio, as indicated by Figure 2c and Figure 2d. This is because more potential employers enter the labor market and post vacancies in anticipation of better matching performance. Figure 2e shows that the frictional wage dispersion, mean-to-minimum wage ratio, increases since more matches will yield higher cash flows and the threshold as a result of higher cash flow growth rate. Since a higher cash flow growth rate pushes more matches away from their endogenous separation thresholds, the average expected job tenure also increases, as shown in Figure 2f.

\[\text{[Insert Figure 2 here.]}\]

5.2.2 Volatility of match-specific cash flow $\sigma$

Since the matching surplus, $S(X)$, is equity-like that resembles a call option of the matching value, an increase of cash flow volatility $\sigma$ increases the worker’s expected payoff. The employer has a higher incentive to issue more debt to divert the matching cash flow from

\textsuperscript{37}Previous explanations are based on underinvestment problem\cite{Myers2001}, free-cash flow problem\cite{Jensen1986}, and competitive industry equilibrium\cite{Miao2005}.
surplus to debt, which the worker cannot claim. Consistent with this reasoning, the firm’s coupon choice, $c$, increases with the matching cash flow volatility, indicated by Figure 3a. This is so-called “coupon effect”. Meanwhile, ceteris paribus, a higher cash flow volatility depresses the debt price (“price effect”) Therefore, the net impact of matching cash flow volatility $\sigma$ on an employer’s leverage ratio is ambiguous. Correspondingly, Figure 3b exhibits a non-monotonic relationship between $\sigma$ and the average market leverage ratio in the economy. Under the current calibration, the average leverage ratio initially increases with cash flow volatility, reflecting the fact that the “coupon effect” overwhelms the “price effect”. For higher values of $\sigma$, the “price effect” dominates, together with the matching surplus increase, leads to a negative relationship between the average leverage ratio and $\sigma$. This is consistent the inconclusive empirical evidence regarding the relationship between economic volatility and leverage. As volatility increases and an employer optimally chooses higher coupon rate, the matching relationship has a higher probability to hit the bankruptcy threshold, causing more workers to return to unemployment every instant and leading to a higher unemployment rate, as in Figure 3c. This is consistent with the well-documented fact that both unemployment rate and economic volatility are countercyclical. The employer’s expected value after a match decreases with $\sigma$ in my calibration. This leads to a lower employer entry to labor search and results in a slack labor market, i.e., lower vacancy-to-unemployment ratio, shown in Figure 3d. The frictional wage dispersion increases with higher cash flow volatility because matching cash flows are more likely to hit higher cash flow levels with larger cash flow volatility, as in Figure 3e. The average job tenure decreases

---

38 Heuristically, as volatility increases, the match has a higher chance to generate large cash flow. By Nash bargaining, the worker will have a larger payoff under this high profitability scenario. Therefore, the employer has more incentive to use debt as a strategic tool in the wage bargaining process with the worker. 39 This is because bankruptcy is costly and the matching cash flow has a higher chance to hit the bankruptcy boundary with higher cash flow volatility. 40 For example, Titman and Wessels (1988) find that volatility is negatively associated with leverage, while Johnson (2018) finds a positive co-movement between aggregate volatility and leverage.
with cash flow volatility because the matching cash flow has a higher chance to hit the boundary of endogenous matching separation when the cash flow volatility is higher, as shown in Figure 3.

5.2.3 Flow unemployment benefit $b$

First, consider a single-employer’s decision process, holding expected value of a job searching worker $U$, constant. An increase in flow value of unemployment benefit, $b$, exogenously increases the expected value of job searching, $U$. Consequently, a potential employer could “afford” to choose a higher leverage while still keeping the expected value from applying for its job vacancy in line with the value required by the labor market before the increase in $b$. Moreover, a positive shock to $U$ also raises the worker’s threat point in the Nash bargaining. These two forces provokes the employer to issue more debt and the average leverage ratio of the economy increases as indicated by Figure 4a and Figure 4b. This is consistent with the empirical findings that employers competing for scarce workforce must compensate workers for their unemployment costs resulting from risky capital structures. Correspondingly, an employer uses higher leverage when the unemployment benefit is more generous (Agrawal and Matsa, 2013). A higher $b$ will reduce the relative value of employment against unemployment. This relative value change, together with the increased coupon choices on the employer side, will increase the threshold for endogenous matching separation, thereby raising the unemployment rate, as shown in Figure 4c. Moreover, an exogenous increase in flow unemployment benefit $b$ and expected value of being unemployed, $U$, will also reduce the asset values (i.e., surplus and debt) associated with a match in the labor

\[41\] I cannot disentangle between the two forces that drive up the employers’ optimal leverage choice. Agrawal and Matsa (2013) find that the compensating for workers’ unemployment risk is likely to be the underlying reason for the observed positive relationship between unemployment benefit and an employer’s leverage ratio.
Consequently, a higher $b$ will discourage a potential employer’s entry to the labor market and results in a slacker labor market, indicated by Figure 4d. A higher $b$ also has a positive selection effect in the sense that the cash flow threshold that warrants an optimal matching separation increases. Therefore, the frictional wage dispersion is reduced, from Figure 4e. This provides a rationale for the empirical evidence that unemployment benefit duration and generosity are negatively associated with wage inequality (e.g., Koeniger et al., 2007). The average job tenure in the economy is also reduced by higher unemployment flow benefit $b$ for similar reasons, shown in Figure 4f.

5.2.4 Worker’s bargaining power $\beta$

An increase in worker’s bargaining power will trigger the employer to use a higher leverage to gain a strategic advantage in the wage bargaining with worker. This will result in higher coupon rate and higher leverage ratio for a higher $\beta$, as shown in Figure 5a and Figure 5b. This is consistent with empirical findings that firms respond to a reduction in workers’ bargaining power by employing lower leverage (Bronars and Deere, 1991; Matsa, 2010). In contrast to standard search models of labor market without endogenous capital structure decisions, a higher bargaining power does not necessarily indicates that an employer’s expect payoff from a match will be lower. In my model, although an employer obtains a smaller share of matching surplus when the worker has a higher bargaining power, it can counteract the higher bargaining power by issuing more debt, thereby diverting more cash flow from matching surplus to debt. Consequently, the impact of $\beta$ on a potential employer’s labor market entry decision is ambiguous. This is demonstrated by Figure 5d, which exhibits a non-monotonic relationship between $\beta$ and labor market tightness. Since an employer employs a higher coupon rate when the worker’s bargaining power is higher, $\beta$ is positively
associated with the threshold of endogenous matching separation, thereby increasing the unemployment rate, as indicated by Figure 5c. A higher $\beta$ generates two opposite forces that influence the mean-min wage ratio: first, it has a positive selection effect that raises the cash flow threshold of matching separation. The first effect will depress the mean-min wage ratio. Meanwhile, a higher $\beta$ means that the worker’s wage will be more closely tied up with the matching performance. This second effect will increase the frictional wage dispersion. In my calibration, the second effect dominates the first, resulting in a positive relationship between the worker’s bargaining power and the mean-min wage ratio, as shown in Figure 5e. Lastly, the average expected job tenure is negatively related to $\beta$, because a higher $\beta$ raises the threshold of endogenous matching separation. This is demonstrated in Figure 5f.

5.2.5 Bankruptcy cost $\alpha$

An increase in bankruptcy cost increases the cost of debt and this will discourage the employers from issuing more debt. Consequently, Figure 6a and Figure 6b show that both the coupon choice and the average leverage ratio are negatively associated with a higher bankruptcy cost. This is standard in single-firm contingent claim models (e.g., Leland, 1994). From the an employer’s perspective, the increase in bankruptcy costs will depress its expected value from a match in the labor market. This is because it is more costly to issue debt for the strategic purpose in wage bargaining process. Consequently, employers are disinclined to enter the labor market and post vacancies. This results in a higher unemployment rate and a slacker labor market, as show in Figure 6c and Figure 6d. These predictions are consistent with the most recent experience of financial crisis, in which the employers disrupted by negative shocks in credit supplies reduced their hiring (e.g., Chodorow-Reich, 2014). Since an employer use a lower coupon when the bankruptcy cost is high. This results in a lower
threshold of endogenous match separation and a lower average cash flow in the economy. Therefore the impact of bankruptcy cost on mean-min wage ratio is mild, as indicated by Figure 6e. As shown in Figure 6f, the average job tenure in the economy is positively correlated with bankruptcy cost since a higher bankruptcy cost dampens the employer’s incentive to issue debt and results in a lower cash flow threshold of endogenous matching separation.

[Insert Figure 6 here.]

6. Conclusion

This paper develops a highly tractable labor market search model that encompasses a the employers’ optimal capital structure policies in the spirit of Leland (1994). I have shown that the wage bargaining between employers and workers, together with the search friction in the labor market, create a novel trade-off that influences the employers’ capital structure decisions. I derive a closed-form solution for the double-Pareto distribution of matching cash flows under the stationary equilibrium, which allows me to analyze the aggregate leverage of the economy and labor market quantities in a highly tractable manner.

The model is able to explain some well-known comparative static results that puzzles the single-firm contingent models of capital structure. For example, high-growth firms do not necessarily employ higher leverage. The model is also able to explain some well-documented empirical regularities between labor market regulations and labor outcomes. For example, a more generous unemployment benefit program is able to reduce the wage inequality in the economy. Moreover, a calibration of the base case model reveals that the model produces a much higher ratio between average and minimum wage and a much more elastic vacancy-to-unemployment ratio with respect to average productivity, compared with most frictional

---

42A higher bankruptcy cost does not change the right tail of the stationary cross-sectional distribution of the cash flow.
search models of labor market. In this sense, my model provides promising directions toward
the reconciliation between the search models of labor market and two well-known puzzles in
the labor economics literature: the excessively low model-implied mean-min wage ratio and
the excessively low model-implied elasticity of vacancy-to-unemployment ratio with respect
to productivity. The trade-off theory from the labor market is able to generate an empirically
plausible leverage ratio without resorting to the tax-benefit of debt.

The framework of the current model is very flexible. In untabulated analysis, I extend the
baseline model to allow unemployed workers to choose the levels of job-searching effort. Thus
the model could potentially explain the incentives of labor force participation. Moreover, I
also apply a similar model solution technique and solve two similar models that encompass
heterogeneous matching performances: one featuring Bayesian learning about the quality of
individual matches and another with an asymmetric information regarding the match-specific
quality.

To maintain tractability, however, the paper overlooks some potentially interesting mod-
eling choices. First, in this paper I assume that in a given match, the employer only has one
opportunity to choose its capital structure at the start of a match. As shown in Goldstein
et al. (2001), allowing the firm to adjust its capital structure over the course of production
greatly alters its initial capital structure decisions. A direct extension of this would be to
examine how the model fares if the employer is allowed to adjust the capital structure over
the course of a match. Second, I assume that an employer could commit a leverage policy
ex ante before the match starts. Recent corporate finance research has shown that the lack
of commitment to a specific leverage policy creates a so-called “leverage ratchet effect” that
distorts a firm’s capital structure decisions (DeMarzo and He, 2016; Admati et al., 2018)
and has profound welfare implications (Johnson et al., 2018). Third, I assume that an un-
employed worker has perfect information about a potential employer’s post-match capital

43
structure policy. A more realistic assumption would be that a potential employer’s past capital structure choices have a reputational effect on the searching worker’s perception about the potential employer’s future choice of its capital structure. Last but not least, to fully address the critique of traditional search models of labor market by Shimer (2005), i.e., the overly low response of vacancy-to-unemployment ratio with respect to productivity shocks, one has to introduce aggregate uncertainty regarding the matching productivity to study the time series of labor market aggregates. I leave these and other interesting extensions for future research.
REFERENCES


A. Appendices

A.1 Proof of Lemma 1

Proof. I speculate \( w(X) \) is linear in \( X \). Notice that \( U \) is independent of \( X \). From equations (3.1.6), (3.1.8) and (3.1.13), and free-entry condition, multiplying both sides of (3.1.6) by \( \beta \), I have

\[
\beta r E(X) = \beta \left[ (X - c - w) + \mu X E'(X) + \frac{1}{2} \sigma^2 E''(X) - s E(X) \right] \tag{A.1.1}
\]

and multiplying the difference between (3.1.8) and (3.1.13) by \( 1 - \beta \): and

\[
(1 - \beta) r [W(X) - U] = (1 - \beta) \left\{ w + \mu X W'(X) + \frac{1}{2} \sigma^2 X^2 W''(X) - s [W(X) - U] \right\}
\]

By (4.1.1), (A.1.1) is equal to (A.1.2), and using (4.1.2) and (4.1.3) in the main text and simplifying, I have proven (4.1.4). Q.E.D.

A.2 Proof of Proposition 3

Proof. The homogeneous part of ODE (4.4.1) is a Cauchy-Euler equation, which admits a general form of solution:

\[
T_{HG}(X) = HX^\rho + \tilde{H}X^{\tilde{\rho}} \tag{A.2.1}
\]

where \( \rho \) and \( \tilde{\rho} \) are negative and positive solutions, respectively, to \( \frac{1}{2} \sigma^2 \tilde{\rho} (\tilde{\rho} - 1) + \mu \tilde{\rho} - s = 0 \), respectively.

\[
\rho = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) - \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2s}{\sigma^2}} \tag{A.2.2}
\]
\[
\hat{\rho} = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) + \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2s}{\sigma^2}} \quad (A.2.3)
\]

Therefore, the general solution of the boundary value problem (4.4.1) subject to (4.4.2) takes the form

\[
T(X) = HX^\rho + \tilde{H}X^{\hat{\rho}} + \frac{1}{s} \quad (A.2.4)
\]

Since \( \lim_{X \to \infty} T(X) = \frac{1}{s} < \infty \), I have \( \tilde{H} = 0 \). \( H \) can be determined by another boundary condition \( T(X) = 0 \):

\[
HX^\rho + \frac{1}{s} = 0 \quad \Rightarrow \quad H = -\frac{1}{s}X^{-\rho} \quad (A.2.5)
\]

Bringing (A.2.5) to (A.2.4) leads to (4.4.3) in Proposition 3. Q.E.D.

### A.3 Proof of Proposition 4

**Proof.** From Gabaix (2009), the general solution of (4.5.2) is

\[
f^i(X) = \zeta^i_+ X^{-m_0-1} + \zeta^i_- X^{-m_1-1} \quad X \neq X_0 \quad (A.3.1)
\]

where \( i \in \{0, 1\} \). \( f^0(X) \) represents the distribution density function for \( X \in [X, X_0) \), and \( f^1(X) \) represents the distribution density function for \( X \in (X_0, \infty) \). In (A.3.1), \( m_0 \) and \( m_1 \) are the negative and positive roots, respectively, for the equation \( \frac{\sigma^2}{2}m(m-1) + \mu m - s = 0 \).

By the definition of the distribution density function, the expression \( \int_X^\infty f(X) \, dX \) must be integrable. This gives

\[
\int_X^{X_0} f^0(X) \, dX + \int_{X_0}^{\infty} f^1(X) \, dX < \infty \quad (A.3.2)
\]
From (A.3.1), I have $\zeta_1 = 0$; otherwise, \( \int_{X_0}^{\infty} f^1(X) \, dX \) explodes as \( X \to \infty \). From (4.5.3a), I have 
$$
\zeta^0 X^{-m_0 - 1} + \zeta^0 X^{-m_1 - 1} = 0, \text{ i.e.,}
$$
$$
\zeta^0_+ = -\zeta^0 X^{-m_0 - 1} X^{m_1 + 1}
= -\zeta^0 X^{m_1 - m_0}
$$

The two remaining unknown coefficients, $\zeta_1$ and $\zeta_0$, are uniquely determined by (4.5.3b) and (4.5.3c). Letting $\zeta = \zeta_1$ and $\tilde{\zeta} = \zeta_0$, we have (4.5.4) in Proposition 4. In Section IA.3, I have solved $\zeta$ and $\tilde{\zeta}$.

Q.E.D.

\[ \int_{X_0}^{\infty} f^1(X) \, dX = \frac{-\zeta_1}{m_0} X^{-m_0} \bigg|_{X_0}^{\infty} - \frac{\zeta_1}{m_1} X^{-m_1} \bigg|_{X_0}^{\infty}, \text{ therefore, if } \zeta_1 \neq 0, \text{ the first term explodes as } x \to \infty. \]
Table 1: Parameter values for baseline calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Interpretation</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Risk-free rate</td>
<td>2%</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Cash flow volatility</td>
<td>0.25</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Cash flow drift</td>
<td>0.0067</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Bankruptcy cost</td>
<td>0.25</td>
</tr>
<tr>
<td>$m(u,v)$</td>
<td>Matching function</td>
<td>$6u^{0.5}v^{0.5}$</td>
</tr>
<tr>
<td>$b$</td>
<td>Flow value of unemployment benefit</td>
<td>0.1693</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Flow cost of a vacancy</td>
<td>0.4</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Worker’s bargaining power</td>
<td>0.72</td>
</tr>
<tr>
<td>$X_0$</td>
<td>Starting cash flow</td>
<td>0.3662</td>
</tr>
<tr>
<td>$s$</td>
<td>Exogenous separation rate</td>
<td>0.0908</td>
</tr>
</tbody>
</table>

Table 2: Baseline model results

<table>
<thead>
<tr>
<th>variable</th>
<th>interpretation</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>Coupon rate</td>
<td>0.1568</td>
</tr>
<tr>
<td>$U$</td>
<td>Unemployment value</td>
<td>19.4756</td>
</tr>
<tr>
<td>$g$</td>
<td>Job-finding rate</td>
<td>3.0500</td>
</tr>
<tr>
<td>$\bar{X}$</td>
<td>Average cash flow</td>
<td>0.5530</td>
</tr>
<tr>
<td>Mm-ratio</td>
<td>Mean-min wage ratio</td>
<td>1.7948</td>
</tr>
<tr>
<td>$\eta_T(\varepsilon)$</td>
<td>Elasticity of labor market tightness to $\bar{X}$</td>
<td>15.4778</td>
</tr>
<tr>
<td>Average leverage</td>
<td>Average market leverage ratio</td>
<td>0.1826</td>
</tr>
<tr>
<td>Average tenure</td>
<td>Average job tenure</td>
<td>4.3500</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Vacancy-employment ratio</td>
<td>0.2584</td>
</tr>
</tbody>
</table>
Figure 1: **Stationary distribution of \( X \)**

All parameters are presented in [Table 1](#)
Figure 2: Matching cash flow drift $\mu$

$\mu$ varies from $-0.03$ to $0.03$. All other parameters are presented in Table 1.
Figure 3: Matching cash flow volatility $\sigma$

$\sigma$ varies from 0.15 to 0.3. All other parameters are presented in Table 1.
Figure 4: Flow value of unemployment $b$

$b$ varies from 0% to 90% of flow product of a labor market match. All other parameters are presented in Table 1.
Figure 5: Worker’s bargaining power $\beta$

$\beta$ varies from 0.4 to 0.9. All other parameters are presented in Table 1.
Figure 6: Bankruptcy cost $\alpha$

$\alpha$ varies from 0 to 0.5. All other parameters are presented in Table 1.
Internet Appendix to
“A Model of Capital Structure under Labor Market Search”

**IA.1. Proof of Proposition 1**

*Proof.* Notice that the homogeneous part of (4.2.4) is a Cauchy-Euler equation, which has a general solution that takes the form

\[ S_{HG}(X) = AX^\vartheta + \tilde{A}X^{\tilde{\vartheta}} \]

where \( \vartheta \) and \( \tilde{\vartheta} \) are negative and positive solutions of the equation

\[ \nu (\nu - 1) + \frac{2\mu}{\sigma^2} \nu - \frac{2\delta}{\sigma^2} = 0. \]

The general solution of (4.2.4) takes the form

\[ S(X) = \Pi(X) - \frac{c + rU + rV}{\delta} + AX^\vartheta \tag{IA.1.1} \]

By the “value-matching” condition,

\[ -AX^\vartheta = \Pi(X) - \frac{c + rU + rV}{\delta} \]

\[ A = -X^{-\vartheta} \left[ \Pi(X) - \frac{c + rU + rV}{\delta} \right] \tag{IA.1.2} \]

\[ ^{44} \text{Notice that } \tilde{A} = 0 \text{ by the “no-bubble” condition.} \]
Bringing (IA.2) to (IA.1), I have obtained (4.2.6) in Proposition 1. From (4.2.6), taking derivatives with respect to $X$ and the “smooth-pasting” condition gives

$$\frac{1}{\delta - \mu} - \left[ \Pi (X) - \frac{c + rU + rV}{\delta} \right] \left( \frac{\vartheta}{X} \right) = 0$$

$$(1 - \vartheta) \Pi (X) = -\vartheta \frac{c + rU + rV}{\delta}$$

$$X = \frac{-\vartheta}{1 - \vartheta} \frac{(\delta - \mu)}{c + rU + rV} \delta$$ (IA.1.3)

(IA.1.3) is (4.2.7) in Proposition 1. In equilibrium, by the free-entry condition of the potential employers, $V = 0$. Bringing the free-entry condition to (4.2.6) and (4.2.7) yields (4.2.8) and (4.2.9) in Proposition 1. Q.E.D.

### IA.2. Proof of Proposition 2

#### IA.2.1 First-order condition

*Proof.* The optimal coupon rate $c^{\text{max}}$ solves the constrained maximization problem defined in Section 4.3. First, I calculate several quantities that will facilitate the calculation of the first-order condition. The derivative of $X$ with respect to $c$, $\frac{\partial X}{\partial c}$, according to (4.2.7), is

$$\frac{\partial X}{\partial c} = \frac{-\vartheta}{1 - \vartheta} (\delta - \mu) \frac{1 + r \frac{\partial V}{\partial c}}{\delta}$$ (IA.2.1)

From (4.2.7),

$$\Pi (X) = \frac{X}{\delta - \mu} = \frac{1}{\delta - \mu} \frac{-\vartheta}{1 - \vartheta} (\delta - \mu) \frac{c + rU + rV}{\delta} = \frac{-\vartheta}{1 - \vartheta} \frac{c + rU + rV}{\delta}$$ (IA.2.2)
and

\[ \Pi(X) - \frac{c + rU + rV}{\delta} = \frac{-\vartheta \cdot c + rU + rV}{1 - \vartheta \cdot \frac{\delta}{\delta}} - \frac{c + rU + rV}{\delta} = \frac{-c + rU + rV}{(1 - \vartheta) \delta} \]  

(IA.2.3)

From (4.2.7) and (IA.2.1)

\[ \frac{\partial}{\partial c} \left( \frac{X}{X} \right)^\vartheta = \left( \frac{X}{X} \right)^\vartheta \frac{1}{1 - \vartheta} \frac{1}{\delta - \mu} \frac{c + rU + rV}{1 - \vartheta} \left( \delta - \mu \right) \frac{1 + r\frac{\partial V}{\partial c}}{\delta} \]

\[ \left( \frac{X}{X} \right)^\vartheta \frac{1}{c + rU + rV} \]

(IA.2.4)

From (4.2.6) and (IA.2.3), the matching surplus function \( S(X) \) becomes

\[ S(X) = \Pi(X) - \frac{c + rU + rV}{\delta} + \frac{c + rU + rV}{(1 - \vartheta) \delta} \left( \frac{X}{X} \right)^\vartheta \]

(IA.2.5)

and from (IA.2.4) and (IA.2.5), the derivative of \( S(X) \) with respect to \( c \) is

\[ \frac{\partial S(X)}{\partial c} = \frac{-1 + r\frac{\partial V}{\partial c}}{\delta} + \frac{c + rU + rV}{(1 - \vartheta) \delta} \left( \frac{X}{X} \right)^\vartheta \frac{-\vartheta \left( 1 + r\frac{\partial V}{\partial c} \right)}{c + rU + rV} + \frac{1 + r\frac{\partial V}{\partial c}}{\delta} \left( \frac{X}{X} \right)^\vartheta \]

\[ = \frac{-1 + r\frac{\partial V}{\partial c}}{\delta} + \frac{1 + r\frac{\partial V}{\partial c}}{\delta} \left( \frac{X}{X} \right)^\vartheta = \frac{-1 + r\frac{\partial V}{\partial c}}{\delta} \left( 1 - \left( \frac{X}{X} \right)^\vartheta \right) \]

(IA.2.6)

Since (4.2.1) defines the relationship between \( g \) and \( c \) for a given \( U \), differentiate both sides of (4.2.1) with respect to \( c \) and the derivative of \( g \) with respect to \( c \) is

\[ \frac{\partial g}{\partial c} = -g \frac{\partial S(X_0)}{\partial c} \]

(IA.2.7)

and the derivative of the potential employer’s hiring rate \( h \) with respect to \( c \) is \( h'(g) \frac{\partial g}{\partial c} \).
From (3.1.4) and (IA.2.3), the value of debt contract $D(X)$ is

$$D(X) = \frac{c}{\delta} - \left( \frac{c}{\delta} - (1 - \alpha) \left( \Pi(X) - \frac{rU + rV}{\delta} \right) \right) \left( \frac{X}{\delta} \right)^{\vartheta}$$

$$= \frac{c}{\delta} - \left( \frac{\alpha c}{\delta} - (1 - \alpha) \left( \Pi(X) - \frac{c + rU + rV}{\delta} \right) \right) \left( \frac{X}{\delta} \right)^{\vartheta}$$

$$= \frac{c}{\delta} - \left( \frac{\alpha c + (1 - \alpha) c + rU + rV}{(1 - \vartheta) \delta} \right) \left( \frac{X}{\delta} \right)^{\vartheta}$$

(IA.2.8)

From (IA.2.4) and (IA.2.8) the derivative of $D(X)$ with respect to $X$ is

$$\frac{\partial D(X)}{\partial c} = \frac{1}{\delta} \left( \frac{\alpha}{\delta} + \frac{(1 - \alpha) (1 + r \frac{\partial V}{\partial c})}{(1 - \vartheta) \delta} \right) \left( \frac{X}{\delta} \right)^{\vartheta}$$

$$- \left( \frac{\alpha c + (1 - \alpha) c + rU + rV}{(1 - \vartheta) \delta} \right) \left( \frac{X}{\delta} \right)^{\vartheta} \left( 1 + \frac{r \frac{\partial V}{\partial c}}{c + rU + rV} \right)$$

$$= \frac{1}{\delta} - \frac{1}{\delta} \left( \alpha + \frac{(1 - \alpha) (1 - \vartheta) \partial}{(1 - \vartheta) \delta} \right) \left( 1 + \frac{r \frac{\partial V}{\partial c}}{c + rU + rV} \right) \left( \frac{X}{\delta} \right)^{\vartheta}$$

$$+ \alpha \vartheta \frac{c (1 + r \frac{\partial V}{\partial c})}{c + rU + rV} \left( \frac{X}{\delta} \right)^{\vartheta}$$

$$= \frac{1}{\delta} \left[ 1 - \left( 1 + (1 - \alpha) r \frac{\partial V}{\partial c} - \alpha \vartheta \frac{c (1 + r \frac{\partial V}{\partial c})}{c + rU + rV} \right) \left( \frac{X}{\delta} \right)^{\vartheta} \right]$$

(IA.2.9)

Taking the derivative of both sides of (4.2.2) with respect to $c$, the first-order condition of the optimal coupon rate is

$$\frac{\partial V}{\partial c} = \frac{h}{g} \left[ \frac{g}{\delta} + \frac{1}{\delta} \left( \alpha \vartheta \frac{c + rU + rV}{c + rU + rV} - \beta \right) \left( \frac{X^{\vartheta}}{\delta} \right)^{\vartheta} \right] + h_{(\cdot)} \left[ (1 - \beta) S(X_0) + D(X_0) \right]$$

$$r \left[ 1 + \frac{h}{g} (1 - \beta) - \frac{h}{g} \left( \alpha \vartheta \frac{c + rU + rV}{c + rU + rV} - \beta + \alpha \right) \left( \frac{X^{\vartheta}}{\delta} \right)^{\vartheta} \right]$$

$$- h'(g) \frac{g}{\delta} \left( 1 - \left( \frac{X^{\vartheta}}{\delta} \right)^{\vartheta} \right) \left( \frac{(1 - \beta) S(X_0) + D(X_0)}{S(X_0)} \right) = 0$$

(IA.2.10)
and

$$h_{(c)} = h'(g) \frac{g}{\delta} \left[ 1 - \left( \frac{X_0}{X} \right)^\vartheta \right] S(X_0)$$  \hspace{1cm} (IA.2.11)$$

where $S(X_0)$ and $D(X_0)$ is given by [IA.2.5] and [IA.2.8] with $X = X_0$, and $X$ is given by (4.2.7). This is equivalent to the numerator of [IA.2.10] being equal to zero—that is (4.3.1).

Q.E.D.

### IA.2.2 Second-order condition

**Proof.** The objective function $V$ is continuously differentiable with respect to $c$, and the set of viable coupon rate $c$ is a closed interval in the set of non-negative real numbers, which is $c \in [0, \bar{c}]$, which is closed and bounded. One candidate for $\bar{c}$ is such that $X = X_0$. In other words, default occurs immediately after matching. Therefore, the maximization problem is well defined and a maximizer $c^{max}$ exists.

From [IA.2.10], the second-order derivative of $V$ with respect to $c$ at the optimal coupon level $c^{max}$ is as follows. First, for ease of exposition I define the following quantities:

$$A = r \left[ 1 + h \left( \frac{1}{\delta} (1 - \beta) - \frac{h}{\delta} \left( \alpha \vartheta \frac{c + rU + rV}{c + rU + rV - \beta} + \alpha \right) \left( \frac{X_0}{X} \right)^\vartheta \right) - h'(g) \frac{g}{\delta} \left( 1 - \left( \frac{X_0}{X} \right)^\vartheta \right) \frac{(1 - \beta)S(X_0) + D(X_0)}{S(X_0)} \right]$$  \hspace{1cm} (IA.2.12)

$$R = h \left[ \frac{\beta}{\delta} + \frac{1}{\delta} \left( \alpha \vartheta \frac{c + rU + rV}{c + rU + rV - \beta} \right) \left( \frac{X_0}{X} \right)^\vartheta \right] + h_{(c)} \left[ (1 - \beta) S(X_0) + D(X_0) \right]$$  \hspace{1cm} (IA.2.13)

where $h_{(c)}$ is given in Proposition 2. The derivative of [IA.2.12] with respect to $c$ does not matter for the determination of the sign of the second-order condition.

Taking the derivative of [IA.2.13] with respect to $c$ and using the fact that $\frac{\partial V}{\partial c}|_{c=c^{max}} = 0$,
I obtain three terms. The first term is

\[ R_1 = h \left[ \frac{1}{\delta} \left( \alpha \vartheta \left( \frac{rU + rV - \vartheta c}{c + rU + rV} \right)^2 + \beta \frac{\vartheta}{c + rU + rV} \right) \left( \frac{X_0}{X} \right)^{\vartheta} \right] \]  

(IA.2.14)

The second term is

\[ R_2 = 2h(c) \left[ \beta + \frac{1}{\delta} \left( \frac{\alpha \vartheta c}{c + rU + rV} - \beta \right) \left( \frac{X_0}{X} \right)^{\vartheta} \right] \]  

(IA.2.15)

\[ h(c) \] is given in Proposition 2. The third term is

\[ R_3 = h^{(c)} \left[ (1 - \beta) S(X_0) + D(X_0) \right] \]  

(IA.2.16)

where \( h^{(c)} \) is \( \frac{\partial^2 h}{\partial c^2} = h_1 h_2^{(c)} + h_2^{(c)} h_2 \). \( h_1 \) and \( h_2 \) are given in Proposition 2. \( h_1^{(c)} = \frac{\partial g}{\partial c} = h''(g) \frac{1 - (\frac{X_0}{X})^{\vartheta}}{S(X_0)} \), and

\[ h_2^{(c)} = \frac{\partial g}{\partial c} \left[ \frac{1 - (\frac{X_0}{X})^{\vartheta}}{S(X_0)} \right] + \frac{g}{\delta} \frac{\left( \frac{X_0}{X} \right)^{\vartheta}}{S(X_0)} \left[ \frac{1 - (\frac{X_0}{X})^{\vartheta}}{S(X_0)} \right]^2 \]

\( \frac{\partial g}{\partial c} \) is given by (IA.2.7).

I have the second order derivative of \( V \) with respect to \( c \) as:

\[ \frac{\partial^2 V}{\partial c^2} = \frac{(R_1 + R_2 + R_3) A - \frac{\partial A}{\partial c} R}{A^2} \]

By the first-order condition, \( R = 0 \). Therefore,

\[ \frac{\partial^2 V}{\partial c^2} = \frac{R_1 + R_2 + R_3}{A} \]  

(IA.2.17)
where $R_1$, $R_2$, $R_3$ and $A$ is given by (IA.2.14), (IA.2.15), (IA.2.16) and (IA.2.12), respectively. Q.E.D.

**IA.3. Solve $\zeta$ and $\tilde{\zeta}$ in Proposition 4**

*Proof.* $\zeta$ and $\tilde{\zeta}$ are determined by (4.5.3b) and (4.5.3c). In this subsection, I will solve $\zeta$ and $\tilde{\zeta}$. The solution procedure consists of several steps.

First, using (4.5.4), I obtain the left and right derivatives of $f(X)$ with respect to $X$ at $X = X_0$.

\[ f' (X_0-) = -\tilde{\zeta} (m_0 + 1) X_0^{-m_0-2} + \zeta (m_1 + 1) X_0^{-m_0-2} \left( \frac{X}{X_0} \right)^{m_1-m_0} \]  
\[ f' (X_0+) = -\tilde{\zeta} (m_1 + 1) X_0^{-m_1-2} \]  

IA.3.1 IA.3.2

Second, using (4.5.4), I obtain the integral of $f(X)$ over $X$:

\[ \int_X^{X_0} f(X) \, dX = \int_X^{X_0} \left( \tilde{\zeta} X^{-m_0-1} - \zeta X^{m_1-m_0} X^{-m_1-1} \right) \, dX \]
\[ = \tilde{\zeta} \left( -\frac{1}{m_0} X^{-m_0} + X^{m_1-m_0} \frac{1}{m_1} X^{-m_1} \right) \bigg|_X^{X_0} \]
\[ = \tilde{\zeta} \left[ \frac{1}{m_1} X^{-m_0} \left( \left( \frac{X}{X_0} \right)^{m_1} - 1 \right) + \frac{1}{m_0} (X^{-m_0} - X_0^{-m_0}) \right] \]  
\[ = \tilde{\zeta} \left[ \frac{1}{m_1} X^{-m_0} \left( \left( \frac{X}{X_0} \right)^{m_1} - 1 \right) + \frac{1}{m_0} (X^{-m_0} - X_0^{-m_0}) \right] \]  

IA.3.3

and

\[ \int_{X_0}^{\infty} f(X) \, dX = \int_{X_0}^{\infty} \zeta X^{-m_1-1} \, dX = \zeta \left( -\frac{1}{m_1} \right) X^{-m_1} \bigg|_{X_0}^{\infty} = \frac{\zeta}{m_1} X_0^{-m_1} \]  

IA.3.4
Third, I obtain the right derivatives of \( f(X) \) with respect to \( X \) at \( X = X_0 \):

\[
\dot{f}'(X+) = \ddot{\zeta} (m_1 - m_0) X^{-m_0 - 2}
\]

(IA.3.5)

Fourth, I express boundary conditions (4.5.3b) and (4.5.3c) from (IA.3.1)–(IA.3.5).

(4.5.3b) becomes

\[
\ddot{\zeta} \Lambda_1 - \zeta \Lambda_2 = s \ddot{\zeta} \Lambda_3 + s \zeta \Lambda_4 + \ddot{\zeta} \Lambda_5
\]

\[\iff \ (\Lambda_1 - s \Lambda_3 - \Lambda_5) \ddot{\zeta} - (\Lambda_2 + s \Lambda_4) \zeta = 0 \] (IA.3.6)

and (4.5.3c) becomes

\[
g - g \left( \ddot{\zeta} \Lambda_3 + \zeta \Lambda_4 \right) = s \ddot{\zeta} \Lambda_3 + s \zeta \Lambda_4 + \ddot{\zeta} \Lambda_5
\]

\[\iff \ [(g + s) \Lambda_3 + \Lambda_5] \ddot{\zeta} + (g + s) \Lambda_4 \zeta = g \] (IA.3.7)

The parameters \( \Lambda_i, i \in \{1, 2, 3, 4, 5\} \) are defined as follows:

\[
\Lambda_1 = \frac{1}{2} \sigma^2 \left[ - (m_0 + 1) + (m_1 + 1) \left( \frac{X}{X_0} \right)^{m_1 - m_0} \right] X_0^{-m_0}
\]

(IA.3.8)

\[
\Lambda_2 = -\frac{1}{2} \sigma^2 (m_1 + 1) X_0^{-m_1}
\]

(IA.3.9)

\[
\Lambda_3 = \int^{X_0} \frac{X}{X} X^{-m_0 - 1} \left[ 1 - \left( \frac{X}{X_0} \right)^{m_1 - m_0} \right] dX = \frac{1}{m_1} X_0^{-m_0} \left( \left( \frac{X}{X_0} \right)^{m_1} - 1 \right) + \frac{1}{m_0} (X^{-m_0} - X_0^{-m_0})
\]

(IA.3.10)

\[
\Lambda_4 = \int^{\infty} X^{-m_1 - 1} dX = \frac{1}{m_1} X_0^{-m_1}
\]

(IA.3.11)

\[
\Lambda_5 = \sigma^2 \frac{1}{2} (m_1 - m_0) X^{-m_0}
\]

(IA.3.12)

(IA.3.6) and (IA.3.7) pin down \( \zeta \) and \( \ddot{\zeta} \). It can be shown that \( f(X_0+) = f(X_0-) \)—that is,
$f(X)$ is continuous at $X_0$. $\zeta$ and $\tilde{\zeta}$ are unique. Q.E.D.