Carry Coals to Newcastle?

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Abstract

Restrictions on short-selling and leverage are common in the mutual fund industry, but we rarely observe they are binding. Are investors carrying coals to Newcastle? In a principal-agent framework in which a risk-averse manager is protected by limited liability, we find that constraints are needed to tackle the manager’s excessive risk-taking behavior. They are not binding because there is an interior position that maximizes the manager’s expected utility locally due to risk aversion, and constraints transform it to a global maximum. Moreover, this “puzzle” is exclusive for “low-type” managers, which suggests a new measure of investment skill.

Keywords: mutual fund, portfolio delegation, non-binding constraints, managerial skill, principal-agent, short-selling, leverage

JEL Classification: C72, D82, D86, G11, J24

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1 Introduction

In light of the Investment Company Institute (2019), the total net assets of mutual funds in the United States was $17.7 trillion in 2018, which was around 86% of the GDP of the whole country in that year, and about 44.8% of the households in the United States invested in mutual funds in 2018.\footnote{According to the World Bank, the GDP of the United States in 2018 was about $20.5 trillion in current US dollars. Source: https://data.worldbank.org/indicator/NY.GDP.MKTP.CD?end=2018&locations=US&start=1960&view=chart; retrieved July 25, 2019.}

Given the huge amount of assets managed by professional managers, starting from the Investment Company Act of 1940, constraints on investment strategies are widespread among mutual funds in the United States. No-short-sale and no-leverage are two common ones among all possible constraints. In fact, leverage is directly prohibited by section 13(a) of the Investment Company Act, unless authorized by investors. Some research in delegated asset management directly assumes fund managers cannot short and/or borrow, e.g. Palomino and Prat (2003) and He and Xiong (2013), which additionally and indirectly confirms the ubiquity of restrictions on short-selling and leverage. As Almazan et al. (2004) have documented, around 20% and 70% of all funds were not permitted to borrow and short respectively.\footnote{The mutual fund data we cite from the Investment Company Institute (ICI) is for our personal use only.}

Why do investors want to do that? Theoretical literature suggests that restrictions are used to prevent managers from gambling (Vayanos, 2018; Buffa, Vayanos, and Woolley, 2019) or migrate the incentive problem (Gómez and Sharma, 2006; Dybvig, Farnsworth, and Carpenter, 2010), because of the well-known irrelevance result discovered by Stoughton (1993) and Admati and Pfleiderer (1997).

The puzzle here is that we rarely observe portfolio positions which are binding at the constraints in reality. For example, Catalyst Growth of Income Fund\footnote{For readers who notice that only 20% of mutual funds were under no-leverage constraints: this is because actively managed equity mutual funds are the main ones which are subject to no-leverage restrictions (Boguth and Simutin, 2018). In our model, the fund manager does actively adjust her portfolio based on the signal she receives.} of the Catalyst Funds is subject to the no-short-selling and no-leverage constraints,\footnote{Without constraints, the performance-based fees are irrelevant for inducing incentives, because the fund manager can always undo the effect by adjusting her original portfolio.} but none of them was binding according to table \ref{table:constraints}. Indeed, given the popularity of the no-short-selling and no-leverage constraints, professional managers have the incentive to always adjust their portfolios in order to comply with constraints.\footnote{Ticker symbols: CGGAX, CGGCX and CGGIX.}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Constraint & Number of Funds & Number of Funds Under Constraint \hline
No-short-sale & 5,000 & 90\% \hline
No-leverage & 5,000 & 85\% \hline
\end{tabular}
\caption{Constraints in Mutual Funds}
\end{table}

\begin{itemize}
\item \footnote{Source: the NSAR-A/A form submitted by Mutual Fund Series Trust to the U.S. Securities and Exchange Commission (SEC); filing date: March 1, 2018; URL: https://www.sec.gov/Archives/edgar/data/1355064/000158064218001244/answer.fil; retrieved from EDGAR, September 17, 2019.}
\end{itemize}
constraints, shouldn’t we expect that many mutual funds either invest zero dollars or a-hundred-percent of their capitals in risky assets, if investors are not doing something redundant? Hence, our research question is: how can non-binding constraints affect managers’ behavior? Are investors and regulators carrying coals to Newcastle? Our answer is no. We consider a standard principal-agent model in a two-asset and binary-signal world in which the principle of limited liability applies. The investor (he) maximizes his expected utility by designing a linear contract with possible trading constraints if necessary. Given any contract, the manager (she) maximizes her expected utility by choosing her effort level and the weight in the risky asset. In our model, as the literature in portfolio delegation shows, restrictions are necessary due to the conflict of interests between the fund manager and the investor. Without them, the manager will take excessive risk which is far from optimal in the view of the investor, due to the protection from limited liability. The reason that constraints are not binding is that when the manager’s payoff positively depends on the performance of the portfolio, her risk aversion creates an interior local maximum before limited liability kicks in, and when the constraints are appropriately chosen, this local maximum becomes “global.” Therefore, the manager would only choose the “interior” portfolio under such contracts. The intuition is illustrated in figure 1. Note that the princi-

\footnote{Historically, Newcastle Upon Tyne in England was famous for being a center of coal industry. Hence, carrying coals to Newcastle was a pointless action.}
ple of limited liability plays an important role here: if the manager was not protected, that “interior” portfolio was also global optimal even without any constraints under the original payoff scheme.

Our model yields a sharp and testable prediction: this “non-binding puzzle” can only happen when the manager is low-quality. The role of “low-quality” has two folds—it jointly determines the necessity of no-short-selling and no-leverage constraints and the location of the local maximum. Note that no-short-selling and no-leverage constraints are quite restrictive indeed. However, they only hurt the efficiency of profiting from the information advantage when the manager is high-quality. If the manager is inferior, the investor knows for sure that any extreme portfolio is due to gambling instead of deep insights about the market. Hence, strict restrictions on low-quality managers are indispensable. In addition, since the private signal received by an inferior manager is too coarse, she knows that any portfolio that is too close to the boundary would backfire with high probability before limited-liability plays a part. With the two points in mind, that is why we may observe a mutual fund’s investment style is too conservative comparing to the trading restrictions imposed by investors. And once this phenomenon is observed, we can infer the quality of this particular manager could not be high.

In fact, according to Barron’s fund family ranking, the top 1 fund family of 2018 based on the performance of actively managed funds was the American Funds, and none of its mutual funds are subject to no-short-sale and/or no-leverage constraints. See table 2. The Catalyst Funds was even not on that list which contains 57 fund families. Moreover, in line with Morningstar Fund Family 150 (Laske, 2019), the American Funds and Catalyst Funds were ranked 3rd and 147th respectively. Finally, the current Morningstar return rating for Catalyst Growth of Income Fund is only 1 star. These facts are consistent with the prediction we made above.

Our paper is related to the literature in various strands. First, it is related to the vast literature on optimal contracts in the delegated portfolio management industry, which traces back to Bhattacharya and Pfleiderer (1985). Recent contributions on this strand of literature include Ou-Yang (2003), Palomino and Prat (2003), Li and Tiwari (2009), Kyle, Ou-Yang, and Wei (2011), Gorton, He, and Huang (2010), Huang (2015), Sato (2016), Sotes-Paladino and Zapatero (2017), Cvitanić and Xing (2018), Huang, Qiu, and Yang (2019) and Sockin and Xi-

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The central contribution of this paper is that, to the best of our knowledge, we are the first ones who identify this “non-binding puzzle” and provide a (rational) rationale. Hence, our paper particularly adds new insights to the literature on the important role of trading constraints as essential ingredients of optimal contracts in the asset management industry. The most influential empirical research in this area is probably the work of Almazan et al. (2004), which provides solid and affluent stylized facts. As Gómez and Sharma (2006) and Dybvig, Farnsworth, and Carpenter (2010) show, restrictions prevent fund managers from offsetting the incentive effects of their compensation contracts by adjusting portfolios. In a continuous-time setting, Dai, Jin, and Liu (2011) numerically show that trading constraints can affect mutual funds’ optimal investment strategy “even they are not currently binding.” He and Xiong (2013) study how investment mandates, which are restrictions on the managers’ positions on outside assets, can weaken the moral hazard problem, while the no-short-selling and no-leverage constraints are exogenously given. Liu (2015) finds that restrictions increase the managers’ incentives to acquire long-term information in a dynamic model. Vayanos (2018) and Buffa, Vayanos, and Woolley (2019) show that risk limits on managers’ portfolio choices can rise due to the heterogeneity of managers’ risk aversions. The main feature that differentiates our paper from the literature is that, in models of those papers, trading restrictions are effective because they are possible to be binding, which is inconsistent with the real world situation. Hence, our paper echoes the point made by previous literature and fills the gap between theory and reality simultaneously.

Second, our paper also points out new directions of empirical research in the asset management industry. A direct application is that empiricists can test the relationship between “non-bindingness” and managerial skill. More importantly, our paper suggests a novel measure of mutual fund managers’ skill, which can isolate “low-quality” mutual funds, in addition to commonly used measures, e.g. the Sharpe ratio (Sharpe, 1966) and Jensen’s alpha (Jensen, 1968). One particular advantage of this “non-binding” measure is that it is asset-pricing-model-free, because it does not depend on the construction of a proper benchmark, and hence avoids the “joint-hypothesis problem” (Fama, 1991), and the “war” on which (factor) model is true (for example, Fama and French (1993, 2015), Carhart (1997) and Hou, Xue, and Zhang (2015)).

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10 Stracca (2006) provides an excellent review of the early theoretical literature.
11 We exclude the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), because (I) it is
Hence, it also contributes to the literature of measuring managerial talent and sheds light on the long debate that if mutual fund managers lack skill. The current evidence on the existence of managerial skill is mixed. The classical measure of managerial skill in academia is the net alpha proposed by Carhart (1997) and he concludes managers do not have investment skill; also see Ferson (2013) for an overview of popular measures and evidence on managerial skill. Recently, Kacperczyk, Nieuwerburgh, and Veldkamp (2014) measure managerial talent as the ability to “pick stocks or time the market;” Berk and van Binsbergen (2015) suggest a value measure—a manager’s skill is the value of her fund “extracts from capital markets.” In contrast, both of the two studies find strong evidence of the existence of managerial skill.

Finally, the setting of He and Xiong (2013) is very similar to ours, but we make three key different assumptions. In He and Xiong (2013), the investor is risk-neutral, the portfolio weight in any asset can only be either 0 or 1 and the no-short-selling and the no-leverage constraints are exogenously given. The first assumption suggests the investor would yearn for extreme positions. The second modeling choice actually rules out “interior” portfolios ad hoc. The third one makes answering if “investors are carrying coals to Newcastle” impossible. Instead, we assume both of the investor and the manager have constant relative risk aversion (CRRA) utility functions\(^\text{12}\); and allow the set of possible portfolio choices to be a continuum; and endogenize the rise of trading constraints. In short, in addition to being more realistic, our modeling choices allow us to answer the key questions of this paper which cannot be answered in the original framework of He and Xiong (2013).

Our model is also similar to the static contracting model considered in Vayanos (2018) and Buffa, Vayanos, and Woolley (2019). In their model, “managers differ in their preferences and the private information they may acquire,” which cannot be directly observed by the investor. As we discussed before, they find that risk limits are necessary to prevent the risk-neutral manager who has no information advantage from gambling. However, we

\(^{12}\text{This assumption also differentiates our paper from numerous papers in the area of delegated portfolio management, e.g., Bhattacharya and Pfleiderer (1985), Stoughton (1993), Admati and Pfleiderer (1997), Gómez and Sharma (2006), Kyle, Ou-Yang, and Wei (2011), Huang (2015), Vayanos (2018), Buffa, Vayanos, and Woolley (2019), Huang, Qiu, and Yang (2019) and Sockin and Xiaolan (2019), which assume constant absolute risk aversion (CARA) utility functions for tractability. As Campbell (2018) summaries, CARA utility has a few serious drawbacks, especially the counter-factual property of “wealth irrelevance for risky investment.” Because of our binomial-tree setting, we can solve the manager's portfolio problem analytically, without assuming CARA preferences. In addition, CRRA preferences are consistent with linear contracts considered in our paper (Dai, Jin, and Liu, 2011, footnote 6).}
show that restrictions are needed even without the existence of uninformed risk-neutral managers, and again, their model cannot explain why restrictions can be effective and non-binding simultaneously.

We organize the rest of the paper as follows. We present our model in section 2. We solve the model, characterize the optimal contracts and demonstrate our main results in section 3. In section 4, we conduct numerically analysis to show how the optimal contract changes as the parameters vary and the “non-binding puzzle” can arise even under non-linear contracts. We make concluding remarks in section 5.

2 Model

Consider a typical principal-agent problem. We have an investor (he) and a mutual fund manager (she). Their Bernoulli (ex-post) utility functions over their terminal wealth are

\[ u_i(w_i) = \begin{cases} -\infty & \text{if } \hat{u}_i(w_i) \text{ is undefined,} \\ \hat{u}_i(w_i) & \text{otherwise,} \end{cases} \]

where

\[ \hat{u}_i(w_i) = \begin{cases} \ln(w_i) & \text{if } \gamma = 1, \\ \frac{w_i^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1, \end{cases} \]

for the investor and

\[ u_m(w_m) = \frac{w_m^{1-\beta}}{1-\beta}, \]

for the manager. Here \( w_j \) is the terminal wealth level of \( j \in \{\text{investor} = i, \text{manager} = m\} \). \( \beta \in (0, 1) \) and \( \gamma \in (0, \infty) \) measure the players’ risk aversions. That is, both of the investor and the manager have CRRA utility functions. And for the investor, if the terminal wealth is out of the natural domain of a CRRA utility function, we define his utility under this situation as \(-\infty\).\(^\text{13}\)

There are two assets: a Lucas (1978) tree (risky asset/”stock”) with net return \( \tilde{R} \) and a risk-free bond (safe asset). We normalize the net return of the bond to 0. Hence, \( \tilde{R} \) is also the

\(^{13}\text{We do not need to do the same thing to the manager, because she is protected by limited liability and her wealth is always non-negative.}\)
risky asset’s excess return. For simplicity, we assume a (symmetric) binomial tree structure:

\[
\tilde{R} = \begin{cases} 
R & \text{with probability } p, \\
-R & \text{with probability } 1-p,
\end{cases}
\]

where \( p \in (1/2, 1) \) and \( R \in (0, 1) \). Note that the expected excess return of the tree is \((2p-1)R > 0\), that is, the ex-ante risk premium is strictly positive.\(^{14}\)

This is a one-period game. At the beginning, only the investor has 1 unit of capital, but he can only access the tree via the manager. In particular, he can transfer all his wealth to the manager and proposes a contract \((\delta, b, L, U)\). Here \( b \geq 0 \) is the base salary and \( \delta \geq 0 \) is the percentage of the market value of assets under management (AUM) that the manager will receive.\(^{15}\) Hence, we restrict our attention to linear contracts. This structure, although seems (too) restrictive at first glance, does capture the dominant (advisory) contract forms in the real world (Deli, 2002; Elton, Gruber, and Blake, 2003; Golec and Starks, 2004).\(^{16}\) In addition, this “reduced-form” contract is also extensively used in the literature.\(^{17}\) Finally, linear contracts do approach the first-best under certain conditions in our setting.

\( L \) and \( U \) are constraints on the manager’s portfolio choice and we assume

\[
-\infty \leq L \leq U \leq +\infty. \quad (1)
\]

\( L = -\infty \) means the manager is not restricted on short-selling. Analogously, \( U = +\infty \) means there is no zero-leverage constraint. Hence, we do allow the investor to “liberate” the manager as long as doing so is optimal.

If the manager rejects the contract, then the game ends and she receives \( w_m = 0 \), which is just her outside option. If she accepts the contract, then she chooses her effort level, \( e \in \{0,1\} \) and receives a signal \( s \in \{0,1\} \). \( s \) contains some information about \( \tilde{R} \). In particular,

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\(^{14}\)This assumption matches the empirical fact. As Damodaran (2019) estimates, the implied equity risk premiums for S&P 500 from 1961 to 2018 were all positive. It also ensures that the “ideal” position in the tree is positive, if no new information arrives.\(^{15}\)The constraint \( \delta \geq 0 \) guarantees a non-decreasing fee structure. As Vayanos (2018) notices: “A non-decreasing fee is economically appealing because it ensures that the manager does not engage in (unmodeled) activities that reduce \( w \), e.g., costly round-trip transactions, so to raise her fee.”\(^{16}\)The situation is different for compensation contracts between investment advisors and individual portfolio managers, which usually include a relative performance (comparing to a benchmark) part (Ma, Tang, and Gómez, 2019).\(^{17}\)For example, Stoughton (1993), Admati and Pfleiderer (1997), Carpenter (2000), Kapur and Timmermann (2005), Gómez and Sharma (2006), Basak, Pavlova, and Shapiro (2007), Cuoco and Kaniel (2011), Kyle, Ou-Yang, and Wei (2011), Sotes-Paladino and Zapatero (2017), Huang, Qiu, and Yang (2019), and Sockin and Xiaolan (2019).
we assume
\[ \Pr[s = 1|\bar{R} = R, e] = \Pr[s = 0|\bar{R} = -R, e] = 1/2 + qe, \]

where \( q \in (0, 1/2) \) captures the ability of the manager. Let \( \hat{p}(s, e) := \Pr[\bar{R} = R|s, e] \) be the manager’s posterior belief about the stock market, then by Bayes’ law, we have
\[
\begin{align*}
\hat{p}(1, e) &= \frac{p(1/2 + qe)}{p_1}, \\
\hat{p}(0, e) &= \frac{p(1/2 - qe)}{p_0},
\end{align*}
\]

where
\[
\begin{align*}
p_1 &:= p(1/2 + qe) + (1 - p)(1/2 - qe), \\
p_0 &:= p(1/2 - qe) + (1 - p)(1/2 + qe).
\end{align*}
\]

Hence, \( p_s \) is the probability of receiving signal \( s \in \{0, 1\} \). Note that if \( e = 0 \), the signal has no value. Effort is costly, and we assume the utility cost of effort is
\[ C(e) = ce, \]

where \( c > 0 \). Hence, the manager’s utility level is given by
\[ \tilde{u}_m(w_m, e) = u_m(w_m) - ce. \]

Based on \( s \), the manager makes the investment decision and forms a portfolio \( \pi \) subject to
\[ L \leq \pi \leq U, \]

where \( \pi \) is the proportion of the money invested by the manager in the tree.

After the realization of \( \bar{R} \), according to the contract \( (\delta, b) \) and the portfolio choice \( \pi \), the payoffs of the investor and manager, \( w_I \) and \( w_m \), are realized. Then the game ends. In addition, the manager is subjected to limited liability. Hence, the terminal wealth of the manager is
\[ w_m = \max\{0, b + \delta(1 + \pi \bar{R})\}, \]
and the terminal wealth of the investor is

$$w_t = 1 + \pi R - w_m.$$  \hfill (8)

Finally, we close the model by presenting two tie-breaking rules. (I) Given any contract and $e$, the manager chooses $\pi$ to maximize her expected utility after the realization of the signal, and if she is indifferent among a few choices, she always chooses the portfolio with the minimal (absolute) weight in the tree. The preference for safe asset can be justified, if investing in the stock market is relatively costly. This rule is mathematically equivalent to assume

$$\pi^* (s; \delta, b, L, U, e) = \arg \min_{\pi \in A(e, s, \delta, b, L, U)} |\pi|,$$  \hfill (9)

where

$$A(e, s, \delta, b, L, U) := \arg \max_{\pi \in [L, U]} \mathbb{E} [u_m(w_m(\pi)) | e, s, \delta, b].$$

Note that $\mathbb{E} [u_m(w_m)| e, s, \delta, b]$ is continuous with respect to $\pi$ and $[L, U]$ is a compact set in $\mathbb{R}$, hence $A(e, s, \delta, b, L, U)$ is non-empty and closed. Therefore, (9) is well-defined. (II) When the manager is in different between working hard and shirking, she works hard. That is,

$$e^* (\delta, b, L, U) = \max \{ \arg \max_{e \in [0, 1]} [\mathbb{E} [u_m(w_m(\pi^* (s; \delta, b, L, U, e = 1)))] - c, \mathbb{E} [u_m(w_m(\pi^* (s; \delta, b, L, U, e = 0)))]}\}. $$  \hfill (10)

None of our main results depends on (9). However, (10) is crucial for the existence of an (interesting) optimal contract.\footnote{The reason is that (10) is necessary for the upper semi-continuity of the investor’s indirect utility function. According to Sappington (1991), similar assumptions are commonly assumed in the literature of incentive theory and they avoid “a technical open-set problem of limited economic interest.”}

## 3 Analysis

We begin by deriving the manager’s trading strategy given the contract and her effort choice. Fortunately, this optimal trading strategy can be concisely described by simple formulas.
3.1 Optimal Trading Strategy

The case in which $\delta = 0$ is trivial. Note that if $\delta = 0$, the manager’s payoff is unrelated to the performance of the portfolio. Since $c > 0$, she would choose $e = 0$. Therefore,

$$
\pi^*(1; 0, b, L, U, e) = \pi^*(0; 0, b, L, U, e) = \pi^*(0; 0, b, L, U, 0) = \begin{cases} 
0 & \text{if } 0 \in [L, U], \\
L & \text{if } L > 0, \\
U & \text{otherwise.}
\end{cases}
$$

(11)

Then what if $\delta > 0$? The following lemma gives us the key building block of this paper. To simplify the notation, let

$$
g(\pi; e, s, \delta, b) := \mathbb{E}[u_m(w_m(\pi))|e, s, \delta, b].
$$

**Lemma 1.** If $\delta > 0$, then

(I) $g(\pi; e, s, \delta, b)$ is strictly decreasing when $\pi \in (-\infty, \pi_1(\delta, b)]$ and strictly increasing when $\pi \in [\pi_2(\delta, b), +\infty)$;

(II) $g(+\infty; e, s, \delta, b) = g(-\infty; e, s, \delta, b) = +\infty$;

(III) $g(\pi; e, s, \delta, b)$ is strictly increasing when $\pi \in (\pi_1(\delta, b), \pi_c(e, s, \delta, b)]$ and strictly decreasing when $\pi \in [\pi_c(e, s, \delta, b), \pi_2(\delta, b))$.

That is, $g(\pi; e, s, \delta, b)$ has a local maximum at $\pi_c(e, s, \delta, b)$ where

$$
\pi_1(\delta, b) := -\frac{\delta + b}{\delta R},
$$

(12)

$$
\pi_2(\delta, b) := -\frac{\delta + b}{\delta R},
$$

(13)

$$
\pi_c(e, s, \delta, b) := \Delta(s, e) \frac{\delta + b}{\delta R},
$$

(14)

$$
\Delta(s, e) := \frac{K(s, e) - 1}{K(s, e) + 1},
$$

(15)

$$
K(s, e) := \left[ \frac{\breve{p}(s, e)}{1 - \breve{p}(s, e)} \right]^{1/\beta}.
$$

(16)

**Proof.** See appendix A.
The following corollary, which is an immediate observation implied by lemma 1, together with (9) and (11), pins down the manager’s optimal trading strategy.

**Corollary 1.** Given any contract \((\delta, b, L, U)\) with \(\delta > 0\),

\[
\max_{\pi \in [L, U]} \mathbb{E}[u_m(w_m)|e, s, \delta, b] = \begin{cases} 
\max\{g(L; e, s, \delta, b), g(U; e, s, \delta, b), g(\pi_c; e, s, \delta, b)\} & \text{if } \pi_c(e, s, \delta, b) \in [L, U], \\
\max\{g(L; e, s, \delta, b), g(U; e, s, \delta, b)\} & \text{otherwise.}
\end{cases}
\]  

\[17\]

**Proof.** Omitted. \(\square\)

Lemma 1 confirms the argument illustrated in figure 1. The first two points echo the insight of proposition 3 in Gollier, Koehl, and Rochet (1997). The intuition is simple. When the manager’s payoff is positively linked to the market value of the portfolio, since the liability is limited and her utility function is unbounded above, she will always take infinite risk if there is no restriction. This creates an incentive for the investor to set investment constraints, and we formalize this intuition in proposition 2. Note that this intuition is quite robust and does not limited to the two-point distribution. The important role of limited liability can also be understood via a thought experiment: if the manager was not protected, \(\pi_c(e, s, \delta, b)\) was also the global solution, and she would only take finite risk. The last point provides the foundation for non-binding constraints.

### 3.2 Optimal Contracts

Now we characterize the optimal contracts and obtain the following low-hanging fruit.

**Lemma 2.** The manager’s individual rational (IR) constraint is always satisfied and the investor’s IR constraint is never binding in equilibrium.

**Proof.** Since the manager's outside option is 0, any contract satisfies her IR constraint due to the principle of limited liability (see (7)). In addition, note that the investor cannot invest in the risky asset directly, if he does the investment by himself, then his expected / reservation utility is \(u_i(1)\). Now consider any contract with the form \((0, 0, \pi_0, U_0)\), where
π_0 := \frac{1}{R} \left( \frac{p}{1-p} \right)^{1/\gamma} - 1, \quad (18)

U_0 \geq \pi_0. \quad (19)

Here \( \pi_0 \) is the solution to the investor's first-order condition as if he can access to the risky asset:

\[
p(1 + \pi_0 R)^{-\gamma} R - (1 - p)(1 - \pi_0 R)^{-\gamma} R = 0. \quad (20)
\]

Under such contract, the investor's expected utility is \( \mathbb{E}[u_i(1 + \pi_0 \tilde{R})] > u_i(1) \). Hence, any optimal contract would not bind the investor's IR constraint.

Note that lemma 2 indicates that the investor's IR constraint is too loose, and its proof suggests that the investor's expected utility is bounded below by \( \mathbb{E}[u_i(1 + \pi_0 \tilde{R})] \). Hence, we define the following “IR” constraint instead:

\[
\mathbb{E}[u_i(w_i)|\delta, b, L, U] \geq \mathbb{E}[u_i(1 + \pi_0 \tilde{R})]. \quad (21)
\]

The next result characterizes the closed-form solution to one type of optimal contracts.

**Proposition 1.** If \( (\delta^*, b^*, L^*, U^*) \) is an optimal contract and \( \delta^* = 0 \), then \( b^* = 0, L^* = \pi_0 \) and \( U^* \geq \pi_0 \).

**Proof.** If \( \delta^* = 0 \), the manager's trading strategy is given by (11), which immediately implies \( b^* = 0, L^* = \pi_0 \) and \( U^* \geq \pi_0 \). \( \square \)

**Proposition 2.** If \( (\delta^*, b^*, L^*, U^*) \) is an optimal contract, then

\[
\delta^* + b^* \leq 1, \quad (22)
0 \leq b^* < 1, \quad (23)
0 \leq \delta^* < 1. \quad (24)
\]

In addition, for any constraint \( \tilde{C} \in \{L^*, U^*\} \), if \( \tilde{C} \) will be binding with non-zero probability, then

\[
-\frac{1}{R} \leq \tilde{C} \leq \frac{1}{R}. \quad (25)
\]
Finally, if $\delta^* > 0$, then

$$-\infty < L^* \leq U^* < +\infty.$$  \hfill (26) \hfill \\

Proof. See appendix A. \hfill \Box

Note that the second half of proposition 2 chimes in with the crowd wisdom–trading restrictions are used to prevent the manager’s from taking excessive risk because of the asymmetry of the payoff induced by limited liability.

By lemma 2 and proposition 2, the optimal contract(s) can be obtained by solving the following program ($P_0$):

$$\max_{(\delta,b,L,U)} \mathbb{E}[u_i(w_i)],$$

subject to constraints (1), (9), (10),

$$\delta + b \leq 1,$$  \hfill (27) \\
$$0 \leq b < 1,$$  \hfill (28) \\
$$0 \leq \delta < 1.$$  \hfill (29) \\

Then we define

$$\Gamma(p,R,\beta,\gamma,q,c) := \{ (\delta,b,L,U) | (\delta,b,L,U) \text{ solves } \mathcal{P}_0 \}. \hfill (30)$$

Hence, $\Gamma(p,R,\beta,\gamma,q,c)$ is the set of all optimal (linear) contracts.

In contrast to the manager’s optimal portfolio selection problem, neat closed-form solutions to the optimal contracts can only be obtained in some special cases, and the details will be showed later.

We then discuss the effort level under optimal contracts. The following result sharply connects the optimal contract to the manager’s optimal effort level.

**Proposition 3.** If $(\delta^*,b^*,L^*,U^*)$ is an optimal contract, and $e^*(\delta^*,b^*,L^*,U^*)$ be the effort level the manager will choose under this contract. Then $\delta^* > 0 \Rightarrow e^* = 1$ and $e^* = 1 \Rightarrow \delta^* > 0$.

Proof. We show it by contradiction. Suppose $\delta^* > 0$ and $e^* = 0$. Then the manager has no information advantage, and this contract is strictly dominated by $(0,0,\pi_0, U_0)$, which contradicts its optimality.
Now suppose \( e^* = 1 \) and \( \delta^* = 0 \). Then the manager’s payoff is independent of the performance of the portfolio. Since effort is costly, she will deviate to \( e = 0 \). \( \square \)

The interpretation of proposition 3 is subtle. It does not mean that \( \delta^* > 0 \) and \( e^* = 1 \) in any equilibrium. It only says that they appear (or disappear) together. See proposition 1. Nevertheless, proposition 3 helps us to establish the next result which highlights the possibility of the existence of non-binding and effective constraints.

**Proposition 4.** If \((\delta^*, b^*, L^*, U^*)\) is an optimal contract and

\[
\begin{align*}
\delta^* &> 0, \\
L^* &\in G(\delta^*, b^*) := (\max\{\hat{L}(1, 1, \delta^*, b^*), \hat{L}(1, 0, \delta^*, b^*)\}, \pi_c(1, 0, \delta^*, b^*)), \\
U^* &\in H(\delta^*, b^*) := (\pi_c(1, 1, \delta^*, b^*), \min\{\hat{U}(1, 1, \delta^*, b^*), \hat{U}(1, 0, \delta^*, b^*)\}),
\end{align*}
\]

then (I) \( L^* \) and \( U^* \) will never be binding; (II) any contract \((\delta^*, b^*, L^{**}, U^{**})\) is also optimal if \( L^{**} \in G(\delta^*, b^*) \) and \( U^{**} \in H(\delta^*, b^*) \). Here

\[
\begin{align*}
\hat{L}(e = 1, s, \delta^*, b^*) &:= \min \mathcal{R}(s), \\
\hat{U}(e = 1, s, \delta^*, b^*) &:= \max \mathcal{R}(s),
\end{align*}
\]

where \( \mathcal{R}(s) \) is the set of all real roots of the equation

\[
g(\pi_c(e = 1, s, \delta^*, b^*); e = 1, s, \delta^*, b^*) = g(\pi; e = 1, s, \delta^*, b^*). \tag{31}
\]

**Proof.** See appendix A. \( \square \)

For convenience and later use, we introduce the concept of “strictly binding” here.

**Definition 1.** For any contract \((\delta, b, L, U)\), we say \( L \) will be strictly binding if \( \exists s \in \{0, 1\} \) and \( \epsilon > 0 \) such that

\[
L = \pi^*(s; \delta, b, L, U, e^*(\delta, b, L, U)), L - \epsilon = \pi^*(s; \delta, b, L - \epsilon, U, e^*(\delta, b, L - \epsilon, U)).
\]

Similarly, \( U \) will be strictly binding if \( \exists s \in \{0, 1\} \) and \( \epsilon > 0 \) such that

\[
U = \pi^*(s; \delta, b, L, U, e^*(\delta, b, L, U)), U + \epsilon = \pi^*(s; \delta, b, L + \epsilon, U + \epsilon, e^*(\delta, b, L, U + \epsilon)).
\]

Note that it is a stronger concept since it implies “binding,” and a constraint can be binding but not strictly binding.
The final result in this subsection points out an important feature of the optimal contract when the investor is more risk averse than the manager.

**Proposition 5.** If $\gamma \geq \beta$ and an optimal contract $(\delta^*, b^*, L^*, U^*)$ exists, then $b^* = 0$.

**Proof.** See appendix B.

The intuition behind proposition 5 is that the higher $b$, the more risk the manager will take, and hence a more risk-averse investor should not give the manager any non-zero base salary. In addition, while keeping the manager’s incentive compatible (IC) constraint satisfied, the investor can reduce the volatility by increasing $\delta$ and decreasing $b$. On the other hand, this result suggests that the difference of the risk tolerance can be solely identified by examining the fee structures in compensation contracts.

### 3.2.1 No Difference of the Risk Tolerance and Low Effort Cost

Propositions 2 and 4 together provide the theoretical foundation for explaining the ”non-binding puzzle.” Note that these results hinge on $\delta^* > 0$. Unfortunately, this is not guaranteed. Intuitively, if the cost of effort $c$ is too large, there is no way to induce the manager to acquire valuable information. Under this situation, the optimal contract is $(0, 0, \pi_0, U_0)$, which is both uninteresting and unrealistic. Hence, in most of the time, we consider cases in which $c$ is small enough such that the optimal contract should link the manager’s payoff to the market value of AUM. The intuition is that when $c$ is small, then a positive $\delta$ will make gathering information profitable for the manager, and with superior information, the portfolio can be further optimized. The additional gain can make both of the investor and the manager better off. The following proposition confirms this intuition under the special case in which there is no difference of the risk tolerance between the investor and the manager.

**Proposition 6.** Suppose $\beta = \gamma$ and

$$c < Y := \left( x_1 - x_0 \right) \frac{1 - \left( \frac{x_0}{x_1} \right)^{1/(1-\beta)}}{1 - \beta},$$

(32)
where

\[ \chi_0 := \mathbb{E}[u_m(1 + \pi_0 \tilde{R})|e = 0] \]
\[ = p \frac{[1 + \Delta(1, 0)]^{1-\beta}}{1 - \beta} + (1 - p) \frac{[1 - \Delta(1, 0)]^{1-\beta}}{1 - \beta}, \quad (33) \]
\[ \chi_1 := \mathbb{E}[\mathbb{E}[u_m(1 + \pi_c(e = 1, s, \delta, 0) \tilde{R})|s]|e = 1] \]
\[ = \sum_{s \in \{0,1\}} p_s \left\{ \tilde{p}(s, 1) \frac{[1 + \Delta(s, 1)]^{1-\beta}}{1 - \beta} + (1 - \tilde{p}(s, 1)) \frac{[1 - \Delta(s, 1)]^{1-\beta}}{1 - \beta} \right\}. \quad (34) \]

If contract \((\delta^*, b^*, L^*, U^*)\) is optimal, then \(\delta^* > 0\).

**Proof.** See appendix A. \(\square\)

The more striking result is that the optimal contract is quite simple if \(\beta = \gamma\) and \(c\) satisfies (32), which allows us to do comparative statics analytically. The following proposition characterizes the optimal linear contract(s) under this special case.

**Proposition 7.** Suppose \(\beta = \gamma\) and (32) holds. The contract \((\delta^*(c) > 0, 0, -1/R, 1/R)\) is optimal, where

\[ \delta^*(c) := \left( \frac{c}{\chi_1 - \chi_0} \right)^{1/(1-\beta)}. \quad (35) \]

Moreover, a contract \((\delta^*, b^*, L^*, U^*)\) is optimal if and only if

\[ \delta^* = \delta^*(c), \quad (36) \]
\[ b^* = 0, \quad (37) \]

and

\[ L^* \in \tilde{G}(\delta^*(c)) := [\max\{\tilde{L}(1, 1, \delta^*(c), 0), \tilde{L}(1, 0, \delta^*(c), 0)\} - \Delta(0, 1)/R], \quad (38) \]
\[ U^* \in \tilde{H}(\delta^*(c)) := [\Delta(1, 1)/R, \min\{\tilde{U}(1, 1, \delta^*(c), 0), \tilde{U}(1, 0, \delta^*(c), 0)\}]. \quad (39) \]

**Proof.** See appendix C. \(\square\)

Note that there are two main features of the optimal (linear) contracts: (I) zero base salary; (II) all constraints will not be strictly binding (see definition 1).\(^{19}\) All the two fea-
tures are driven by the condition $\beta = \gamma$. Note that increasing the base salary encourages the manager’s risk-taking behavior. Since there is no difference of the risk tolerance, any $b > 0$ will lead to a suboptimal trading strategy which would hurt the investor. And any strictly binding constraint will also lead to suboptimal portfolio choice, which hurts the investor and the manager.

Now we compare the allocation rule under the optimal linear contracts to the first-best allocation. Here the first-best means the maximal expected utility the investor can achieve if he can dictate the manager’s choice of effort and portfolio. The first-best is obtained by solving a social planner (she/he)’s problem. Note that (32) guarantees that taking effort is efficient. Her/his belief about the world is characterized by the probability space

$$(\Omega := \{0, 1\} \times \{-R, R\}, \mathcal{F} := 2^\Omega, \mathbb{P}),$$

where

$$\mathbb{P}(s = 1, \tilde{R} = R) = p_1 \tilde{p}(1, 1) =: \rho_{11},$$

$$\mathbb{P}(s = 1, \tilde{R} = -R) = p_1 (1 - \tilde{p}(1, 1)) =: \rho_{10},$$

$$\mathbb{P}(s = 0, \tilde{R} = R) = p_0 \tilde{p}(0, 1) =: \rho_{01},$$

$$\mathbb{P}(s = 0, \tilde{R} = -R) = p_0 (1 - \tilde{p}(0, 1)) =: \rho_{00}.$$

Let $\pi(s)$ be the social planner’s trading strategy. Her/his problem is

$$\max_{(\alpha_{11}, \alpha_{10}, \alpha_{01}, \alpha_{00}, \pi(0), \pi(1)) \in [0, 1]^4 \times \mathbb{R}^2} \sum_{s \in \{0, 1\}} \left[ \rho_{s1} u_i(\alpha_{s1} (1 + \pi(s) R)) + \rho_{s0} u_i(\alpha_{s0} (1 - \pi(s) R)) \right],$$

such that

$$\sum_{s \in \{0, 1\}} \left[ \rho_{s1} u_m((1 - \alpha_{s1})(1 + \pi(s) R)) + \rho_{s0} u_m((1 - \alpha_{s0})(1 - \pi(s) R)) \right] \geq c. \quad (44)$$

The first-order conditions imply

$$\alpha_{11} = \alpha_{10} = \alpha_{01} = \alpha_{00} =: 1 - x^*(c).$$

Hence, the first-best involves a constant proportional sharing-rule, which confirms the find-

---

20This definition of the first-best is adapted from Dybvig, Farnsworth, and Carpenter (2010). But they model the manager’s portfolio choice differently—“we model the manager’s choice of trading strategy as a report of the signal.”
ing of Dybvig, Farnsworth, and Carpenter (2010). Since $\beta = \gamma$, the social planner’s optimal portfolio choice is given by $\Delta(s,1)/R$. Therefore, the proportion given to the manager is

$$x^*(c) = \left( \frac{c}{\chi_1} \right)^{1/(1-\beta)}.$$  \hspace{1cm} (45)

Comparing (35) with (45), we have

$$\delta^*(c) > x^*(c),$$

since $\chi_1 = \mathbb{E}[u_m(1 + \pi_0 \tilde{R})|e = 0] > 1 > 0$. This result is intuitive, because the manager’s incentive compatible (IC) constraint forces the investor to share more with the manager. In addition, the optimal linear contracts approach the first-best when $q \to 1/2^-$, in the sense that

$$\lim_{q \to 1/2^-} \frac{\delta^*(c)}{x^*(c)} = 1,$$

due to the fact that

$$\lim_{q \to 1/2^-} \frac{\chi_1}{\chi_0} = +\infty.$$ 

Hence, even if we allow more general forms of the contract, since the investor’s expected utility is bounded by the first-best, the gap between an optimal linear contract and an optimal “universal” contract is small if the manager’s ability $q$ is large. This additional justifies our restriction on the contract space and may also explain the popularity of linear contracts in practice.

Finally, since we have closed-form solutions under this special setting, we can do comparative statics analytically, and the results are summarized in proposition 8.

For convenience, we introduce a few new notations first. If $\beta = \gamma$ and $c$ satisfies (32), then

\begin{itemize}
\item[21] The result of Dybvig, Farnsworth, and Carpenter (2010) is obtained under the assumption that both the investor and the manager have the log utility function. Hence, actually we generalize their result by showing that this is also true for any CRRA utility function with a relative risk-aversion coefficient between 0 and 1 in our model. As they point out, these results build on “Ross’s (1974,1979) work on preference similarity.”
\item[22] If we use a different metric, say $|\delta^*(c) - x^*(c)|$, to measure “distance,” then we get the similar result: $\lim_{q \to 1/2^-} |\delta^*(c) - x^*(c)| = 0$. Also note that $\lim_{c \to 0^+} |\delta^*(c) - x^*(c)| = 0$. So, under this metric, the optimal linear contract does approach the first-best when the cost of effort goes to zero.
\end{itemize}
by proposition 7, \( \Gamma(p, R, \beta, \gamma, q, c) \) is not empty and the following functions are well-defined.

\[
\tilde{L}(p, R, \beta, \gamma, q, c) := \sup_{(\delta, L, U) \in \Gamma(p, R, \beta, \gamma, q, c)} L, \quad (46)
\]

\[
\tilde{U}(p, R, \beta, \gamma, q, c) := \inf_{(\delta, L, U) \in \Gamma(p, R, \beta, \gamma, q, c)} U. \quad (47)
\]

We introduce (46) and (47) because, under the condition of proposition 7, we have a continuum of optimal contracts only differ in their choices of constraints, but \( \tilde{L}(p, R, \beta, \gamma, q, c) \) and \( \tilde{U}(p, R, \beta, \gamma, q, c) \) are uniquely determined by the exogenous parameters of the model. Note that \([\tilde{L}(\cdot), \tilde{U}(\cdot)]\) can be interpreted as the “effective” position limits, and in this special case,

\[
\tilde{L}(p, R, \beta, \gamma, q, c) = \Delta(0, 1)/R, \\
\tilde{U}(p, R, \beta, \gamma, q, c) = \Delta(1, 1)/R.
\]

Hence, they will be binding but none of them will be strictly binding.

**Proposition 8.** Suppose \( \beta = \gamma =: \theta \) and \( c \) satisfies (32).

(I) \( \frac{\partial \delta^*(c)}{\partial c} > 0, \frac{\partial \tilde{L}(\cdot)}{\partial \theta} = \frac{\partial \Delta(0, 1)/R}{\partial \theta} = 0 \) and \( \frac{\partial \tilde{U}(\cdot)}{\partial \theta} = \frac{\partial \Delta(1, 1)/R}{\partial \theta} = 0; \)

(II) \( \frac{\partial \delta^*(c)}{\partial q} < 0, \frac{\partial \tilde{L}(\cdot)}{\partial q} = \frac{\partial \Delta(0, 1)/R}{\partial q} < 0 \) and \( \frac{\partial \tilde{U}(\cdot)}{\partial q} = \frac{\partial \Delta(1, 1)/R}{\partial q} > 0; \)

(III) \( \frac{\partial \delta^*(c)}{\partial \theta} \neq 0, \)

\[
\frac{\partial \tilde{L}(\cdot)}{\partial \theta} = \frac{\partial \Delta(0, 1)/R}{\partial \theta} \begin{cases} < 0 & \text{if } q < p - 1/2, \\ = 0 & \text{if } q = p - 1/2, \\ > 0 & \text{if } q > p - 1/2, \end{cases}
\]

and \( \frac{\partial \tilde{U}(\cdot)}{\partial \theta} = \frac{\partial \Delta(1, 1)/R}{\partial \theta} < 0; \)

(IV) \( \frac{\partial \delta^*(c)}{\partial R} = 0, \)

\[
\frac{\partial \tilde{L}(\cdot)}{\partial R} = \frac{\partial \Delta(0, 1)/R}{\partial R} \begin{cases} < 0 & \text{if } q < p - 1/2, \\ = 0 & \text{if } q = p - 1/2, \\ > 0 & \text{if } q > p - 1/2, \end{cases}
\]

and \( \frac{\partial \tilde{U}(\cdot)}{\partial R} = \frac{\partial \Delta(1, 1)/R}{\partial R} < 0; \)
(I) \((p,R,\beta,\gamma,q,c) = (0.55,0.5,0.02,\theta,\theta,5 \times 10^{-5})\) \((p,R,\beta,\gamma,q,c) = (p,0.5,0.5,0.02,5 \times 10^{-5})\)

Figure 2: Left panel: \(\delta^*\) and \(\Upsilon - c\) as functions of \(\theta(=\beta = \gamma)\); right panel: \(\delta^*\) and \(\Upsilon - c\) as functions of \(p\).

\[
(V) \quad \frac{\partial \delta^*(c)}{\partial p} > 0, \quad \frac{\partial \bar{f}(\cdot)}{\partial p} \frac{\partial \Delta(0,1)/R}{\partial p} > 0 \quad \text{and} \quad \frac{\partial \bar{g}(\cdot)}{\partial p} \frac{\partial \Delta(1,1)/R}{\partial p} > 0.
\]

Proof. See appendix A. \(\square\)

We briefly discuss the intuition behind proposition 8. (I) is intuitive: a higher \(c\) enforces the manager’s IC constraint, and the investor has to share more with her to induce effort, but \(c\) changes neither the “effective” position limits nor the trading strategy. In (II), the effect of \(q\) on \(\delta^*(c)\) is a little bit counter-intuitive. This is because we assume there is only one investor and he has the right to propose the contract: a better signal makes the “pie” bigger, but the manager is always paid the minimal money that retains her IC constraint. Naturally, when \(q\) increases the investor relaxes the position limits to utilize the manager’s information advantage, and the manager trades more aggressively because she faces less uncertainty conditional on a better signal. In (III), when the players’ risk aversions increases, they prefer more safe assets, hence both the “effective” position limits and the trading strategy are shrunk to zero. In (IV), \(\delta^*(c)\) does not depend on \(R\) because the optimal trading strategy always offsets the effect of \(R\) so that \(\chi_1\) and \(\chi_0\) are fixed. (V) is also obvious—a more optimistic prior about the risky asset “shifts” the trading strategy to the right.

Note that we use notations \(\frac{\partial \delta^*(c)}{\partial \theta} > 0\) and \(\frac{\partial \delta^*(c)}{\partial p} > 0\). This is because the expressions for \(\frac{\partial \delta^*(c)}{\partial \theta}\) and \(\frac{\partial \delta^*(c)}{\partial p}\) are too complicated to have neat answers about their signs. However, our numerical results suggest that \(\delta^*(c)\) is decreasing in the relative risk aversion coefficient of the manager (investor) and increasing in \(p\). See blue lines in figure 2.\(^{23}\) The intuition behind\(^{23}\) Orange lines have been drawn to track (32).
2(I) may be that the more risk averse of the players, the more valuable the signal, and \( \delta^* \) decreases according to (35); a similar intuition behind figure 2(II) holds: when \( p \) increases, the ex-ante uncertainty about the risky asset decreases and the value of the signal–\( \chi_1 - \chi_0 \)–decreases, and then \( \delta^* \) increases according to (35), as long as (32) is satisfied. When the value of the signal becomes too low, inducing the manager’s effort is no longer profitable for the investor, and the optimal \( \delta^* \) is 0. Again, such counter-intuitive patterns in figure 2 are due to our assumption that the investor has all the bargaining power.

Proposition 8, together with figure 2, provides a benchmark for our numerical experiments in section 4.1.

3.3 The “Non-binding Puzzle” and Managerial Talent

As we discussed in the introductory section, no-short-sale and no-leverage are commonly imposed by investors in the real world, and are extensively assumed as exogenous assumptions by (financial) economists. Therefore, under what conditions would we observe constraints on short-selling and leverage which will never be binding? Using the results in section 3.2.1, we identify the following sufficient condition.

**Proposition 9.** If \( \beta = \gamma \), (32) holds, and

\[
\frac{1}{2} < p < \eta, \quad \text{(48)} \\
0 < q < \min \left\{ p - \frac{1}{2} \frac{\eta - p}{2(p + \eta - 2\eta p)} \right\}, \quad \text{(49)}
\]

where

\[
\eta := \frac{(1+R)^\beta}{(1-R)^\beta + 1},
\]

then the contract \((\delta^*(c),0,0,1)\) is optimal and the manager’s portfolio choice will always be in \((0,1)\).

**Proof.** See appendix A. \(\square\)

The economic meaning of (48) is that neither the investor nor the manager wants to leverage or short, when new information hasn’t come. This condition matches the real world, and technically guarantees \(0, \min \left\{ p - \frac{1}{2} \frac{\eta - p}{2(p + \eta - 2\eta p)} \right\} \neq \emptyset\). Hence, the key driver
behind proposition 9 is actually (49), which suggests a tight connection between the “non-binding puzzle” and the manager’s ability. The “low-quality (ability)” has the dual role of (I) inducing the restrictions on short-selling and leverage and (II) controlling the manager’s trading strategy. Note that constraints on short-selling and leverage are very restrictive. For example, if the manager’s ability is high, then the expected excess return of the tree conditional on a bad signal \( s = 0 \) is negative, and a short-selling strategy will make both the investor and the manager better off. Since the manager’s ability is common knowledge, the investor would only impose such restrictions when the manager is low-quality, because he knows that no matter what signal the manager will receive, it would not be good enough to justify any extreme portfolio choice. In addition, when such restrictive constraints are imposed, the quality of the signal also prevents the manager from approaching the boundary, because even with the new information, the stock market is still chaos for the manager, and any portfolio position on the boundary is too risky. The following example illustrates proposition 9.

**Example 1.** Suppose

\[(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.5, 0.02, 5 \times 10^{-5}).\]

Then we have

\[
\begin{align*}
\beta &= \gamma = 0.5, \\
\chi_1 &\approx 2.01152, \\
\chi_0 &\approx 2.00998, \\
c &= 5 \times 10^{-5} < \chi_1 - \chi_0 \\
0 < p &= 0.5 < \eta = \frac{\sqrt{3}}{\sqrt{3} + 1} \approx 0.633975, \\
0 < q &< \min \left\{ p - \frac{1}{2}, \frac{\eta - p}{2(p + \eta - 2\eta p)} \right\} \approx 0.0862866.
\end{align*}
\]

Hence, the condition of proposition 9 is satisfied and we have an optimal contract

\[(\delta^*, b^*, L^*, U^*) = (\delta^*(c), 0, 0, 1)\] (50)

with a sharing rule \( (\delta^*(c) \approx 0.0010481, b^* = 0) \) and constraints on short-selling and lever-
The manager’s expected utility as a function of $\pi$ for $s \in \{0, 1\}$, under the optimal contract $(\delta^*(c) \approx 0.00104811, 0, 0, 1)$. The values of parameters are chosen as $(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.5, 0.02, 5 \times 10^{-5})$.

Under this contract, the manager’s trading strategy is

$$\pi^*(s; \delta^*(c), 0, 0, 1, e^* = 1) \approx \begin{cases} 0.54713 & \text{if } s = 1, \\ 0.24009 & \text{if } s = 0. \end{cases}$$

With this strategy, neither the no-short-selling constraint nor the no-leverage constraint will be binding. We also plot the manager’s expected utility (without the effort cost) as a function of her portfolio choice given the optimal contract and the realization of the signal in figure 3, which has the same pattern demonstrated in figure 1 and confirms the trading strategy predicted by theory.

The next example starkly demonstrates that the “non-binding puzzle” can arise even when the investor and the manager have different risk-aversion coefficients. Hence, the condition in proposition 9 is not necessary.

**Example 2.** Suppose

$$(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$$.
Figure 4: The manager’s expected utility as a function of $\pi$ for $s \in \{0, 1\}$, under the optimal contract $(0.00084815, 0.00019895, 0, 1)$. The values of parameters are chosen as $(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$.

Note that the settings of examples 1 and 2 just differ in $\gamma$–the manager is more risk averse than the investor in this example. Now we have to rely on numerical methods to solve the optimal contract(s), since $\beta \neq \gamma$. The details of our numerical methods are discussed in section 4.

The optimal contract reported by MATLAB is

$$(\delta^*, b^*, L^*, U^*) \approx (0.00084915, 0.00019895, -2.4686, 2.4686).$$

Under this contract the manager’s trading strategy is

$$\pi^*(s; \delta^*, b^*, L^*, U^*, e^* = 1) \approx \begin{cases} 0.67532 & \text{if } s = 1, \\ 0.29634 & \text{if } s = 0. \end{cases}$$

Therefore, none of the constraints will be binding and the contract

$$(\delta^*, b^*, 0, 1) \approx (0.00084815, 0.00019895, 0, 1).$$
is also optimal by proposition 4. See figure 4 for an illustration. Hence, we generate the “non-binding puzzle” even if $\beta \neq \gamma$.

Comparing example 2 with example 1, we can see that a less risk-averse investor “shifts” the trading strategy to the right, by lowing $\delta$ and providing non-zero base salary. The intuition is straightforward: in the investor’s opinion, the manager is too risk-averse, and increasing $b$ will induce her to be more aggressive. In addition, from the perspective of risk sharing, the more risk-averse party should bear less share and compensated by more “fixed” payments.

If an empiricist (he/she) decides to test proposition 9, he/she has to estimate the difference of the risk-tolerance and the cost of effort first, which is not a trivial task, though taking proposition 5 into account. Even if he/she can observe $\beta$, $\gamma$ and $c$ perfectly, the requirement that $\beta = \gamma$ and $c < (\chi_1 - \chi_0) \left[ 1 - \left( \frac{\chi_0}{\chi_1} \right)^{1/(1-\beta)} \right]^{1-\beta}$ would reduce the sample size dramatically. Hence, a more down-to-earth question is: what can be inferred if we do observe non-binding constraints on short-selling and leverage? The next proposition answers this question.

**Proposition 10.** If we observe a contract $(\delta^*, b^*, 0, 1)$ in which $\delta^* > 0$ and the constraints are not binding in any state, then (48) and (49) hold.

**Proof.** See appendix A.

Comparing with proposition 9, proposition 10 is much easier to test. It also predicts that non-binding constraints on short-selling and leverages only arise when the manager’s talent is below a threshold which merely depends on the fundamentals of the economy. Hence, it motivates a fresh new measure of managerial talent: wherever we observe the “non-binding puzzle,” we know that fund’s manager is “low-type.”

Traditionally, measuring a mutual fund’s performance involving in constructing a benchmark: practitioners usually chooses an index that matches the fund’s “style;” financial

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24 Readers may notice that we describe the risky asset by the distribution of its return instead of the payoff, and returns should be determined by a pricing kernel, which involves the households’ preferences in the economy. Our model is a partial equilibrium model, and both the investor and the manager are price takers. Hence, the fundamentals of the economy in this model, $(p, R)$, is equivalent to the fundamentals of a general equilibrium model, $(F, \tilde{M})$, where $F$ is the distribution of the payoff of the Lucas tree and $\tilde{M}$ is the pricing kernel, and both of them are not affected by a single investor and/or a single manager.

25 Barber, Huang, and Odean (2016) and Berk and van Binsbergen (2016) find the evidence that “sophisticated” investors may use the CAPM to assess mutual fund performance.
economists construct a benchmark in a “more sophisticated” way—choosing an(a) (reduced-form) asset pricing model\(^{26}\) and estimating the net alpha.

Hence, practitioners’ approach is actually subjective and arbitrary. Financial economists’ approach, however, is also constrained by the correctness of the asset pricing model they choose. Note that a promising feature of the measure suggested by this paper, is that it provides a nature way to group different mutual funds, which is somewhat similar to the seminal work of Barras, Scaillet, and Wermers (2010). However, the later method still depends on the right choice of an asset pricing model. In addition, even the threshold is determined by the economy’s fundamentals, funds can still be grouped *without* knowing these parameters.

### 4 Numerical Analysis

When \(\beta \neq \gamma\), we solve the optimal contracts numerically. Instead of solving \(\mathcal{P}_0\), we solve another program (\(\mathcal{P}_1\)) suggested by lemma 1 and (25) of proposition 2:

\[
\max_{(\delta, b, L, U)} \mathbb{E}[u_i(w_i)],
\]

subject to constraints (9), (10), (27), (28),(29) and

\[
-\frac{1}{R} \leq L \leq U \leq \frac{1}{R}. \tag{52}
\]

In addition, by proposition 5, we can impose one more constraint:

\[
b_{\gamma \geq \beta} = 0. \tag{53}
\]

The advantage of \(\mathcal{P}_1\) is that we can search for \(L^*\) and \(U^*\) in a (much) smaller space. The disadvantage is that: if an optimal contract with non-binding constraints exists, \(\mathcal{P}_1\) may fail to identify its variants which only differ in the choices of constraints that are outside of \([-1/R, 1/R]\). This disadvantage is not so important and whenever solving the optimal contract numerically, we always stick to \(\mathcal{P}_1\).

To further improve the performance of our numerical methods, we encode the two necessary conditions—(21), which is implied by lemma 2, and proposition 3—into our code.

\(^{26}\) As noted in the introductory section, factor models developed by Fama and French (1993, 2015), Carhart (1997) and Hou, Xue, and Zhang (2015), are frequently used.
Moreover, we consider the contract \( \left( \tilde{\delta}, \tilde{b}, \frac{\tilde{\delta} + \tilde{b}}{\delta R}, \frac{\tilde{\delta} + \tilde{b}}{\delta R} \right) \), where 

\[
\tilde{\delta} \in \mathcal{C} := \arg \max_{\delta \in \left( \frac{\chi^1 - \chi^0}{1-\beta} \right)^{1/(1-\beta)}} E \left[ u_i(w_i) \left| \delta, \frac{c}{\chi^1 - \chi^0} \left( \frac{\chi^1 - \chi^0}{1-\beta} \right)^{1/(1-\beta)} - \delta, -\frac{\delta + b}{\delta R}, \frac{\delta + b}{\delta R} \right. \right],
\]

\[
\tilde{b} = \left( \frac{c}{\chi^1 - \chi^0} \right)^{1/(1-\beta)} - \tilde{\delta}.
\]

Such contract is the best among contracts which induce the effort and only utilize “interior” maxima. Note that \( \mathcal{C} \) can be empty, and under this situation we set \( \left( \tilde{\delta}, \tilde{b}, \frac{\tilde{\delta} + \tilde{b}}{\delta R}, \frac{\tilde{\delta} + \tilde{b}}{\delta R} \right) = (0, 0, \pi_0, U_0) \). Hence, \( \left( \tilde{\delta}, \tilde{b}, -\frac{\tilde{\delta} + \tilde{b}}{\delta R}, \frac{\tilde{\delta} + \tilde{b}}{\delta R} \right) \) is always well-defined. Any solution \( (\delta^*, b^*, L^*, U^*) \) to \( P_1 \) should satisfy

\[
E[u_i(w_i)|\delta^*, b^*, L^*, U^*] \geq E \left[ u_i(w_i) \left| \tilde{\delta}, \tilde{b}, \frac{\tilde{\delta} + \tilde{b}}{\delta R}, \frac{\tilde{\delta} + \tilde{b}}{\delta R} \right. \right] .
\]

Certainly, as is well-known, numerical optimization suffers from “traps” created by local optima, and there is no single algorithm that guarantees the identification of global optima under any circumstance. However, the optimality of the contracts in example 1 is guaranteed by proposition 9 (or 7). Hence, example 1 provides a perfect benchmark to (cross-)check our numerical methods (and theory). We plot the investor’s expected utilities around contract (50) in figure 5, which is consistent with our theory.

Under the setting of example 1, the numerical optimal contract reported by MATLAB (without taking (53) and (54) into account\(^{27} \)) is

\[
(\delta^*_n, b^*_n, L^*_n, U^*_n) = (0.001048099744037, 0, 0.183001784092694, 1.002978645648959).
\]

There are three points can be made by this simple experiment. First, since the constraints of this contract are slack, we can consider the algorithm did find the right constraints. Second, even without using (53), our numerical algorithm correctly identified the optimal value of \( b \). Third, since the theoretical value of the optimal \( \delta \) is

\[
\delta_t^* \approx 0.001048099743067,
\]

\(^{27}\text{By proposition 7, using (54) will lead to perfect identification of the true optimal contract predicted by theory.}\)
Figure 5: The investor’s expected utility given different choices of the contract. The values of parameters are chosen as $(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.5, 0.02, 5 \times 10^{-5})$. Note: in subfigure (VI), we define the investor’s expected utility as $u_i(1)$ if $L > U$. 
Figure 6: The investor’s expected utility given different choices of the contract. The values of parameters are chosen as \((p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})\). Note: in sub-figure (VI), we define the investor’s expected utility as \(u_i(1)\) if \(L > U\).
$\delta^*_n$ differs from $\delta^*_t$ until the 12th digit after the decimal point.\footnote{If we add (53), $\delta^*_n = 0.001048099746296$, which also differs from $\delta^*_t$ until the 12th digit after the decimal point. In this example, considering (53) yields a slightly worse result. This may be due to “luck.” However, (53) does improve the efficiency of searching optimal contracts when $\gamma \geq \beta$.}

The above experiment suggests that our numerical methods are reliable. In addition, the numerical optimal contract in example 2 passes the “plot” check in figure 6.

4.1 Numerical Comparative Statics

Proposition 8 and figure 2 summarize the results of comparative statics when $\beta = \gamma$ and $c$ is small.\footnote{Recall that if the cost of effort is too large, the optimal contract is given by proposition 1, and there is no interesting comparative statics under this situation.} Hence, the numerical experiments here are conducted under the settings $\beta > \gamma$ or $\beta < \gamma$. Note that for any optimal contract with a non-binding constraint, we can always find another optimal contract in which the corresponding constraint is binding but not strictly binding.

Figures 7, 8, 10 and 11 show that even $\beta \neq \gamma$, the comparative statics of $\delta^*$ and $\pi^*$ is almost identical to results in section 3.2.1. Hence, the intuition we obtained from the special case in which $\beta = \gamma$ holds in general. In particular, figures 8(V) and 8(VI) confirm proposition 10. That is, the non-binding no-short-sale and no-leverage constraints can only be observed when the manager’s ability is low.

There are also some new insights generated from our numerical comparative statics. The most important graph of this subsection is figure 9. Because in section 3.2.1, we assume $\beta = \gamma$. So, we can only change $\beta$ and $\gamma$ together. A few points should be noticed from figure 9. First, if $|\beta - \gamma|$ is too large, then there is no way to induce effort. Second, when the investor becomes more risk-averse, $\delta^*$ increases and $b^*$ decreases as long as $e^* = 1$. There are two factors that drive this: (I) risk-sharing; (II) “shrinking” the “local” portfolios to zero due to risk-aversion, as the investor desires—which can also be seen in figure 9(V). Third, the effects of $\beta$ on $\delta^*$ and $b^*$ are non-monotonic. However, the situation here is subtle: given the manager’s IC constraint is satisfied, increasing $\beta$ will decrease $\delta^*$, which can be explained by risk-sharing between the two players; however, increasing $\beta$ may increase or decrease $b^*$, which is illustrated in figure 9(IV).

In addition, propositions 1 and 5 say that if $\gamma \geq \beta$, then $b^* = 0$. Hence, there is no in-
Figure 7: The optimal contract and the manager’s trading strategy as functions of the cost of effort. $p, R$ and $q$ are fixed at 0.55, 0.5 and 0.02 respectively. Left panel: $\beta = 0.5 > \gamma = 0.4$; right panel: $\beta = 0.48 < \gamma = 0.5$. 

(I) $(p, R, \beta, \gamma, q) = (0.55, 0.5, 0.4, 0.02)$

(II) $(p, R, \beta, \gamma, q) = (0.55, 0.5, 0.48, 0.5, 0.02)$

(III) $(p, R, \beta, \gamma, q) = (0.55, 0.5, 0.4, 0.02)$

(IV) $(p, R, \beta, \gamma, q) = (0.55, 0.5, 0.48, 0.5, 0.02)$

(V) $(p, R, \beta, \gamma, q) = (0.55, 0.5, 0.4, 0.02)$

(VI) $(p, R, \beta, \gamma, q) = (0.55, 0.5, 0.48, 0.5, 0.02)$
(I) \((p, R, \beta, \gamma, c) = (0.55, 0.5, 0.4, 5 \times 10^{-5})\)

(II) \((p, R, \beta, \gamma, c) = (0.55, 0.5, 0.48, 5 \times 10^{-5})\)

(III) \((p, R, \beta, \gamma, c) = (0.55, 0.5, 0.4, 5 \times 10^{-5})\)

(IV) \((p, R, \beta, \gamma, c) = (0.55, 0.5, 0.48, 5 \times 10^{-5})\)

(V) \((p, R, \beta, \gamma, c) = (0.55, 0.5, 0.4, 5 \times 10^{-5})\)

(VI) \((p, R, \beta, \gamma, c) = (0.55, 0.5, 0.48, 5 \times 10^{-5})\)

Figure 8: The optimal contract and the manager’s trading strategy as functions of the ability of the manager. \(p, R\) and \(c\) are fixed at 0.55, 0.5 and \(5 \times 10^{-5}\) respectively. Left panel: \(\beta = 0.5 > \gamma = 0.4\); right panel: \(\beta = 0.48 < \gamma = 0.5\).
Figure 9: The optimal contract and the manager’s trading strategy as functions of the investor’s or the manager’s risk-aversion coefficient. \( p, R, q \) and \( c \) are fixed at 0.55, 0.5, 0.02 and \( 5 \times 10^{-5} \) respectively. Left panel: \( \beta = 0.5 \); right panel: \( \gamma = 0.5 \).
Figure 10: The optimal contract and the manager’s trading strategy as functions of $R$. $p$, $q$ and $c$ are fixed at $0.55, 0.02$ and $5 \times 10^{-5}$ respectively. Left panel: $\beta = 0.5 > \gamma = 0.4$; right panel: $\beta = 0.48 < \gamma = 0.5$. 

(I) $(R, \beta, \gamma, q, c) = (0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$

(II) $(R, \beta, \gamma, q, c) = (0.5, 0.48, 0.5, 0.02, 5 \times 10^{-5})$

(III) $(R, \beta, \gamma, q, c) = (0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$

(IV) $(R, \beta, \gamma, q, c) = (0.5, 0.48, 0.5, 0.02, 5 \times 10^{-5})$

(V) $(R, \beta, \gamma, q, c) = (0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$

(VI) $(R, \beta, \gamma, q, c) = (0.5, 0.48, 0.5, 0.02, 5 \times 10^{-5})$
Figure 11: The optimal contract and the manager’s trading strategy as functions of the prior belief of the players. $R$, $q$ and $c$ are fixed at $0.5, 0.02$ and $5 \times 10^{-5}$ respectively. Left panel: $\beta = 0.5 > \gamma = 0.4$; right panel: $\beta = 0.48 < \gamma = 0.5$. 

(I) $(R, \beta, \gamma, q, c) = (0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$

(II) $(R, \beta, \gamma, q, c) = (0.5, 0.48, 0.5, 0.02, 5 \times 10^{-5})$

(III) $(R, \beta, \gamma, q, c) = (0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$

(IV) $(R, \beta, \gamma, q, c) = (0.5, 0.48, 0.5, 0.02, 5 \times 10^{-5})$

(V) $(R, \beta, \gamma, q, c) = (0.5, 0.5, 0.4, 0.02, 5 \times 10^{-5})$

(VI) $(R, \beta, \gamma, q, c) = (0.5, 0.48, 0.5, 0.02, 5 \times 10^{-5})$
teresting comparative statics of $b^*$ when the investor is more risk-averse than the manager. However, the left panels of figures 7, 8, 10 and 11 show that when $\gamma < \beta$, the optimal base salary can be positive and suggest

$$\frac{\partial b^*}{\partial c} > 0, \quad (55)$$
$$\frac{\partial b^*}{\partial q} < 0, \quad (56)$$
$$\frac{\partial b^*}{\partial R} = 0, \quad (57)$$
$$\frac{\partial b^*}{\partial p} > 0, \quad (58)$$

as long as the optimal contract induces $e^* = 1$. We conjecture the intuitions behind (55), (56), (57) and (58) are same to the intuitions behind their counter-parts—(I), (II) and (IV) of proposition 8 and figure 2(II).

### 4.2 “Quadratic” Contracts

As we discussed in previous sections, linear contracts are mainstream contracts in reality and optimal linear contracts should approach the “general” optimal contracts because they do approach the first-best results under certain conditions.

However, is this “non-binding puzzle” driven by the linearity of compensation contracts? We now consider a more general contract form: $(\delta, \hat{\delta}, b, L, U)$. The only difference here is that we allow the investor to choose an additional parameter $\hat{\delta}$ for the additional quadratic term. More precisely, under this contract, the manager’s payoff is given by

$$w_m = \max\{0, b + \delta (1 + \pi \hat{R}) + \hat{\delta} (1 + \pi \hat{R})^2\}, \quad (59)$$

instead of (7). Note that this is different from the “quadratic contract” considered in the literature, for example, Bhattacharya and Pfleiderer (1985), Stoughton (1993) and Gómez and Sharma (2006).\textsuperscript{30} We still assume

$$b \geq 0,$$

\textsuperscript{30}In their settings, the “quadratic contract” does not have a linear term. Moreover, under such contract, the manager chooses the effort level and reveals her signal to the investor who does the investment by himself. That is, the manager is actually an analyst.
as in the linear contract case. In addition, we require that

\[ z'(w) = \delta + 2w\hat{\delta} \geq 0, \tag{60} \]

where

\[ z(w) := b + \delta w + \hat{\delta}w^2, \]
\[ w := 1 + \pi \tilde{R}. \]

Note that (60) also guarantees a non-decreasing fee structure, so that we can rule out some behavior such as “costly round-trip transactions,” as Vayanos (2018) points out. Since the market value of AUM cannot be negative in any equilibrium, (60) is equivalent to

\[ \delta \geq 0, \tag{61} \]
\[ \hat{\delta} \geq 0. \tag{62} \]

Then we have the following result which is an analogy to proposition 2.

**Proposition 11.** If \((\delta^*, \hat{\delta}^*, b^*, L^*, U^*)\) is an optimal contract, then

\[ (\delta^*, \hat{\delta}^*, b^*) \in \Theta := \{(\delta, \hat{\delta}, b)|b \geq 0, \delta \geq 0, \hat{\delta} \geq 0, b + \delta + \hat{\delta} \leq 1\}. \tag{63} \]

Moreover, for any \(\tilde{C} \in \{L^*, U^*\}\), if \(\tilde{C}\) will be binding with non-zero probability, then

\[ \tilde{C} \in [-1/R, 1/R]. \tag{64} \]

**Proof.** See appendix A. \(\square\)

Note that the set of linear contract considered in previous sections is a proper subset of the set of “quadratic” contracts considered here.

We also assume the same tie-breaking rules:

\[ \pi^*(s; \delta, \hat{\delta}, b, L, U, e) = \arg \min_{\pi \in \hat{\Pi}(e, s, \delta, \hat{\delta}, b, L, U)} |\pi|, \tag{65} \]

\(31\) For details, see footnote 15.
where

\[ \hat{A}(e, s, \delta, \hat{\delta}, b, L, U) := \arg \max_{\pi \in [L, U]} \mathbb{E}[u_m(w_m(\pi))]|e, s, \delta, \hat{\delta}, b]; \]

and

\[ e^* (\delta, \hat{\delta}, b, L, U) = \max \{\arg \max_{e \in \{0, 1\}} [\mathbb{E}[u_m(w_m(\pi^* (s; \delta, \hat{\delta}, b, L, U, e = 1)))] - c, \]

\[ \mathbb{E}[u_m(w_m(\pi^* (s; \delta, \hat{\delta}, b, L, U, e = 0)))]\} \]. (66)

Hence, the following program, \( \mathcal{P}_2 \), will yield the optimal “quadratic” contract:

\[ \max_{(\delta, \hat{\delta}, b, L, U)} \mathbb{E}[u_i(w_i)], \]

subject to (52), (65), (66) and

\[ (\delta, \hat{\delta}, b) \in \Theta. \] (67)

Unfortunately, unlike the linear contract case, even the manager’s trading strategy under a non-trivial “quadratic” contract has to be solved numerically, due to the additional non-linearity introduced by the quadratic term. Therefore, we have to solely rely on numerically methods to solve \( \mathcal{P}_2 \).

The following two examples show that the “non-binding puzzle” can arise even under “quadratic” contracts. Hence, the mechanism that generates this puzzle is robust.

**Example 3.** Suppose

\[ (p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.5, 0.02, 5 \times 10^{-5}). \]

Hence, the values of the exogenous parameters are the same as in example 1. The optimal contract reported by MATLAB is

\[ (\delta^*, \hat{\delta}^*, b^*, L^*, U^*) \approx (0.00077195, 2.9461 \times 10^{-5}, 3.8385 \times 10^{-7}, 0.10002, 1.1170). \]
Figure 12: The manager’s expected utility as a function of $\pi$ for $s \in \{0, 1\}$, under the optimal contract $(0.00077195, 2.9461 \times 10^{-5}, 3.8385 \times 10^{-7}, 0, 1)$. The values of parameters are chosen as $(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.5, 0.02, 5 \times 10^{-5})$.

Under this contract, the manager’s trading strategy is

$$\pi^*(s; \delta^*, \hat{\delta}^*, b^*, L^*, U^*) \approx \begin{cases} 0.60580 & \text{if } s = 1, \\ 0.26792 & \text{if } s = 0. \end{cases}$$

Therefore, none of the constraints will be binding and the contract

$$(\delta^*, \hat{\delta}^*, b^*, L^*, U^*) \approx (0.00077195, 2.9461 \times 10^{-5}, 3.8385 \times 10^{-7}, 0, 1)$$  \quad (68)$$
yields the same expected utility for the investor. Also see figure 12. Comparing figure 3 with figure 12, the manager’s expected utility is lower under the “quadratic” contract. As expected, the investor expected utility under (68) $\approx 2.0106$, which is greater than his expected utility under the optimal linear contract ($\approx 2.0105$). The “plot” check for the optimality of (68) is presented in figure 13.
Figure 13: The investor’s expected utility given different choices of the “quadratic” contract. The values of parameters are chosen as \((p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.02, 5 \times 10^{-5})\). Note: in sub-figure (IV), we define the investor’s expected utility as \(u_i(1)\) if \(L > U\).

**Example 4.** Suppose

\[(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.5, 0.45, 0.02, 5 \times 10^{-5})\].

Comparing it with example 3, the manager and the investor have different risk-averse coefficients. The optimal contract reported by MATLAB is

\[(\delta^*, \hat{\delta}^*, b^*, L^*, U^*) \approx (0.00065606, 4.0402 \times 10^{-5}, 0.084739, 1.0124)\].
Figure 14: The manager’s expected utility as a function of $\pi$ for $s \in \{0, 1\}$, under the optimal contract $(0.000645606, 4.0402 \times 10^{-5}, 0, 0, 1)$. The values of parameters are chosen as $(p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.45, 0.5, 0.02, 5 \times 10^{-5})$.

Under this contract, the manager’s trading strategy is

$$\pi^*(s; \hat{\delta}^*, \hat{\delta}^*, b^*, L^*, U^*) \approx \begin{cases} 0.64311 & \text{if } s = 1, \\ 0.28612 & \text{if } s = 0. \end{cases}$$

Therefore, none of the constraints will be binding and the contract

$$(\delta^*, \hat{\delta}^*, b^*, L^*, U^*) \approx (0.00064506, 4.0402 \times 10^{-5}, 0, 0, 1)$$

(69)

yields the same expected utility for the investor. Also see figure 14. The “plot” check for the optimality of (69) is presented in figure 15.

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Figure 15: The investor’s expected utility given different choices of the “quadratic” contract. The values of parameters are chosen as \((p, R, \beta, \gamma, q, c) = (0.55, 0.5, 0.45, 0.5, 0.02, 5 \times 10^{-5})\).

Note: in sub-figure (IV), we define the investor’s expected utility as \(u_i(1)\) if \(L > U\).

# 5 Conclusion

A long time puzzle in the mutual fund industry is the predominately existence of non-binding trading restrictions. Previous literature shows that restrictions are used to discipline fund managers’ behavior, but their effectiveness relies on the possibility of being materialized in some state(s), which is difficult to explain the “non-binding” phenomenon observed in reality. However, the naive answer to this puzzle, “investors are carrying coals to Newcastle,” is far from satisfactory.

We address this puzzle, by modeling the interaction between an investor and a mutual fund manager in a standard principal-agent framework: all players have CRRA preferences;
the manager is protected by limited liability; trading strategies are not exogenously limited. In our model, position restrictions arise endogenously as part of the optimal contract(s), due to the protection of the principle of limited liability; otherwise, the manager would take too much risk by gambling the investor's money on the stock market. More importantly, these restrictions may not be binding in any possible state. Because the manager is risk-averse, there is a portfolio position maximizes her expected utility locally before the protection of limited liability kicks in. When the position limits are precisely chosen, this “local” portfolio becomes global optimum.

In particular, we examine the cases involve constraints on short-sale and leverage, which are commonly used in practice. We find that when such restrictions are imposed by the investor, and they would be not binding in any state, the manager’s ability must be below a threshold; if not, either the investor should relax the constraints, or at least one constraint would be binding with non-zero probability. A direction for future research is to test our theory by testing this sharp prediction. In addition, since this result connects the non-binding constraints to the fund manager’s talent, it suggests a novel asset-pricing-model-free measure of managerial skill and may contribute to the debate on the existence of the skill of professorial managers, which can only be answered empirically with the guidance of theory.

Finally, since our model is a partial equilibrium model, it should be interesting to embed it to a general equilibrium asset pricing model, and new insights may be obtained, as, for example, Kapur and Timmermann (2005), Gorton, He, and Huang (2010), Cuoco and Kaniel (2011), Kyle, Ou-Yang, and Wei (2011), Huang (2015), Sato (2016), Cvitanić and Xing (2018), Buffa, Vayanos, and Woolley (2019), Huang, Qiu, and Yang (2019) and Sockin and Xiaolan (2019).
References


Table 1: The portfolio of Catalyst Growth of Income Fund on March 31, 2019. Data is obtained from the NPORT-EX form submitted by Mutual Fund Series Trust to the SEC; filing date: May 28, 2019; period of report: March 31, 2019; URL: https://www.sec.gov/Archives/edgar/data/1355064/000158064218001244/answer.fil; retrieved from EDGAR, September 17, 2019.

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Table 2: Short-selling and leverages constraints on the mutual funds of the *American Funds* (AF). Data is obtained from the most recent N-SAR forms they submitted to the SEC; retrieved from EDGAR, September 16-18, 2019. Note: the list of mutual funds here is obtained from [https://www.capitalgroup.com/individual/products/mutual-funds.html](https://www.capitalgroup.com/individual/products/mutual-funds.html) (retrieved September 16, 2019), and according to this webpage, it does not include the mutual funds in “American Funds Portfolio SeriesSM, American Funds Retirement Income Portfolio SeriesSM and American Funds College Target Date Series®.”

<table>
<thead>
<tr>
<th>Fund Name</th>
<th>Short-sale allowed</th>
<th>Leverage allowed</th>
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<tbody>
<tr>
<td>AF Corporate Bond Fund</td>
<td>Y</td>
<td>Y</td>
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<tr>
<td>AF Developing World Growth &amp; Income Fund</td>
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<tr>
<td>AF Emerging Markets Bond Fund</td>
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<tr>
<td>AF Global Balanced Fund</td>
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<tr>
<td>AF Inflation Linked Bond Fund</td>
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<tr>
<td>AF Mortgage Fund</td>
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<tr>
<td>AF Short-Term Tax-Exempt Bond Fund</td>
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<tr>
<td>AF Strategic Bond Fund</td>
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<tr>
<td>AF Tax-Exempt Fund of New York</td>
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<td>AF U.S. Govt Money Market Fund</td>
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<td>Y</td>
</tr>
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<td>AMCAP Fund</td>
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<td>American Balanced Fund</td>
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<td>American High-Income Municipal Bond Fund</td>
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<td>EuroPacific Growth Fund</td>
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<td>The New Economy Fund</td>
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<td>The Tax-Exempt Fund of California</td>
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<tr>
<td>U.S. Government Securities Fund</td>
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<td>Y</td>
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<tr>
<td>Washington Mutual Investors Fund</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>
Appendix A  Short Proofs

Proof of Lemma 1.  Note that

\[
g(\pi; e, s, \delta, b) = \begin{cases} 
(1 - \bar{p}(s, e)) \frac{(b + \delta(b - 1 - \pi R))^{1 - \beta}}{1 - \beta} & \text{if } \pi < \pi_1(\delta, b), \\
\bar{p}(s, e) \frac{(b + \delta(b - 1 - \pi R))^{1 - \beta}}{1 - \beta} + (1 - \bar{p}(s, e)) \frac{(b + \delta(b - 1 - \pi R))^{1 - \beta}}{1 - \beta} & \text{if } \pi_1(\delta, b) \leq \pi \leq \pi_2(\delta, b), \\
\bar{p}(s, e) \frac{(b + \delta(b - 1 - \pi R))^{1 - \beta}}{1 - \beta} & \text{if } \pi > \pi_2(\delta, b). 
\end{cases}
\]

(I) and (II) hold obviously. If \( \pi_1(\delta, b) \leq \pi \leq \pi_2(\delta, b) \),

\[
g'(\pi; e, s, \delta, b) = \delta R \bar{p}(s, e) (b + \delta(1 + \pi R))^{-\beta} - \delta R (1 - \bar{p}(s, e)) (b + \delta(1 - \pi R))^{-\beta}. 
\]

In this interval, it is easy to verify that \( g'(\pi; e, s, \delta, b) < (>) 0 \) if and only if \( \pi > (<) \pi_c(e, s, \delta, b) \).
Since \( K(s, e) \in (0, \infty) \) and \( \Delta(s, e) = \frac{K(s, e) - 1}{K(s, e) + 1} \in (-1, 1) \), we must have \( \pi_c(e, s, \delta, b) \) is in between \( \pi_1(\delta, b) \) and \( \pi_2(\delta, b) \). \( \square \)

Proof of Proposition 2.  Since \( u_t(w_t) = -\infty \) for \( w_t < 0 \), in equilibrium we have

\[
1 + \pi^* R - w^*_m \geq 0, \forall \tilde{R} \in \{-R, R\}. 
\]

Hence,

\[
1 + \pi^* R \geq b^* + \delta^*(1 + \pi^* R), \quad (70)
\]

\[
1 - \pi^* R \geq b^* + \delta^*(1 - \pi^* R), \quad (71)
\]

\[
1 + \pi^* R \geq 0, \quad (72)
\]

\[
1 - \pi^* R \geq 0. \quad (73)
\]

(70) + (71) implies (22).  Combining (22) with the assumption that \( b^* \geq 0 \) and \( \delta^* \geq 0 \), we have \( b^* \in [0, 1] \) and \( \delta^* \in [0, 1] \).  If \( b^* = 1 \), then \( \delta^* = 0 \).  However, any contract \((0, 1, L^*, U^*)\) is strictly dominated by \((0, 0, L^*, U^*)\).  Hence, \( b^* < 1 \).  If \( \delta^* = 1 \), then \( b^* = 0 \), and any such contract is strictly dominated by \((0, 0, 0, 0)\).  Hence, \( \delta^* < 1 \).

Note that (72) and (73) imply that under any circumstances, the weight on the risky asset should be between \( -1 \tilde{R} \) and \( \frac{1}{\tilde{R}} \).  Since \( \tilde{C} \) will be binding, we must have (25).

Finally, we show (26) by contradiction.  When \( \delta^* > 0 \), if \( L^* = -\infty \) and/or \( U^* = +\infty \), by lemma 1, the manager’s portfolio choice \((-\infty \text{ or } +\infty)\) will be out of the range \([ -\frac{1}{\tilde{R}}, \frac{1}{\tilde{R}} ] \). \( \square \)

Proof of Proposition 4.  By proposition 3, \( \delta^* > 0 \) implies \( e^* = 1 \).  By lemma 1 and the continu-
ity of \( g(\cdot) \), equation (31) has 3 real roots. Hence, \( R(\cdot), \hat{L}(\cdot) \) and \( \hat{U}(\cdot) \) are well-defined. Now we fix \( s \). By construction, for any \( \pi \in (\hat{L}(e = 1, s, \delta^*, b^*), \hat{U}(e = 1, s, \delta^*, b*)) \), we have

\[
g(\pi; e = 1, s, \delta^*, b^*) \leq g(\pi_c(e = 1, s, \delta^*, b^*); e = 1, s, \delta^*, b^*).
\]

The equality holds if and only if \( \pi = \pi_c(e = 1, s, \delta^*, b^*) \). Hence, if \( L^* \in (\hat{L}(e = 1, s, \delta^*, b*), \pi_c(e = 1, s, \delta^*, b^*)) \) and \( U^* \in (\pi_c(e = 1, s, \delta^*, b^*), \hat{U}(e = 1, s, \delta^*, b*)) \), the manager will choose \( \pi_c(e = 1, s, \delta^*, b^*) \in (L^*, U^*) \) after receiving signal \( s \). Therefore, if

\[
L^* \in (\hat{L}(e = 1, s = 0, \delta^*, b*), \pi_c(e = 1, s = 0, \delta^*, b^*)) \cap (\hat{L}(e = 1, s = 1, \delta^*, b*), \pi_c(e = 1, s = 1, \delta^*, b^*)),
\]

\[
U^* \in (\pi_c(e = 1, s = 0, \delta^*, b^*), \hat{U}(e = 1, s = 0, \delta^*, b*)) \cap (\pi_c(e = 1, s = 1, \delta^*, b^*), \hat{U}(e = 1, s = 1, \delta^*, b*)),
\]

then \( L^* \) and \( U^* \) will never be binding. Also note that \( \tilde{p}(s = 1, e = 1) > \tilde{p}(s = 0, e = 1) \), we have

\[
K(s = 1, e = 1) > K(s = 0, e = 1).
\]

Since the function \( f(\lambda) := \frac{1 - \lambda}{1 + \lambda} \in (-1, 1) \) is strictly increasing for \( \lambda \in (0, \infty) \), we have \( \pi_c(e = 1, s = 0, \delta^*, b^*) < \pi_c(e = 1, s = 1, \delta^*, b^*) \). Hence,

\[
\begin{align*}
(\hat{L}(e = 1, s = 0, \delta^*, b*), \pi_c(e = 1, s = 0, \delta^*, b^*)) \cap (\hat{L}(e = 1, s = 1, \delta^*, b*), \pi_c(e = 1, s = 1, \delta^*, b^*)) &= G(\delta^*, b^*), \\
(\pi_c(e = 1, s = 0, \delta^*, b*), (e = 1, s = 0, \delta^*, b^*)) \cap (\pi_c(e = 1, s = 1, \delta^*, b*), \hat{U}(e = 1, s = 1, \delta^*, b*)) &= H(\delta^*, b^*).
\end{align*}
\]

Since \( L^* \) and \( U^* \) are slack, any contract \( (\delta^*, b^*, L^{**}, U^{**}) \) which satisfies the “slackness” condition would not change the manager’s behavior and the investor’s expected utility. Then by the optimality of the original contract, \( (\delta^*, b^*, L^{**}, U^{**}) \) is also optimal. \( \square \)

**Proof of Proposition 6.** Note that if we restrict \( \delta = 0 \), the best contract for the investor is \( (0, 0, \pi_0, U_0) \). And under this contract the investor’s expected utility is \( E[u_t(1 + \pi_0 \hat{R})] \).

Now consider a contract \( (\delta > 0, 0, -1/R, 1/R) \). We first pin down the manager’s portfolio choice under this contract. By lemma 1, for any contract \( (\delta > 0, 0, -1/R, 1/R) \), if the manager does not exert effort \( (e = 0) \), she will choose \( \pi_0 \) regardless of the signal; if the manager works hard \( (e = 1) \), she will choose

\[
\pi_c(s) = \frac{1}{R} \Delta(s, e), \tag{74}
\]

which is independent of \( \delta \). Also note that this portfolio is the one the investor will choose as if he receives \( s \) and does the investment by himself, since \( \beta = \gamma \).
Then we consider the manager's IC constraint. She works hard if and only if

$$\delta^{1-\beta}(E[u_m(1 + \pi_c(e = 1, s, \delta, 0)\tilde{R})|s|e = 1] - E[u_m(1 + \pi_0\tilde{R})|e = 0]) \geq c. \quad (75)$$

Since $\beta = \gamma$, $E[u_m(1 + \pi_c(e = 1, s, \delta, 0)\tilde{R})|s|e = 1]$ can be interpreted as the expected utility of the investor as if he does the investment optimally with the information advantage and $E[u_m(1 + \pi_0\tilde{R})|e = 0]$ is his maximal expected utility without the additional information. Since $q > 0$,

$$E[u_m(1 + \pi_c(e = 1, s, \delta, 0)\tilde{R})|s|e = 1] - E[u_m(1 + \pi_0\tilde{R})|e = 0]) > 0.$$

Now consider the investor’s expected utility under this contract and the manager works hard, which is $(1 - \delta)^{1-\beta}E[u_m(1 + \pi_c(e = 1, s, \delta, 0)\tilde{R})|s|e = 1]$. Hence, this contract improves his expected utility if we can find a $\delta > 0$ such that (75) and

$$(1 - \delta)^{1-\beta}E[u_m(1 + \pi_c(e = 1, s, \delta, 0)\tilde{R})|s|e = 1] - E[u_m(1 + \pi_0\tilde{R})|e = 0]) > 0 \quad (76)$$

hold. This can be done since $c$ satisfies (32).

Hence, we have proved any contract with $\delta = 0$ is not optimal, which implies any optimal contract must have a positive $\delta$ if it exists. \qed

Proof of Proposition 8. The key equations we use in the proof are (15), (16), (33), (34) and (35).

(I) This is obviously and omitted.

(II) Since $\chi_0$ is independent of $q$ and $\chi_1$ is strictly increasing in $q$, we have $\frac{d\delta^*(q)}{dq} < 0$. Then note that $K(1, 1)$ ($K(0, 1)$) is strictly increasing (decreasing) in $q$, and the monotonicity of $f(\lambda) = \frac{1}{\lambda + 1}$ for $\lambda > 0$, hence the rest of (ii) has been proved.

(III) Note that $K(0, 1) \leq 1$ if and only if $p \leq q + 1/2$. Hence,

$$\frac{\partial K(0, 1)}{\partial \theta} \leq 0$$

if and only if $p \leq q + 1/2$. Then the monotonicity of $f(\lambda)$ implies the sign of $\frac{\partial \tilde{L}(\lambda)}{\partial \theta}$. By similar arguments, $\frac{\partial \tilde{U}(\lambda)}{\partial \theta} < 0$. 

53
(IV) By (33), (34) and (35), $\delta^*(c)$ does not depend on $R$. The signs of $\frac{\partial \tilde{L}(\cdot)}{\partial R}$ and $\frac{\partial \tilde{U}(\cdot)}{\partial \theta}$ are obtained, once we notice that $\Delta(s,e)$ is independent of $R$; $\Delta(1,1) > 0$; and $\Delta(0,1) \gtrless 0$ if and only if $q \gtrless p - 1/2$.

(V) It directly follows from the fact that $\frac{\partial \Delta(s,e)}{\partial p} > 0$. □

Proof of Proposition 9. Since $R \in (0,1)$, $\eta > 1/2$. Hence, (48) and (49) are well-defined. We first prove the optimality of $(\delta^*(c),0,0,1)$. By proposition 7, it boils down to check

\begin{align*}
0 &\in \tilde{G}(\delta^*(c)), \quad (77) \\
1 &\in \tilde{H}(\delta^*(c)). \quad (78)
\end{align*}

(49) implies $q < p - 1/2$, and then by (3),

$$\Delta(0,1)/R > 0, \quad (79)$$

Combing (79) with the fact that $0 > -1/R$, we conclude (77) holds.

Similarly, (2), (48) and (49) imply

$$\Delta(1,1)/R < 1. \quad (80)$$

Combing (80) with the fact that $1 < 1/R$, we conclude (78) holds. Hence, $(\delta^*(c),0,0,1)$ is optimal.

Under this contract, the trading strategy is $\Delta(s,1)/R$. By (79) and (80), the constraints will never be binding. □

Proof of Proposition 10. By proposition 3, $e^* = 1$. Since the no-short-selling and no-leverage constraints are not binding in any state, we have

\begin{align*}
\Delta(0,1) \frac{\delta^* + b^*}{\delta^* R} &> 0, \quad (81) \\
\Delta(1,1) \frac{\delta^* + b^*}{\delta^* R} &< 1. \quad (82)
\end{align*}

(81) implies

$$q < p - 1/2. \quad (83)$$
By (22) and (24),
\[
\Delta(1,1) \frac{\delta^* + b^*}{\delta^* R} \leq \Delta(1,1) \frac{1}{\delta^* R} < \Delta(1,1) \frac{1}{R}.
\]
Hence, (82) implies
\[
p < \eta, \quad (84)
\]
\[
q < \frac{\eta - p}{2(p + \eta - 2\eta p)}. \quad (85)
\]
Combing (83), (84) and (85) with our setting–1/2 < p < 1 and 0 < q < 1/2, we obtain (48) and (49). \(\square\)

**Proof of Proposition 11.** Using the similar argument in the proof of proposition 2, we have (72), (73),
\[
1 + \pi^* R \geq b^* + \delta^* (1 + \pi^* R) + \hat{\delta}^* (1 + \pi^* R)^2, \quad (86)
\]
and
\[
1 - \pi^* R \geq b^* + \delta^* (1 - \pi^* R) + \hat{\delta}^* (1 - \pi^* R)^2. \quad (87)
\]
Again, (72) and (73) imply
\[
-1/R \leq \pi^* \leq 1/R. \quad (88)
\]
(86) + (87) implies
\[
1 \geq b^* + \delta^* + [1 + (\pi^*)^2 R^2/2] \hat{\delta}^*. \quad (89)
\]
By (88),
\[
0 \leq (\pi^*)^2 R^2 \leq 1.
\]
Hence,
\[
[1 + (\pi^*)^2 R^2/2] \hat{\delta}^* \geq \delta^* \text{ if } \delta^* > 0. \quad (90)
\]
Combining (89) with (90), we obtain (63).
given by \( \pi \) proposition 3, \( e \in E \)

Then for any contract \((\delta^*, b^*, L^*, U^*)\) is optimal and \( b^* > 0 \). By proposition 3, \( e^* = 1 \). Hence, the manager’s IC constraint is satisfied:

\[
\mathbb{E}[\mathbb{E}[u_m(\delta^*(1 + \pi^*(s; \delta^*, b^*, L^*, U^*, e = 1)\tilde{R}) + b^*)|s]|e = 1] \\
- \mathbb{E}[u_m(\delta^*(1 + \pi^*(s; \delta^*, b^*, L^*, U^*, e = 0)\tilde{R}) + b^*)|e = 0] \geq c.
\]

Since constraints \( L^* \) and \( U^* \) will not be strictly binding, the manager’s trading strategy is given by \( \pi_c(e,s,\delta^*,b^*) \) and then

\[
\mathbb{E}[\mathbb{E}[u_m(\delta^*(1 + \pi^*(s; \delta^*, b^*, L^*, U^*, e = 1)\tilde{R}) + b^*)|s]|e = 1] \\
- \mathbb{E}[u_m(\delta^*(1 + \pi^*(s; \delta^*, b^*, L^*, U^*, e = 0)\tilde{R}) + b^*)|e = 0]) \\
= (\delta^* + b^*)^{1-\beta}(\mathbb{E}[\mathbb{E}[u_m(1 + \pi^*(s; 1,0, L^*, U^*, e = 1)\tilde{R})|s]|e = 1] - \mathbb{E}[u_m(1 + \pi^*(s; 1,0, L^*, U^*, e = 0)\tilde{R})|e = 0]).
\]

Note that

\[
\mathbb{E}[\mathbb{E}[u_m(1 + \pi^*(s; 1,0, L^*, U^*, e = 1)\tilde{R})|s]|e = 1] - \mathbb{E}[u_m(1 + \pi^*(s; 1,0, L^*, U^*, e = 0)\tilde{R})|e = 0]
\]

is independent of \( \delta^*, b^*, L^* \) and \( U^* \). Let

\[
\phi_0 := \delta^* + b^* \in (0,1).
\]

Then for any contract \((\delta, b, L^* - \tau, U^* + \tau)\) in which \((\delta, b)\) is in a small neighborhood of \((\delta^*, b^*)\), \( \delta + b = \phi_0 \) and \( \tau \geq 0 \) is chosen such that the constraints will still not be strictly binding, the manager’s IC constraint holds. Under this contract, the investor’s expected utility can be written as a function of \( 1/\delta \), that is,

\[
h(1/\delta) := \sum_{s \in \{0,1\}} p_s \left\{ \tilde{p}(s,1) \frac{[1 - \phi_0 + \phi_0 \Delta(s,1)(1/\delta - 1)]^{1-\gamma}}{1-\gamma} + (1 - \tilde{p}(s,1)) \frac{[1 - \phi_0 + \phi_0 \Delta(s,1)(1 - 1/\delta)]^{1-\gamma}}{1-\gamma} \right\}.
\]
Taking the derivative of \( h(\cdot) \) with respect to \( 1/\delta \), we have

\[
h'(1/\delta) = \phi_0 \sum_{s \in \{0,1\}} p_s \Lambda(1/\delta; s),
\]

where

\[
\Lambda(1/\delta; s) := \Delta(s, 1) \left\{ \tilde{p}(s, 1) \left[ 1 - \phi_0 + \phi_0 \Delta(s, 1) \left( \frac{1}{\delta} - 1 \right) \right]^{-\gamma} - (1 - \tilde{p}(s, 1)) \left[ 1 - \phi_0 + \phi_0 \Delta(s, 1) \left( 1 - \frac{1}{\delta} \right) \right]^{-\gamma} \right\}.
\]

Note that \( \Delta(s, 1) \) can be zero for some \( s \in \{0,1\} \), but it is not possible that \( \Delta(s, 1) = 0 \) for every \( s \).

Now consider any \( s \) such that \( \Delta(s, 1) \neq 0 \), then the solution to

\[
\Lambda(1/\delta; s) = 0
\]

is

\[
\delta^{**} = \frac{[1 + \tilde{K}(s, 1)][K(s, 1) - 1]\phi_0}{\tilde{K}(s, 1)[K(s, 1) + 1 - 2\phi_0] + K(s, 1)(2\phi_0 - 1) - 1},
\]

where

\[
\tilde{K}(s, e) := \left[ \frac{\tilde{p}(s, e)}{1 - \tilde{p}(s, e)} \right]^{1/\gamma}.
\]

Since \( \gamma \geq \beta \),

\[
\delta^{**} \geq \phi_0.
\]

The equality holds if and only if \( \gamma = \beta \). Taking the derivative of \( m(\cdot; s) \) with respect to \( 1/\delta \), we have

\[
\Lambda'(1/\delta; s) = -\gamma \phi_0 \Delta^2(s, 1) \left\{ \tilde{p}(s, 1) \left[ 1 - \phi_0 + \phi_0 \Delta(s, 1) \left( \frac{1}{\delta} - 1 \right) \right]^{-(\gamma + 1)} + (1 - \tilde{p}(s, 1)) \left[ 1 - \phi_0 + \phi_0 \Delta(s, 1) \left( 1 - \frac{1}{\delta} \right) \right]^{-(\gamma + 1)} \right\} < 0.
\]

Since \( b^* > 0 \), \( 1/\delta^* > 1/\phi_0 \) and \( h'(1/\delta^*) = \phi_0 \sum_{s \in \{0,1\}} p_s \Lambda(1/\delta^*) < h'(1/\phi_0) \leq 0 \). Hence, this contract is strictly dominated by \((\delta^* + \epsilon, b^* - \epsilon, L^*, U^*)\) where \( \epsilon > 0 \) and is small enough. This contradicts the optimality of \((\delta^*, b^*, L^*, U^*)\). Hence, choosing \( b^* > 0 \) is not optimal if the
Claim 2. If at least one of the constraints will be strictly binding, then \( b^* = 0 \).

Proof. Again, we show it by contradiction. Keep in mind that

\[
\pi_c(e=1, s=0, \delta^*, b^*) < \pi_c(e=0, s=1, \delta^*, b^*) = \pi_c(e=0, s=1, \delta^*, b^*) < \pi_c(e=1, s=1, \delta^*, b^*). 
\]

Therefore, we have the following cases to consider.

1. \( L^* < U^* \leq \pi_c(e=1, s=0, \delta^*, b^*) \). In this case, the manager will always choose \( \pi^* = U^* \), regardless of her private information. However, this contract is strictly dominated by \((0, 0, U^*, U^*)\).

2. \( L^* \leq \pi_c(e=1, s=0, \delta^*, b^*) < U^* < \pi_c(e=0, s=1, \delta^*, b^*) = \pi_c(e=0, s=1, \delta^*, b^*) \). Under this contract, the manager's IC constraint is

\[
\begin{align*}
\Psi(\delta^*, b^*, U^*) := & \ p_0(\delta^* + b^*)^{1-\beta} \left\{ \tilde{\rho}(0,1) \frac{[1 + \Delta(0,1)]^{1-\beta}}{1-\beta} + (1 - \tilde{\rho}(0,1)) \frac{[1 - \Delta(0,1)]^{1-\beta}}{1-\beta} \right\} \\
+ & \ p_1(1,1) \frac{[(\delta^* + b^*) + \delta^* U^* R]^{1-\beta}}{1-\beta} + (1 - \tilde{\rho}(1,1)) \frac{[(\delta^* + b^*) - \delta^* U^* R]^{1-\beta}}{1-\beta} \\
- & \ p \frac{[(\delta^* + b^*) + \delta^* U^* R]^{1-\beta}}{1-\beta} + (1 - p) \frac{[(\delta^* + b^*) - \delta^* U^* R]^{1-\beta}}{1-\beta} \geq c.
\end{align*}
\]

Since

\[
\frac{\partial \Psi}{\partial U^*} = -p(1/2 - q)\delta^* R [(\delta^* + b^*) + \delta^* U^* R]^{-\beta} + (1 - p)(1/2 + q)\delta^* R [(\delta^* + b^*) - \delta^* U^* R]^{-\beta} < 0
\]

if and only if

\[ U^* \prec \pi_c(e=1, s=0, \delta^*, b^*), \]

the manager's IC constraint would still hold if the investor increases \( U^* \) a little bit.
Note that the investor’s expected utility is
\[
E[\mu(w_i)|\delta^*, b^*, L^*, U^*] = p_0\left\{\bar{p}(0,1)\left[\frac{1 - (\delta^* + b^*) + (\delta^* + b^*)\Delta(0,1)(1 - \frac{1}{\delta^*})}{1 - \gamma}\right]
\right.
\]
\[
+ (1 - \bar{p}(0,1))\left[\frac{1 - (\delta^* + b^*) + (\delta^* + b^*)\Delta(0,1)(1 - \frac{1}{\delta^*})}{1 - \gamma}\right]^{1 - \gamma}
\right\}
\]
\[
+ p_1\left\{\bar{p}(1,1)\left[\frac{1 - (\delta^* + b^*) + (1 - \delta^*)U^* R}{1 - \gamma}\right]^{1 - \gamma}
\right.
\]
\[
+ (1 - \bar{p}(1,1))\left[\frac{1 - (\delta^* + b^*) - (1 - \delta^*)U^* R}{1 - \gamma}\right]^{1 - \gamma}
\right\}.
\]

Hence,
\[
\frac{\partial E[\mu(w_i)|\delta^*, b^*, L^*, U^*]}{\partial U^*} \leq 0,
\]

which implies
\[
U^* \geq \frac{\hat{\Delta}(1,1) 1 - \delta^* - b^*}{R} > 0
\]

where
\[
\hat{\Delta}(s,e) := \frac{\hat{K}(s,e) - 1}{\hat{K}(s,e) + 1}.
\]

Otherwise, the investor can do better by increasing \( U^* \) a little bit.

(a) If \( \Delta(0,1) = 0 \), then \( p = 1/2 + q \) and

\[
\frac{\partial \Psi}{\partial b^*} = p_0(\delta^* + b^*)^{-\beta} - \left\{p(1/2 - q)[(\delta^* + b^*) + \delta^* U^* R]^{-\beta}
\right.
\]
\[
+ (1 - p)(1/2 + q)(\delta^* + b^*) - \delta^* U^* R]^{-\beta}\}
\]
\[
= p(1/2 - q)(\delta^* + b^*)^{-\beta} + (1 - p)(1/2 + q)(\delta^* + b^*)^{-\beta}
\]
\[
- \left\{p(1/2 - q)[(\delta^* + b^*) + \delta^* U^* R]^{-\beta} + (1 - p)(1/2 + q)[(\delta^* + b^*) - \delta^* U^* R]^{-\beta}\}
\right. < 0.
\]

The last inequality holds because (I) \( p = 1/2 + q \) implies \( p(1/2 - q) * [(\delta^* + b^*) + \delta^* U^* R] + (1 - p)(1/2 + q)[(\delta^* + b^*) - \delta^* U^* R] = p(1/2 - q)(\delta^* + b^*)^{-\beta} + (1 - p)(1/2 + q)(\delta^* + b^*)^{-\beta} \), (II) \( \beta \in (0,1) \) implies that function \( n(\lambda) := \lambda^{-\beta} \) is strictly convex for \( \lambda > 0 \); (iii) then the last inequality follows from Jensen’s inequality.

Hence, the manager’s IC constraint would still hold if the investor decreases \( b^* \) a
little bit. It is also easy to check that
\[ \frac{\partial \mathbb{E}[u_i(w_i)|\delta^*, b^*, L^*, U^*]}{\partial b^*} < 0. \]
Therefore, \((\delta^*, b^*, L^*, U^*)\) is dominated by \((\delta^*, b^* - \epsilon, L^*, U^*)\) for \(\epsilon > 0\) and small enough.

(b) If \(\Delta(0, 1) \neq 0\), then we consider the contract \((\delta, b, L^* - \tau, U^*)\), where \((\delta, b)\) is in a small neighborhood of \((\delta^*, b^*)\) and \(\delta + b = \phi_0\), which is defined by \((91)\); \(\tau \geq 0\) is chosen such that the manager will always choose \(\pi_c(e = 1, s = 0, \delta, b)\) instead of \(L^* - \tau\). Then
\[ \frac{\partial \Psi(\delta, b, U^*)}{\partial \delta} \bigg|_{(\delta, b) = (\delta^*, b^*)} > 0, \]
since \(U^* > \pi_c(e = 1, s = 0, \delta^*, b^*)\). Hence, \((\delta^* + \epsilon, b^* - \epsilon, L^* - \tau, U^*)\) would not break the manager’s IC constraint for \(\epsilon > 0\) and small enough.

Then, it can be verified that
\[ \frac{\partial \mathbb{E}[u_i(w_i)|\delta, b, L^* - \tau, U^*]}{\partial \delta} \bigg|_{(\delta, b) = (\delta^*, b^*)} > 0. \]
Therefore, \((\delta^*, b^*, L^*, U^*)\) is strictly dominated by \((\delta^* + \epsilon, b^* - \epsilon, L^* - \tau, U^*)\).

3. All other cases can be handled by similar arguments introduced above. \(\square\)

**Appendix C    Proof of Proposition 7**

We prove it in three main steps. Keep in mind that we always assume \(\beta = \gamma\) and \((32)\) holds in this section.

**C.1 The Optimality of \((\delta^*(c), 0, -1/R, 1/R)\) and the “If” Part**

**Claim 3.** Let \((\delta^*, b^*, L^*, U^*)\) be any contract that satisfies \((36), (37), (38)\) and \((39)\). Under this contract, the manager’s effort level is \(e^* = 1\) and her trading strategy is given by \((74)\), which is independent of \(L^*\) and \(U^*\). Moreover, the investor’s expected utility is \((1 - \delta^*(c))^{1-\gamma} \chi_1\), which is also independent of \(L^*\) and \(U^*\).
Proof. Using arguments which are almost identical to those in the proof of proposition 6, we can show that \( e^* = 1 \) and the portfolio choice is (74). Then we plug these results into the objective function of the investor and get

\[
\mathbb{E}[u_i(w_i)|\delta^*, b^*, L^*, U^*] = (1 - \delta^*(c))^{1-\gamma} \chi_1 .
\]

Claim 4. There is no contract with the form \((\delta^*, 0, L^*, U^*)\) better than \((\delta^*(c), 0, -1/R, 1/R)\). Moreover, if it does not satisfy at least one of the conditions –\(\{(36), (38), (39)\}\), then it is strictly dominated by \((\delta^*(c), 0, -1/R, 1/R)\).

Proof. Suppose not, say we can find a better contract \((\delta^*, 0, L^*, U^*)\). Then we know \(\delta^* > 0\) and \(e^* = 1\) under this contract: if \(\delta^* = 0\), this contract is no better than \((0, 0, \pi_0, U_0)\), which is strictly dominated by \((\delta^*(c), 0, -1/R, 1/R)\); if \(\delta^* > 0\) but \(e^* = 0\), this contract is strictly dominated by \((0, 0, \pi_0, U_0)\). Hence, to find the best contract with the form \((\delta^*, 0, L^*, U^*)\), we only need to consider the following program:

\[
\max_{(\delta, 0, L, U)} (1 - \delta)^{1-\gamma} \mathbb{E}[u_i(1 + \pi^*(s; \delta, 0, L, U, e = 1) \tilde{R})|s]|e = 1],
\]

such that

\[
\delta^1 - \beta [\mathbb{E}[u_m(1 + \pi^*(s; \delta, 0, L, U, e = 1) \tilde{R})|s]|e = 1] - \mathbb{E}[u_m(1 + \pi^*(s; \delta, 0, L, U, e = 0) \tilde{R})|e = 0] \geq c.
\]

Note that if \(\pi_0 \in [L, U]\), then \(\pi^*(s; \delta, 0, L, U, e = 0) = \pi_0\) and \(\mathbb{E}[u_m(1 + \pi^*(s; \delta, 0, L, U, e = 0) \tilde{R})|e = 0] = \mathbb{E}[u_m(1 + \pi^*(s; \delta, 0, L, U, e = 1) \tilde{R})|s]|e = 1] \)

is maximized. This means (38) and (39) need to be satisfied. Under this situation, the optimal \(\delta\) is \(\delta^*(c)\) and this contract is only as good as \((\delta^*(c), 0, -1/R, 1/R)\). But then (36) is also satisfied. Hence, if \((\delta^*, 0, L^*, U^*)\) is better, it must be the case that \(\pi_0 \notin [L^*, U^*]\).

Since \(\pi_c(0) < \pi_0 < \pi_c(1)\) (see (74)), \(\pi_0 \notin [L^*, U^*]\) implies that at least one boundary is strictly binding in the sense that either \(\pi_c(1) > U^*\) or \(\pi_c(0) < L^*\) or both. Therefore, such
contract violates (38) or (39). And under this contract

\[
\mathbb{E}[\mathbb{E}[u_i(1 + \pi^*(s; \delta^*, 0, L^*, U^*, e = 1) \tilde{R})|s]|e = 1] < \mathbb{E}[\mathbb{E}[u_i(1 + \pi_c(s) \tilde{R})|s]|e = 1],
\]

(92)

\[
\mathbb{E}[\mathbb{E}[u_m(1 + \pi^*(s; \delta^*, 0, L^*, U^*, e = 1) \tilde{R})|s]|e = 1] - \mathbb{E}[\mathbb{E}[u_m(1 + \pi^*(s; \delta^*, 0, L^*, U^*, e = 0) \tilde{R})|e = 0]
\]

\[
< \mathbb{E}[\mathbb{E}[u_m(1 + \pi_c(s) \tilde{R})|s]|e = 1] - \mathbb{E}[u_m(1 + \pi_0 \tilde{R})|e = 0].
\]

(93)

The intuition behind (92) and (93) is that any strictly binding constraint will break the optimality of the portfolio and lower the value of information.

Since \((\delta^*, 0, L^*, U^*)\) is a (weakly) better contract, (92) implies \(\delta^* < \delta^*(c)\). However, (93) suggests that to make the manager’s IC constraint still hold, \(\delta^* > \delta^*(c)\). A contradiction.

\[\Box\]

**Claim 5.** Among contracts with zero base salary, a contract \((\delta^*, 0, L^*, U^*)\) maximizes the investor’s expected utility if and only if it satisfies (36), (38) and (39).

**Proof.** Note that \((\delta^*(c), 0, -1/R, 1/R)\) satisfies (36), (37), (38) and (39). Then by claims 3 and 4, a contract \((\delta^*, 0, L^*, U^*)\) maximizes the investor expected utility if and only if it satisfies (36), (38) and (39). \[\Box\]

**Claim 6.** There is no contract with the form \((\delta^*, b^* > 0, L^*, U^*)\) strictly better than \((\delta^*(c), 0, -1/R, 1/R)\).

**Proof.** Suppose not, i.e., there is a contract \((\delta^*, b^* > 0, L^*, U^*)\) strictly better than \((\delta^*(c), 0, -1/R, 1/R)\). Using the same argument in the beginning of the proof of claim 4, any contract that can not induce \(e^* = 1\) is strictly dominated by \((\delta^*(c), 0, -1/R, 1/R)\). Hence, \(\delta^* > 0\) and \(e^* = 1\). Therefore, the manager’s IC constraint is satisfied:

\[
\mathbb{E}[\mathbb{E}[u_m(\delta^*(1 + \pi^*(s; \delta^*, b^*, L^*, U^*, e = 1) \tilde{R}) + b^*)|s]|e = 1]
\]

\[
-\mathbb{E}[u_m(\delta^*(1 + \pi^*(s; \delta^*, b^*, L^*, U^*, e = 0) \tilde{R}) + b^*)|e = 0] \geq c.
\]

(94)

Note that we drop the "max\{\, 0\}" operator because \(b^* > 0\) implies that any contract that may cause the interference of limited liability is strictly dominated by \((0, 0, \pi_0, U_0)\). Now we derive
an upper bound of the left-hand-side (LHS) of (94):

\[
(\delta^* + b^*)^{1-\beta}(\chi_1 - \chi_0) = \sup_{L \leq U} \mathbb{E}\left[\mathbb{E}[u_m(\delta^*(1 + \pi^*(s;\delta^*,b^*,L,U,e=1)\tilde{R}) + b^*)|s]| e=1\right]
\]

\[
-\mathbb{E}[u_m(\delta^*(1 + \pi^*(s;\delta^*,b^*,L,U,e=0)\tilde{R}) + b^*)|e=0])
\]

\[
\geq \mathbb{E}[\mathbb{E}[u_m(\delta^*(1 + \pi^*(s;\delta^*,b^*,L^*,U^*,e=1)\tilde{R}) + b^*)|s]| e=1]
\]

\[
-\mathbb{E}[u_m(\delta^*(1 + \pi^*(s;\delta^*,b^*,L^*,U^*,e=0)\tilde{R}) + b^*)|e=0].
\]

Therefore, (94) implies

\[
\delta^* + b^* \geq \left(\frac{c}{\chi_1 - \chi_0}\right)^{1/(1-\beta)} = \delta^*(c). \tag{95}
\]

Since \((\delta^*, b^* > 0, L^*, U^*)\) is a strictly better contract,

\[
\mathbb{E}[\mathbb{E}[u_i((1 - \delta^*)(1 + \pi^*(s;\delta^*,b^*,L^*,U^*,e=1)\tilde{R}) - b^*)|s]| e=1] > 
\]

\[
\mathbb{E}[\mathbb{E}[u_i((1 - \delta^*)(1 + \pi^*(s;\delta^*(c),0,-1/R,1/R,e=1)\tilde{R}) - b^*)|s]| e=1] = (1 - \delta^*(c))^{1-\gamma}\chi_1. \tag{96}
\]

Then we derive an upper bound of the LHS of (96) by assuming the investor can invest by himself with the valuable signal under the sharing rule \((\delta^*, b^*)\):

\[
(1 - \delta^* - b^*)^{1-\gamma}\chi_1 = \sup_{[\pi(s)] \in [0,1]} \mathbb{E}[\mathbb{E}[u_i((1 - \delta^*)(1 + \pi(s)\tilde{R}) - b^*)|s]| e=1] \geq 
\]

\[
\mathbb{E}[\mathbb{E}[u_i((1 - \delta^*)(1 + \pi^*(s;\delta^*,b^*,L^*,U^*,e=1)\tilde{R}) - b^*)|s]| e=1].
\]

Therefore, (96) implies

\[
\delta^*(c) > \delta^* + b^*. \tag{97}
\]

(95) and (97) cannot hold simultaneously. A contradiction.

\[\square\]

**Claim 7.** \((\delta^*(c), 0, -1/R, 1/R)\) is optimal and if a contract \((\delta^*, b^*, L^*, U^*)\) satisfies (36), (37), (38) and (39), then it is optimal.

**Proof.** The optimality of \((\delta^*(c), 0, -1/R, 1/R)\) follows from claims 4 and 6. The second part follows from claim 5.

Hence, we have proved the optimality of \((\delta^*(c), 0, -1/R, 1/R)\) and the sufficiency of (36), (37), (38) and (39) for optimality.
C.2 The “Only If” Part

Note that this part can be directly implied by proposition 5. We provide an alternative proof here since it is more concise.

Claim 8. Any contract \((\delta^*, b^*, L^*, U^*)\) in which \(b^* > 0\) is not optimal.

Proof. Suppose \((\delta^*, b^*, L^*, U^*)\) is optimal and \(b^* > 0\). Again, by propositions 3 and 6, \(\delta^* > 0\) and \(e^* = 1\). If no constraint will be strictly binding, then by claim 1, \(b^* = 0\). A contradiction.

Hence, if a contract \((\delta^*, b^* > 0, L^*, U^*)\) is optimal, then at least one of the constraints will be strictly binding with non-zero probability. Note that if \(L^* = U^*\), then this contract is strictly dominated by \((0,0, L^*, U^*)\). Hence, \(L^* < U^*\). Note that the manager’s IC constraint needs to be satisfied, then by the similar argument in the proof of claim 6, we have

\[
\delta^* + b^* > \delta^*(c). \tag{98}
\]

Comparing (98) with (95), the reason that the equality does not hold is because at least one of the constraints will be strictly binding with non-zero probability.

Since \((\delta^*, b^*, L^*, U^*)\) is optimal, following the similar logic behind (97), we have

\[
\delta^*(c) \geq \delta^* + b^*. \tag{99}
\]

Note that (99) contradicts (98) \(\square\)

Hence, we have showed than any contract with non-zero base salary cannot be optimal.

C.3 The “If and Only If” Condition

Claim 9. A contract \((\delta^*, b^*, L^*, U^*)\) is optimal if and only if it satisfies (36), (37), (38) and (39).

Proof. This is directly implied by claims 5, 7 and 8. \(\square\)