Optimal Time-Consistent Debt Policies

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Abstract

This paper studies a trade-off model of leverage dynamics with a jump-diffusion process for cash flows in which the firm can freely issue new debt and repurchase existing debt over time. We call a debt policy time-consistent if in any state, the equity holders prefer to adhere to it rather than to deviate from it but lose credibility in sustaining any discipline in debt management in the future. The optimal time-consistent debt policy consists of an interest coverage ratio (ICR) target and two regions for the ICR: the stable and the distress regions. When in the stable region, the firm actively manages liabilities to the ICR target by issuing/repurchasing debt. Sufficiently large negative shock push the firm into the distress region, where it abandons the target and waits until either cash flows recover or further negative shocks trigger bankruptcy. The optimal policy captures realistic features of leverage management and dynamics: repurchases, interior optimal debt maturity, firms’ transition to/recovery from “falling angels,” differences in debt price dynamics between investment-grade and speculative bonds. The time-consistency requirement limits firm’s leverage by restricting the maximal repurchase that the equity holders can credibly promise, which in turn, affects the direction of comparative statics.

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1 Introduction

A fundamental question in corporate finance is what determines the firm’s leverage and its dynamics. Although much work has been done in a static framework on determinants of the initial debt level, we still know little about drivers of firm’s leverage dynamics. As Peter DeMarzo stresses in his 2019 presidential address to the American Financial Association, the firm’s choice of leverage over time is fundamentally a commitment problem: The equity holders have limited commitment to refrain themselves from diluting the current bond holders in the future, which depreciates the price of the current bond issues. In this paper, we analyze the optimal debt policy when the firm has absolutely no commitment to future debt issuances and repurchases.

We consider the classical trade-off model of leverage. The firm’s debt is associated with tax benefits, but it also makes (endogenous) default more likely, which destroys value. The equity holders control the firm’s leverage dynamics and can issue new debt and repurchase existing debt costlessly at any moment in time. They can also default at any point in time. Importantly, the equity holders cannot commit to their future debt or default policy. New debt issues/repurchases are priced by the debt holders based on their expectations about the firm’s future leverage choices and default likelihood.

We introduce two novel features into the classical model. First, we depart from the assumption that the cash flows follow the diffusion process and introduce Poisson downward jumps. The literature so far introduced jumps in the cash flow process to improve the quantitative fit of the classical model (e.g., Cremers et al. (2008), Chen and Kou (2009)). Our goal is different and we are interested in obtaining novel qualitative predictions for leverage dynamics, such as realistic debt repurchases, leverage ratio targeting, interior optimal maturity, etc.

The second novel feature is our focus on the optimal debt policy that is time-consistent. Specifically, a debt policy is a rule that for each debt level and interest coverage ratio (henceforth, ICR) prescribes a certain amount of debt to be issued or repurchased. We consider a rich class of policies, in which the equity holders issue new debt when the state reaches the issuance boundary and repurchase debt when the state falls into a certain repurchase region. Since the equity holders can freely adjust their future leverage, they face temptation of deviating from a particular debt policy.

We analyze time-consistent debt policies, which are policies immune to any such deviations. We specify that if such a deviation occurs, then the equity holders lose credibility in sustaining discipline in debt management in the future and the play switched to the
Markov Perfect equilibrium (henceforth, MPE) of the debt issuance game. Formally, the debt policy should satisfy *credibility constraints* requiring that in any state, the equity value under the policy exceeds that in the MPE. DeMarzo and He (2019) characterize this MPE and show that it serves as a natural candidate for deterring deviations. In it, an extreme form of the ratchet effect materialized: the equity holders issue debt continuously, yet, they do not capture any tax benefits from new debt issuances, as they are significantly underpriced due to future dilutions.

We characterize the debt price and the equity value under different debt policies in closed form and derive the optimal time-consistent debt policy. The optimal policy consists of a particular ICR target and two regions for the ICR: the stable and the distress regions. The equity holders issue initial debt to reach the ICR target. As long as shocks to cash flows are small so that the ICR remains in the stable region, the equity holders issue or repurchase debt in order to compensate these shocks and stay at the chosen ICR target. In these times, the debt price is stable and does not react to shocks to cash flows, even though, debt is not risk-free and there is a positive credit spread. After sufficiently large downward jumps in cash flows, the equity holders enter into the distress region. In this region, they wait until the cash flows either recover, at which point they repurchase a bulk of debt to reach the target again, or the default occurs. After such a large drop in cash flows, the debt price drops and becomes sensitive to further shocks to cash flows.

Importantly, (some of) the credibility constraints might bind in the optimum. This allows us to study in what way the equity holders’ lack of commitment constrains their debt policy choices, and how various parameters affect the severity of commitment issue, and through it, the firm’s optimal leverage. To study the former issue, we compare the optimal time-consistent policy to the optimal policy with commitment, when the equity holders can commit to the debt policy, and hence, the credibility constraints are not relevant. Such an optimal policy with commitment also takes the form of targeted ICR, however, it compensates larger drops in cash flows with repurchases. This allows the firm to borrow more in the optimum with commitment. While these larger repurchases make the debt safer and increase the overall firm value, they are too costly for the equity holders to execute, and hence, are not credible in the absence of commitment. Therefore, credibility constraints limit the firm’s leverage by restricting the size of repurchases that the equity holders can credibly promise.

Interestingly, credibility constraints do not bind, when the equity holders merely promise to refrain from debt issuance close to default. Specifically, if we consider only policies that allow for debt issuances, but not repurchases, then the credibility constraints
do not bind in the optimal time-consistent debt policy (which also coincides with the optimal policy with commitment). Thus, in order to explain how lack of commitment limits firm’s leverage, it is important to have large repurchases in the optimum, which is attained in our model by allowing for both repurchases and large drops in cash flows.

Taking into consideration binding credibility constraints reveals the economic mechanisms behind the comparative statics, which are less apparent when one simply compares the MPE and the optimal policy with commitment. As an example, the comparative statics with respect to the volatility of Brownian shocks is quite nuanced and depends on whether the credibility constraints bind or not.

When the credibility constraints do not bind, higher volatility leads to higher leverage. To see this, recall that in the classical Leland (1994) model, higher volatility has an ambiguous effect on the firm value. On the one hand, when the firm is close to default, it increases the chances of escaping the default. On the other hand, when the firm is far from the default, it increases the chances of cash flows deteriorating and the firm sliding to default. While the former force is present under the targeted ICR policy, the latter is attenuated, because the equity holders compensate all negative Brownian shocks with debt repurchases when they are at the target ICR. Because of that, higher diffusion volatility tends to increase leverage in the case when the credibility constraints do not play a role.

When the credibility constraints bind, the comparative statics with respect to the volatility of diffusion is reversed. Higher volatility of diffusion increases the equity value in the MPE, which makes the credibility constraints more restrictive. This, in turn, reduces the leverage and makes the debt more risk, because the maximal credible repurchase that the equity holders can make is reduced.

The comparatives statics is unambiguous with respect to intensity/frequency of Poisson jumps. Naturally, more severe or more frequent downward jumps in cash flows reduce the leverage ratio and increase the ICR target. Interestingly, lower leverage does not compensate completely for the increased riskiness of cash flows, which results in higher credit spreads at the ICR target.

The optimal time-consistent policy captures many realistic features of debt management by companies, namely, (i) switches in debt dynamics between normal and distress periods, (ii) realistic repurchases, and (iii) interior optimal maturity. Importantly, these features arise within a classical trade-off theory and do not require introduction of any additional frictions in the model apart from limited commitment.

First, our model captures the following evolution of company’s leverage. Companies
often announce at the initiation targets for financial policies and subsequently try to stay close to them. However, following significant negative shocks to cash flows, the company’s debt can be downgraded and the company becomes a “fallen angel.” Further, such companies sometimes return to the rank of investment grade by showing good performance. Interestingly, our theory predicts that fallen angels limit borrowing in the turbulent region not because of increased costs of borrowing, but rather because such a policy improves the pricing of issued bonds in the normal region when the firm’s leverage is at the target level.

Another implication of our theory is that debt price dynamics is qualitatively very different in normal times from when the company becomes a fallen angel. While in normal times, the debt price is stable despite the presence of a positive credit spread, in distress times, the credit spreads increase significantly and the company’s debt price becomes sensitive to news about underlying cash flows.

Second, our analysis provides a justification for debt repurchases often observed in practice, but so far ruled out as unrealistic by existing trade-off theories. The key to this result is the addition of downward jumps in cash flows. Specifically, if cash flows follow the diffusion process (as is assumed in many existing trade-off theories), a repurchase boundary completely removes default risk, which trivializes the problem of finding an optimal debt policy and leads to a counter-factual prediction that corporate bonds are risk-free. For this reason, the literature on leverage dynamics so far ruled out repurchases either by assumption or by arguing that they are too costly (e.g., because of high costs of raising equity as in Benzoni et al. 2019). This is no longer the case in our model with Poisson negative jumps, because after a significant downward jump, the equity holders may find it optimal not to repurchase and even to default. Under the optimal time-consistent policy, the equity holders repurchase debt to compensate for relatively small drops in cash flows and stay at the target ICR.

Third, our theory predicts interior optimal debt maturity when the equity holders can also choose debt maturity at the firm’s origination. Intuitively, unlike debt interest, debt principal is not tax-deductible, and so, as in standard trade-off theory (Leland and Toft 1996), debt of longer maturities better captures tax benefits. At the same time, as we argued above, the equity holders cannot credibly promise debt repurchases in the distress region. Debt of shorter maturities is valuable, as it essentially commits them to reduce the debt burden in the times of distress. As a result, optimal maturity can be interior.

Our model provides a benchmark for what is the likely leverage dynamics when the equity holders have absolutely no commitment to future debt policies. DeMarzo and He
(2019) propose to consider the MPE as an alternative benchmark. Given that the MPE exhibits a particularly severe form of the ratchet effect, it is expected that collateralized borrowing or various contractual features, such as state-contingency or covenants, improve on the MPE outcome, and hence, can be justified as optimal in resolving the firm’s commitment problem.

In this paper, we challenge this view by showing that even without any exogenous commitment devices, certain debt policies are credible if they are supported by a threat of reverting to the MPE. Thus, in order to justify certain contractual features, one needs to either show that they improve on the optimal time-consistent debt policy analyzed in the current paper or argue why the endogenous commitment that we propose is not possible. Our analysis of the optimal debt maturity is an example of such an exercise, where we show that shorter maturity serves as an exogenous commitment device in times when the repurchases are not credible.

**Literature Review**  Our paper contributes to the recent literature on leverage dynamics without commitment (Admati et al. 2013, DeMarzo and He 2019, DeMarzo 2019, DeMarzo et al. 2019). We ask the same research question as these papers: What will be the equilibrium dynamic financial policy in the trade-off theory when the firm cannot commit to future financial policies? Our contribution is to analyze this question under a different solution concept. Because we use a different equilibrium concept, we obtain a very different answer about how the equilibrium dynamic financial policy looks like. The equilibrium has features of both adjustment to the target leverage ratio through issuances of new debt and repurchases of existing debt, as in the textbook static trade-off theory of capital structure, and the leverage ratchet effect when the firm does not reduce its debt even though such a reduction is a positive-NPV project. The former happens in normal times when the firm is at or close to its target. The latter happens only after sufficiently severe negative cash flow shocks when the firm is close enough to default, so that debt repurchase is not time-consistent. Intuitively, the leverage ratchet effect is not irrelevant in normal times, because the firm’s reputation effects due to the continuation game are powerful, but becomes very important in financial distress when promises to reduce the debt burden are no longer credible. Because the probability of getting into the financial distress region is affected only by large (jump) shocks, but not by small (diffusion) shocks, an implication of the paper is that the effects of the two types of shocks on equilibrium financial policies are quite different.

The specific relation to existing papers on leverage dynamics without commitment is
as follows. Admati et al. (2013), DeMarzo and He (2019), DeMarzo (2019), DeMarzo et al. (2019) use the Markov Perfect Equilibrium solution concept. The equilibrium has a stark property: The firm never repurchases debt regardless of its leverage, issues debt continuously, and the expectations of future debt issuances depress the price of debt so much that the equity holders do not get any net tax benefits. In contrast, we look for the optimal time-consistent policy, assuming that investors will believe promises of the firm if the firm did not break its promises in the past. Thus, while the MPE rules out reputation effects by assuming that the investors’ beliefs do not depend on the firm’s past issuance and repurchase choices, our solution concept focuses on reputation effects. Informally, one can draw an analogy to equilibria in the repeated prisoners dilemma. The MPE is akin to the equilibrium in which the static Nash equilibrium is played every period, while our equilibrium is more akin to looking for the most cooperative outcome that can be sustained.

Benzoni et al. (2019) is perhaps the most closely related paper, because they also study how promises to better capital structure policies can be sustained by creditors punishing deviations by pricing debt at the level given in the MPE. While Benzoni et al. (2019) and our paper use the same equilibrium concept, they differ significantly in assumptions and thus lead to different insights. Specifically, Benzoni et al. (2019) assume that there are fixed issuance costs and that all cash flow shocks are small (the cash flow process is a diffusion). The firm in their model does not repurchase debt, and the equilibrium financing policy has the firm issuing more debt in lumpy amounts when its cash flows hit higher thresholds. In contrast, we assume that there are no issuance costs (as in DeMarzo and He 2019), but the cash flow process can experience both small (diffusion) and large (jump) negative shocks. Thus, in equilibrium the firm actively manages liabilities by issuing and repurchasing debt to the ICR target, but does not only in normal times: A big enough negative jump puts the firm into a distress region in which it follows the “wait and see” policy. Because the credibility constraint often binds in the optimum, the optima with and without commitment are generally different in our setting, while the two coincide in Benzoni et al. (2019).

Finally, the paper is related to earlier papers that study optimal dynamic capital structure decisions based on the trade-off between tax benefits of debt and costs of financial distress. An incomplete list of these papers include Fischer et al. (1989), Leland (1994, 1998), Leland and Toft (1996), Goldstein et al. (2001), Strebulaev (2007), He (2011). These papers typically assume that the firm must retire all existing debt before issuing new debt (e.g., Fischer et al. 1989, Goldstein et al. 2001) or that the firm issues debt at
the initial date only (e.g., Leland (1994), He (2011)).

In Arellano and Ramanarayanan (2012), a sovereign optimally chooses a portfolio of short- and long-term bonds by trading off the dilution-resistance of the short-term debt and better hedging properties of long-term debt for the risk-averse sovereign. They show that the portfolio is biased towards shorter maturities during the crisis periods when spreads are higher, while the duration of bond portfolio lengthens during “good” times. The forces behind optimal interior maturity in our model are different than in Arellano and Ramanarayanan (2012). The equity holders are risk-neutral, hence, there are no hedging benefits of long-term debt, rather, long-term debt allows them to better capture tax benefits due to the tax deductibility of interest but not principal payments. More importantly, the benefits of short term debt in our model stem from the commitment problem: shorter maturity debt serves as a commitment device for the equity holders to lower debt burden in states when large repurchases are not credible. In contrast, the dilution-resistance of the short-term debt in Arellano and Ramanarayanan (2012) is analogous to that in Bizer and DeMarzo (1992).

The structure of the paper is as follows. Section 2 presents the model. Section 3 derives debt and equity values under debt policies. Section 4 derives the optimal time-consistent debt policy. Section 5 presents empirical implications. Section 6 concludes.

2 The Model

Time \( t \) is continuous. The firm’s operating cash flow \( Y_t \) follows the geometric Brownian motion with downward Poisson jumps

\[
\frac{dY_t}{Y_{t-}} = \hat{\mu} dt + \sigma dZ_t + d \left( \sum_{i=1}^{N_t} (S_i - 1) \right),
\]

where \( Z_t \) is the standard Brownian motion, \( dN_t \) is the Poisson process with constant intensity \( \lambda > 0 \), and the size of the downward jumps \( \tilde{S}_i \equiv -\ln S_i \) is exponentially distributed with parameter \( \eta > 0 \). The expected jump size is \( \zeta \equiv \mathbb{E}[S_i - 1] = -1/(\eta + 1) \). The expected cash flow growth is \( \mu \equiv \hat{\mu} + \lambda \zeta \).

At any time \( t \), the firm can issue/repurchase debt. The outstanding debt has face value \( F_{t-} \). The debt pays coupon \( c \) and matures exponentially at rate \( \xi \). Specifically, each instant \( \xi F_{t-} dt \) is the principal repayment from maturing bonds and \( c F_{t-} dt \) is the coupon payment on outstanding bonds. Let \( y_t \equiv Y_t / F_{t-} \) and note that \( y_t / c \) is the interest
coverage ratio (ICR), i.e., the ratio of pre-tax operating cash flows to the promised coupon payment at time $t$. The firm’s taxes at time $t$ equal $\pi(Y_t - cF_t)\,dt$, $\pi \in (0,1)$.

All agents are risk neutral and discount time at rate $r > 0$. At any $t$, the equity holders observe realizations of cash flows up to and including time $t$, $(Y_s)_{s \leq t}$, and past leverage dynamics up to time $t$, $(F_s)_{s < t}$. They decide how much debt to issue/repurchase, $dF_t$, and whether to default or not. For simplicity, there is zero recovery after default, which allows us to abstract from the seniority of different debt issuances.\footnote{Positive recovery after default can be incorporated using the infinite dilution at default argument in DeMarzo and He (2019).}

At any $t$, the debt holders observe past cash flows and leverage dynamics up to and including time $t$, $(Y_s, F_s)_{s \leq t}$. That is, they observe everything that the equity holders observe at time $t$, and in addition, how much debt was issued/repurchased at time $t$. Given this information, they determine competitively the price of the newly issued debt, $p_t$ (described below).

We study debt policies that are time-consistent in the sense that the equity holders prefer to stick to them rather than deviate and lose credibility in exerting discipline in debt issuance/repurchase in the future. We next formalize this definition.

**Debt Policies** A debt policy $\Sigma$ is a Markov process that specifies (i) the initial debt issuance $F_0$ for a given $Y_0$, (ii) for any $y_t$ and $F_{t-}$, the debt issuance/repurchase amount, $d\Gamma_t$. Then,

$$dF_t = d\Gamma_t - \xi F_{t-} \,dt.$$ 

Without loss of generality, the equity holders choose directly $dF_t = d\Sigma(y_t, F_{t-})$ at any $t$.

We focus on the class of debt policies $\mathcal{S}$ that are characterized by the issuance boundary $y_i$ and repurchase region $[y_r, \bar{y}_r]$, $\bar{y}_r \leq y_i$, as well as target ICRs $y_i^*$ and $y_r^*$ such that $y_i^* \leq y_i$ and $y_r^* \geq \bar{y}_r$. When the firm’s ICR reaches $y_i$, the firm issues debt to lower the ICR to $y_i^*$; when the firm’s ICR falls into the interval $[y_r, \bar{y}_r]$, the firm repurchases debt to increase...
Figure 1: **Dynamics under a generic policy in \( S \)**

Top panel depicts evolution of ICR \( y_t/c \) and bottom panel depicts evolution of cash flows \( Y_t \) and debt \( F_t \).

the ICR to \( y_r^* \). Formally,

\[
d\Sigma(y_t, F_t) = \begin{cases} 
0, & \text{if } y_r < y_t < y_i \text{ or } y_t < y_r, \\
F_{t-}(y_t - y_i^*)/y_i^*, & \text{if } y_t \geq y_i, \\
F_{t-}(y_t - y_r^*)/y_r^*, & \text{if } y_r \leq y_t \leq y_r. 
\end{cases} 
\] (1)

It is without loss of generality to suppose that \( y_r^* \leq y_i \) (by (1)). We include in \( S \) limits of policies as \( y_i^* \to y_i \) (reflecting issuance boundary) and/or \( y_r^* \to y_r \) (reflecting repurchase boundary) and/or \( y_r \to y_i \) (targeted ICR). Figure 1 depicts an example of ICR and debt dynamics under a generic debt policy in \( S \). The equity holders default at the first time \( \tau_b \) when \( y_t < y_b \) for some fixed \( y_b \).

The debt holders who expect the equity holders to follow policy \( \Sigma \) and default at time

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2The class \( F \) excludes certain potentially interesting policies, for example, ones specifying debt issue/repurchase at a certain rate: \( dF_t = g(y_t)F_{t-} dt \) for some function \( g(\cdot) \). We conjecture that our results would not be affected by allowing for such more complex debt policies, yet, the verification of this conjecture requires a significant generalization of the techniques developed in the present paper and is left for future research.

3Note issuance only debt policies are a special case of (1) when \( y_i = 0 \).
\[
p(y_t|\Sigma) = \mathbb{E} \left[ \int_t^{\tau_m \wedge \tau_b} ce^{-r(s-t)}dt + e^{-r(\tau_m - t)}1\{\tau_m \leq \tau_b\} | y_t, \Sigma \right],
\]

where \(\tau_m\) is the stopping time when the bond matures. Note that the debt price depends on \(\Sigma\) indirectly through its effect has on the endogenous default time \(\tau_b\). Given this debt pricing, the equity holders' value from following the debt policy \(\Sigma\) and defaulting at time \(\tau_b\) is given by

\[
E(Y_t, F_t|\Sigma) = \mathbb{E} \left[ \int_t^{\tau_b} e^{-r(s-t)} [(1 - \pi)(Y_s - cF_s)ds - \xi F_s ds + p(y_s|\Sigma)dF_s] | Y_t, F_t, \Sigma \right].
\]

The equity holders default strategically, which is captured by the smooth-pasting condition at the default boundary: \(\partial E(Y, F)/\partial Y = 0\) for all \((Y, F) \in B\) where \(B\) is the boundary of the default region.

Let

\[
W(\Sigma) = \max_{F_0 \geq 0} \{p(Y_0/F_0|\Sigma) F_0 + E(Y_0, F_0|\Sigma)\}
\]

is the equity holders' revenue from issuing \(F_0\) at \(t = 0\) and expected continuation value from following policy \(\Sigma\) in the future. Observe that it coincides with the maximal firm value.

**Leverage Dynamics without Credibility**  We suppose that the first time the equity holders deviate from the debt policy, the dynamics switches to the Markov Perfect Equilibrium (henceforth, MPE) of the game, which takes the following form. In the MPE, the equity holders default if and only if \(y_t < y_{bm}\) for some \(y_{bm}\), and the debt issuance process is given by some \(\Sigma^m\). We derive \(y_{bm}\) and \(\Sigma^m\) in the next section. The debt holders expect the equity holders to follow this issuance and default strategy and price the newly issued debt accordingly at

\[
p_m(y_t) = \mathbb{E} \left[ \int_t^{\tau_{bm} \wedge \tau_m} ce^{-r(s-t)} ds + e^{-r(\tau_m - t)}1\{\tau_m \leq \tau_b\} | y_t, \Sigma^m \right],
\]

where \(\tau_{bm}\) is the stopping time when \(y_t < y_{bm}\) for the first time. The distribution of \(\tau_{bm}\) depends on the MPE issuance strategy \(\Sigma^m\) through its effect on the evolution of \(y_t\).
equity value $E_m(Y, F)$ satisfies

$$
E_m(Y_t, F_{t-}) = (1 - \pi)(Y_t - cF_{t-})dt - \xi F_{t-}dt \\
+ \max_{dF \in \mathbb{R}} \{p_m(Y_t/(F_{t-} + dF))dF + (1 - rdt)\mathbb{E}[E_m(Y_t + dY_t, F_{t-} + dF)]\}
$$

The first two terms are the post-tax profit during $dt$ and the payment to maturing debt. The third term is the optimal revenue from issuing or repurchasing new debt plus the continuation equity value. The debt price equals the value of debt after the equity holders adjust the debt level by $dF$. The last term is the continuation value of the equity holders given the new debt level.

The equity holders do not have credibility in exerting discipline in managing debt, hence, they choose $dF$ in every state without taking into account that their debt management affects the pricing of debt in different states. Further, the equity holders default strategically, which is captured by the smooth-pasting condition at the default boundary: $\partial E_m(Y, F)/\partial Y = 0$ for all $(Y, F)$ such that $Y/F = y_{bm}$. In Section 3, we derive the MPE and show that, as in DeMarzo and He (2019), the equity holders’ lack of credibility depresses the price of the current debt issue so that the equity holders do not capture any tax benefits of any issuances beyond $t = 0$.

**Time-Consistent Debt Policies** Let $\mathcal{R}(\Sigma)$ be the set of all states $(Y, F)$ that can be reached from the initial state $(Y_0, F_0)$ under the debt policy $\Sigma$.

**Definition 1.** A debt policy $\Sigma$ is time-consistent if

$$
E(Y, F|\Sigma) \geq \sup_{\hat{F} \geq 0} \{(\hat{F} - F)p_m(Y/\hat{F}) + E_m(Y, \hat{F})\}, \text{ for all } (Y, F) \in \mathcal{R}(\Sigma), \quad (4)
$$

We refer to conditions (4) as the credibility constraints. We denote by $\mathcal{S}_{TC}$ the class of all time-consistent debt policies.

The idea behind credibility contraints (4) is that the debt policy should be supported by a threat to revert to the MPE. Initially, the debt holders believe that the equity holders will stick to the debt policy $\Sigma$. As long as the equity holders continue following the policy $\Sigma$, the debt holders continue trusting the equity holders to do so in the future, and so, they price the debt accordingly at $p(y|\Sigma)$. If the equity holders deviate from the debt policy in some state $y$ and issue amount $d\Gamma_t = \hat{F} - F$, then the state transitions from $Y/F$ to $Y/\hat{F}$. After this deviation, the equity holders lose credibility in exerting any debt.
discipline in the sense that the debt holders expect the debt issuance to be as in the MPE. They price the debt issuance $\hat{F} - F$ at $p_m(Y/\hat{F})$ and the continuation value of the equity holders is equal to $E_m(Y, \hat{F})$, which is the right-hand side of (4).

Definition 1 states that the debt policy is time-consistent if the equity holders never have incentives to deviate from it, if this entails that they lose credibility in exerting the debt discipline in the future. We are interested in characterizing the optimal time-consistent debt policies defined as follows:

**Definition 2.** A debt policy $\Sigma^*$ is the optimal time-consistent debt policy if

$$\Sigma^* \in \arg\max_{\Sigma \in \mathbb{S}^r \cap \mathbb{S}^T_C} W(\Sigma).$$

(5)

3 Debt and Equity Values

In this section, we characterize debt and equity values under different debt policies and in the MPE. This will allow us to write more explicitly the program (5).

For notational convenience, we will omit in the notations dependence on the debt policy $\Sigma$, and write, $p(y)$ instead of $p(y|\Sigma)$, $E(Y, F)$ instead of $E(Y, F|\Sigma)$, and so on. For $y \in (y_b, y_r) \cup (y_r, y_i)$, $p(y)$ satisfies the HJB equation

$$(r + \lambda + \xi)p(y) = c + \xi + (\hat{\mu} + \xi) y p'(y) + \frac{1}{2}\sigma^2 y^2 p''(y) + \lambda \mathbb{E}[p(Sy)].$$

(6)

We conjecture that

$$p(y) = \begin{cases} 0, & y < y_b \\ \frac{c + \xi}{r + \xi} \left(1 + \sum_{k=1}^{3} b_k(y/y_b)^{-\gamma_k}\right), & y \in [y_b, y_r], \\ p^*, & y \in (y_r, \bar{y}_r), \\ \frac{c + \xi}{r + \xi} \left(1 + \sum_{k=1}^{3} B_k(y/y_b)^{-\gamma_k}\right), & y \in [\bar{y}_r, y_i], \\ p^*, & y > y_i, \end{cases}$$

(7)

where $\gamma_k$s are roots of the characteristic equation

$$\frac{1}{2}\sigma^2 \gamma^2 - \left(\hat{\mu} + \xi - \frac{1}{2}\sigma^2\right) \gamma + \frac{\lambda \eta}{\eta - \gamma} = r + \lambda + \xi.$$

(8)

At the default boundary,

$$p(y_b) = 0.$$

(9)
The debt holders anticipate that when the ICR reaches \( \frac{y_i}{c} \), the equity holders issue debt to lower the ICR to \( \frac{y_i^*}{c} \), thus,

\[
p(y_i) = p_i^* \equiv p(y_i^*).
\] (10)

Similarly, the debt holders anticipate that if \( y_t \) falls into the repurchase region \([y_r, \overline{y}_r]\), then the equity holders repurchase debt to increase the ICR to \( \frac{y_r^*}{c} \), thus,

\[
p(y) = p_r^* \equiv p(y_r^*), \text{ for all } y \in [y_r, \overline{y}_r].
\] (11)

If \( y_i \) or \( \overline{y}_r \) are reflecting boundaries, then conditions (10) and (11) are replaced by their limits as \( y_i^* \to y_i \) and \( y_r^* \to \overline{y}_r \). These conditions are \( p'(y_i) = 0 \) and \( p'([\overline{y}_r]) = 0 \), respectively (see Online Appendix B.2).

Coefficients \( b_k \)s and \( B_k \)s are determined by the boundary conditions (9) – (11) and the following two additional conditions:

\[
\mathbb{E} \left[ \frac{c + \xi}{r + \xi} \left( 1 + \sum_{k=1}^{3} b_k S^{-\gamma_k} \right) \right] = 0,
\] (12)

\[
\mathbb{E} \left[ \frac{c + \xi}{r + \xi} \left( 1 + \sum_{k=1}^{3} B_k (S\overline{y}_r/y_b)^{-\gamma_k} \right) \right] = \mathbb{E} \left[ p(S\overline{y}_r) \right].
\] (13)

Equations (12) and (13) arise because of the presence of downward jumps in cash flows. They require that even if the conjectures for \( p \) on \([y_b, y_r]\) and \([\overline{y}_r, y_i]\) were applied beyond these ranges, this would not change debt pricing on \([y_b, y_r]\) and \([\overline{y}_r, y_i]\). Indeed, by the memoryless property of the exponential distribution of downward jumps, the debt price for \( y \in [y_b, y_r] \) would not change if we changed \( p(y) \) on \( y < y_b \) (from the specification in (7)) as long as \( \mathbb{E}[p(Sy_b)] = 0 \). Condition (12) requires that \( \mathbb{E}[p(Sy_b)] = 0 \) even if the conjecture for \( p \) on \([y_b, y_r]\) is extended to \( y_s \) below \( y_b \).

Similarly, the memoryless property implies that the debt price in the region \([\overline{y}_r, y_i]\) is not affected by a change in \( p(y) \) below \( \overline{y}_r \) (from the specification in (7)) as long as \( \mathbb{E}[p(S\overline{y}_r)] \) stays the same. Condition (13) requires that this is indeed the case if the conjecture for \( p \) on \([\overline{y}_r, y_i]\) is extended to \( y_s \) below \( \overline{y}_r \).\(^4\)

Due to the homogeneity of the setup, the equity value takes the form \( E(Y, F) = e(y) F \).

\(^4\)Online Appendix B.1 provides explicit expressions for (12) and (13) and derives them.
The value of equity satisfies the following HJB on \((y_b, y_r) \cup (\bar{y}_r, y_i)\),

\[
(r + \lambda + \xi)e(y) = (1 - \pi)(y - c) - \xi + (\mu + \xi)ye'(y) + \frac{1}{2}\sigma^2 y^2 e''(y) + \lambda \mathbb{E}[e(Sy)]. \tag{14}
\]

The boundary and smooth-pasting conditions at the default boundary \(y_b\) imply

\[
e(y_b) = 0, \tag{15}
\]

\[
e'(y_b) = 0. \tag{16}
\]

At the issuance boundary \(y_i\), \(e(y_i)F_{i-} = e(y_i^*)F_i + p(y_i)(F_i - F_{i-})\), or using \(p(y_i) = p(y_i^*) = p_i^*\),

\[
\frac{e(y_i) + p_i^*}{y_i} = \frac{e(y_i^*) + p_i^*}{y_i^*}. \tag{17}
\]

Analogously, boundary conditions at repurchase boundaries \(y_r\) and \(\bar{y}_r\) imply

\[
\frac{e(y_r) + p_r^*}{y_r} = \frac{e(y_r^*) + p_r^*}{y_r^*}, \tag{18}
\]

\[
\frac{e(\bar{y}_r) + p_r^*}{\bar{y}_r} = \frac{e(\bar{y}_r^*) + p_r^*}{\bar{y}_r^*}. \tag{19}
\]

Since the equity holders repurchase debt at any \(y \in [y_r, \bar{y}_r]\), for any such \(y\) it holds \(\frac{e(y) + p_r^*}{y} = \frac{e(y^*) + p_r^*}{y^*}\). Note that \((e(y) + p(y))/y\) is the EV/EBIT multiple (enterprise value divided by pre-tax earnings). Equations (17) – (19) state that the EV/EBIT multiple does not change when there is an expected adjustment of the leverage.

If \(y_i\) or \(\bar{y}_r\) are reflecting boundaries, then conditions (17) and (19) are replaced by their limits as \(y_i^* \to y_i\) and \(y_r^* \to \bar{y}_r\). These conditions are \(e'(y_i) = (e(y_i) + p(y_i))/y_i\) and \(e'(y_r^*) = (e(y_r^*) + p(y_r^*))/y_r^*\), respectively (see Online Appendix B.2).

We conjecture that the equity value per unit of debt takes the form:

\[
e(y) = \begin{cases} 
0, & y < y_b; \\
\phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\
\frac{e(y_i^*) + p_i^*}{y_i^*} - p_i^*, & y \in (y_r, \bar{y}_r), \\
\phi y - \rho + \sum_{k=1}^{3} C_k (y/y_b)^{-\gamma_k}, & y \in [\bar{y}_r, y_i], \\
\frac{e(y_i^*) + p_i^*}{y_i^*} - p_i^*, & y > y_i;
\end{cases} \tag{20}
\]

where \(\phi \equiv \frac{1 - \pi}{r - \mu}\) and \(\rho \equiv \frac{c(1 - \pi) + \xi}{r + \xi}\), and \(\gamma_i\)s solve the characteristic equation (8). Coefficients
cks and Cks and the default boundary $y_b$ are pinned down by (15) – (19), and in addition,

$$
\mathbb{E} \left[ \phi y - \rho + \sum_{k=1}^{3} S^{-\gamma_k} \right] = 0 \tag{21}
$$

$$
\mathbb{E} \left[ \phi S \overline{y}_r - \rho + \sum_{k=1}^{3} C_k (S \overline{y}_r/y_b)^{-\gamma_k} \right] = \mathbb{E} [e(S \overline{y}_r)] . \tag{22}
$$

Similarly to conditions (12) and (13) for debt pricing, conditions (21) and (22) require that even if the conjectures for $e$ on $[y_b, y_r]$ and $[\overline{y}_r, y_i]$ were applied beyond these ranges, this would not change the equity value on $[y_b, y_r]$ and $[\overline{y}_r, y_i]$. By the memoryless property of the exponential distribution of downward jumps, this would be the case if $\mathbb{E}[e(Sy_b)]$ and $\mathbb{E}[e(S\overline{y}_r)]$ did not change, which are the conditions (21) and (22).

### 3.1 Credibility Constraints

To make the credibility constraints (4) more explicit, we will derive the debt price and equity value in the MPE, which serves as a punishment to equity holders for deviating from the debt policy.

We can extend the analysis in DeMarzo and He (2019) to characterize the MPE in the case of downward jumps in cash flows. The equity value $E_m(Y, F)$ satisfies the HJB equation:

$$(r + \lambda)E_m(Y, F) = \max_G \left\{ (1 - \pi) (Y - cF) - \xi F + G p_m(Y/F) + (G - \xi F) \frac{\partial}{\partial F} E_m(Y, F) 
+ \mu Y \frac{\partial}{\partial Y} E_m(Y, F) + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2}{\partial Y^2} E_m(Y, F) + \lambda \mathbb{E} [E_m(SY, F)] \right\},$$

where $G$ denotes the debt issuance/repurchase amount. DeMarzo and He (2019) argue in Proposition 1 that in the MPE, the equity holders are indifferent between issuing or repurchasing any amount of debt, which implies that for all $y = Y/F > y_{bm}$,

$$p_m(Y/F) + \frac{\partial}{\partial F} E_m(Y, F) = 0. \tag{23}$$

Thus,

$$
(r + \lambda + \xi) E_m(Y, F) = (1 - \pi) (Y - cF) - \xi F
+ (\mu + \xi) Y \frac{\partial}{\partial Y} E_m(Y, F) + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2}{\partial Y^2} E_m(Y, F) + \lambda \mathbb{E} [E_m(SY, F)] . \tag{24}
$$

---

*Online Appendix B.1 provides explicit expressions for (21) and (22) and derives them.*
We conjecture that $E_m(Y, F) = e_m(y)F$ and $e_m(y) = \phi y - \rho + \sum_{k=1}^{3} c_{km}(y/y_{bm})^{-\gamma_k}$. By (24), $e_m$ satisfies the HJB equation (14) with boundary conditions $e_m(y_{bm}) = e'_m(y_{bm}) = 0$, transversality condition $\lim_{y \to \infty} e_m(y) - (\phi y - \rho) = 0$, and condition (21), which pin down coefficients $c_{km}$s and the default boundary $y_{bm}$. The debt price in the MPE is determined from (23):

$$p_m(y) = ye'_m(y) - e_m(y).$$

In the MPE, the equity holders issue debt continuously at rate $g(y) \equiv \pi e_{yp_m}(y)$ so that the newly issued debt is priced exactly at $p_m(y)$.

An important property of the MPE showed by DeMarzo and He (2019) is that whenever $e_m$ is strictly convex, deviations to large debt issuances/repurchases (of order larger than $dt$) are not profitable. This property allows us to simplify the credibility constraints (4):

**Proposition 1.** $e_m$ is strictly convex on $[y_{bm}, \infty)$. Further, credibility constraints (4) are equivalent to

$$e(y|\Sigma) \geq e_m(y), \text{ for all } y \in [y_b, y_i]. \quad (25)$$

To prove Proposition 1, we first show that $p_m(y)$ is strictly increasing in $y$ on $[y_{bm}, \infty)$. Let us differentiate (24) with respect to $F$ and use (23) to get

$$(r + \lambda + \xi)p_m(y) = c(1 - \pi) + \xi + (\hat{\mu} + \xi)yp_m(y) + \frac{1}{2}\sigma^2 y^2 p''_m(y) + \lambda E[p_m(Sy)]. \quad (26)$$

The boundary conditions for the debt price are $p_m(y_{bm}) = 0$ and $\lim_{y \to \infty} p_m(y) = \rho$. From (26), we get that the debt price can be obtained from the auxiliary environment in which the equity holders do not issue any debt, the debt pays coupon $1 - \pi$, and the default is triggered when the interest coverage ratio $y_t$ reaches $y_{bm}$. Thus,

$$p_m(y) = E\left[\int_{t}^{\tau_m \wedge \tau_m} e^{-r(s-t)} c(1 - \pi) ds \right] 1\{\tau_m \leq \tau_{bm}\} | y_t = y, \Sigma^0 \right], \quad (27)$$

where $\Sigma^0$ denotes the debt policy in which the equity holders do not issue/repurchase debt at any $t > 0$. In Appendix, we show that (27) implies that $p_m$ is strictly increasing on $y \geq y_{bm}$. Intuitively, for larger $y$ the default is less likely, and so, the debt holders expect to receive the coupon for longer.

By $p'_m(y) = ye''_m(y)$, $e_m$ is strictly convex on $[y_{bm}, \infty)$. Proposition 3 in DeMarzo and He (2019) show that when $E_m$ is strictly convex in $F$, no global deviations from the strategy $g$ are profitable in the MPE, i.e., $E_m(Y, F) \geq \max_F \left\{\left(\hat{F} - F\right) p_m(Y/\hat{F}) + E_m(Y, \hat{F})\right\}$
for all \( Y \), which implies that for time-consistent debt policies (4) are equivalent to \( E(Y, F) \geq E_m(Y, F) \). Coupled with the homogeneity of \( E \) and \( E_m \), this implies that the credibility constraints (4) are equivalent to (25).

4 Optimal Time-Consistent Policy

4.1 Targeted ICR

We first describe the debt policy, which as we show in the next subsection is the optimal time-consistent policy.

The targeted ICR policy is characterized by the ICR target \( \hat{y} \) and the repurchase boundary \( y_r \). Denote the class of such policies by \( \hat{S} \). The equity holders issue or repurchase the debt to compensate small shocks to the ICR for which \( y_t \geq y_r \) to ensure that \( y_t \) stays at the target level \( \hat{y} \). For larger shocks to \( y_t \) for which \( y_t \in (y_b, y_r) \), the equity holders do not issue or repurchase any debt and wait until either \( y_t \) hits \( y_r \), at which point they repurchase debt to restore the ICR to the target level \( \hat{y} \), or \( y_t \) drops below \( y_b \), at which point the equity holders default.\(^6\)

Let us derive debt price and equity value under the targeted ICR policy. As before, the debt price satisfies the HJB equation (6) for \( y \in [y_b, y_r] \). We conjecture that the debt price is given by
\[
p(y) = \begin{cases} 
0, & y \in (0, y_b], \\
\hat{p}, & y \in [y_b, y_r], \\
\hat{p}, & y \in [y_r, \infty);
\end{cases}
\]
where \( \hat{p} \equiv p(\hat{y}) \). The coefficients \( b_i \)'s satisfy the boundary conditions \( p(y_b) = 0 \) and \( p(y_r) = \hat{p} \equiv p(\hat{y}) \), as well as condition (12). Further, the price of debt \( \hat{p} \) at the target ICR \( \hat{y} \) is given by
\[
\hat{p} = (c+\xi)dt + (1-rdt-\xi dt) \begin{cases} 
(1-\lambda dt)\hat{p} + \lambda dt \left( \int_0^{\ln(\hat{y}/y_r)} \eta e^{-\eta s} ds \right) \hat{p} + \lambda dt \left( \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} p(e^{-\xi s}) \eta e^{-\eta s} ds \right), & \text{Brownian shocks} \\
\text{small Poisson shock} \\
\text{large Poisson shock}
\end{cases}
\]

\(^6\)Note that the targeted ICR debt policy can be obtained as the limit of the debt issuance/repurchase policy in \( F \) if we allow \( \bar{y}_r, y_r^*, y_i^* \), and \( y_i \) all converge to \( \hat{y} \).
The bond pays a unit flow payoff. With probability $1 - \lambda dt$, only Brownian shocks occur and they are compensated by the equity holders’ issuance or repurchase of debt so that the ICR still equals $\hat{y}$. With probability $\lambda dt$, the Poisson negative shock to $y$ occurs. Then, it is compensated by the equity holders only if it is sufficiently small. In this case, the price of debt continues to be equal to $\hat{p}$. Otherwise, when the shock is sufficiently large, the price of debt drops to $p(e^{-\tilde{s}}\hat{y})$. We can rewrite this equation as the HJB equation:

$$(r + \lambda + \xi)\hat{p} = c + \xi + \lambda \hat{p} \int_{0}^{\ln(\hat{y}/y_r)} \eta e^{-\eta \tilde{s}} d\tilde{s} + \lambda \int_{\ln(\hat{y}/y_b)}^{\ln(\hat{y}/y_r)} p(e^{-\tilde{s}}\hat{y}) \eta e^{-\eta \tilde{s}} d\tilde{s}. \quad (28)$$

These four conditions pin down $b_k$s as well as price $\hat{p}$.

Further, we conjecture that the equity value per unit of debt equals $e(y) = \begin{cases} 0, & y \in (0, y_b], \\ \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\ \left(\frac{y}{y_b} - 1\right) \hat{p} + \frac{y}{y_b} \hat{e}, & y \in [y_r, \infty), \end{cases} \quad (29)

where $\hat{e} \equiv e(\hat{y})$. As before, the coefficients $c_k$s satisfy $e(y_b) = e'(y_b) = 0$ and equation (21). Further, the value of equity at the target $\hat{y}$ is equal to

$$E(\hat{y}F_{t-}, F_t) = (1 - \pi)(\hat{y}F_{t-} - cF_{t-})dt - \xi F_{t-}dt + \mathbb{E}[\hat{p}dF_t]$$

flow payoff

$$+ (1 - rd t - \lambda dt)\mathbb{E}[E(\hat{y}F_t, F_t)|dN_t = 0]$$

issuance/repurchase revenue

$$+ (1 - rd t) \begin{cases} \lambda dt \int_{0}^{\ln(\hat{y}/y_r)} E(\hat{y}F_t, F_t)e^{-\eta \tilde{s}} d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/y_b)}^{\ln(\hat{y}/y_r)} E(e^{-\tilde{s}}\hat{y}F_{t-}, F_{t-})e^{-\eta \tilde{s}} d\tilde{s}, & small Poisson shocks \\ \end{cases}$$

only Brownian shocks

$$+ (1 - rd t) \begin{cases} \lambda dt \int_{\ln(\hat{y}/y_b)}^{\ln(\hat{y}/y_r)} E(\hat{y}F_t, F_t)e^{-\eta \tilde{s}} d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} E(e^{-\tilde{s}}\hat{y}F_{t-}, F_{t-})e^{-\eta \tilde{s}} d\tilde{s}, & large Poisson shocks \end{cases}.$$  \quad (30)

In equation (30), the first term is the (flow) payoff from cash flows after coupon payments and taxes. The second term is the revenue from debt issuance or costs of debt repurchases. The third term is the continuation value when no downward jumps occur. The forth term is the continuation value when the jump is sufficiently small $\exp(-\hat{S}_t) > y_r/\hat{y}$. The fifth term is the continuation value when the jump is large, but not so large to trigger the default, $\exp(-\hat{S}_t) \in (y_b/\hat{y}, y_r/\hat{y})$. In Appendix A.2, we show that equation (30) can be
re-written as the HJB equation:

\[
(r + \lambda - \hat{\mu})\hat{e} = (1 - \pi)(\hat{y} - c) - \xi + \hat{p}(\hat{\mu} + \xi) + \lambda \int_{0}^{\ln(\hat{y}/y_r)} \left((e^{-s} - 1) \hat{p} + e^{-s}\hat{e}\right)\eta e^{-\eta s} d\tilde{s} + \lambda \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} e(\hat{y})\eta e^{-\eta s} d\tilde{s}.
\]

(31)

To pin down \((c_1, c_2, c_3, y_b, \hat{e})\), we need an additional boundary conditions at \(y_r\):

\[
\frac{e(y_r) + \hat{p}}{y_r} = \frac{\hat{e} + \hat{p}}{\hat{y}}.
\]

Targeted ICR policies have an important property that for them, the credibility constraint is particularly easy to check: We only need to check that the default boundary \(y_b\) is below that in the MPE, and the policy improves on the MPE at \(\hat{y}\).

Proposition 2. A targeted ICR policy satisfies the credibility constraints (25) if and only if \(y_b \leq y_{bm}\), \(e(y) \geq 0\) for some \(y \in (y_b, y_{bm}]\), and \(e(\hat{y}) \geq e_m(\hat{y})\).

Proposition 2 greatly simplifies the problem of finding the optimal time-consistent targeted ICR policy. In the optimal time-consistent targeted ICR policy, \(e(\hat{y}) + \hat{p} \geq e_m(\hat{y}) + p_m(\hat{y})\), which implies that \(e(\hat{y}) \geq e_m(\hat{y})\).\(^*\) Thus, to find the optimal time-consistent targeted ICR policy, we only need to check that \(y_b \leq y_{bm}\) and \(e(y)\) is non-negative for at least one point in \((y_b, y_{bm}]\).

4.2 Optimality of Targeted ICR

Given the debt and equity values, we can compute the firm value per unit of debt:

\[
v(y) \equiv e(y) + p(y) = \begin{cases}
0, & y \in (0, y_b], \\
\phi y + \frac{c r + \xi}{r + \xi} + \sum_{k=1}^{3} a_k (y/y_b)^{-\gamma_k}, & y \in [y_b, y_r], \\
\phi y + \frac{c r + \xi}{r + \xi} + \sum_{k=1}^{3} A_k (y/y_b)^{-\gamma_k}, & y \in [y_r, y_i], \\
\frac{e(y_r^*) + p_r^*}{y_r^*} y, & y \in [y_i, \infty).
\end{cases}
\]

(32)

Let \(w(y) \equiv v(y)/y\). Then, the objective function in (5) can be written as \(W(\Sigma) = Y_0 \max_{y \geq 0} w(y)\).

\(^*\)Indeed, it must be either \(e(\hat{y}) \geq e_m(\hat{y})\) or \(\hat{p} \geq p_m(\hat{y})\). If \(\hat{p} \geq p_m(\hat{y})\), then the equity holders issue new debt at price above that in the no-credibility outcome, which implies that \(e(\hat{y}) \geq e_m(\hat{y})\).
Auxiliary Program  We first consider an auxiliary program in which \( y_b \) is fixed at the optimal level \( y_{b0} \), and the credibility constraints are ignored:

\[
\max_{\Sigma \in \mathcal{S}} \{ W(\Sigma) : y_b = y_{b0} \}.
\] (33)

To solve this program, we suppose that the smooth-pasting and super-contact principles hold. Say that the smooth-pasting principle holds if whenever the optimal issuance (repurchase) boundary is an impulse control, i.e., \( y_i > y^*_i \) (\( \overline{y}_r < y^*_r \), respectively), \( w'(y_i) = w'(y^*_i) = 0 \) (\( w'(\overline{y}_r) = w'(y^*_r) = 0 \), respectively). Say that the super-contact principle holds if whenever the optimal issuance (repurchase) boundary is an instantaneous control, i.e., \( y_i = y^*_i \) (\( \overline{y}_r = y^*_r \), respectively), \( w''(y_i) = 0 \) (\( w''(\overline{y}_r) = 0 \), respectively). These principles hold in the theory of optimal control of Brownian motion (see Dixit 1991, Dumas 1991 or for a more rigorous treatment Harrison et al. (1983), Harrison and Taksar (1983)). We can use the standard argument in Dixit 1991, Dumas 1991 to show that these principles hold in our setup without Poisson shocks. We verify numerically that these principles hold in all our parameter specifications.8

Intuitively, the optimality principles state that the marginal value of control before and after it is applied should equal to marginal costs. In our environment, while the debt adjustment is costly/beneficial for the equity holders, as they repurchase or issue new debt at prices \( p^*_r \) or \( p^*_i \), for the firm as a whole, debt adjustment involves neither lump sum nor proportional adjustment costs. At the same time, marginal value of controls are \( w'(\overline{y}_r) \) and \( w'(y_i) \) before the leverage adjustment, and \( w'(y^*_i) \) and \( w'(y^*_r) \) after the leverage adjustment. Thus, the smooth-pasting principle requires \( w'(y_i) = w'(y^*_i) = 0 \) and \( w'(\overline{y}_r) = w'(y^*_r) = 0 \). The intuition for the super-contact principle is analogous when applied to the limit case \( y_{i} - y^*_i \rightarrow 0 \) and \( \overline{y}_r - y^*_r \rightarrow 0 \).

Proposition 3. The debt policy solving (33) is the targeted ICR policy.

The proof of Proposition 3 is provided in Appendix and it boils down to determining appropriate boundary conditions for the function \( v \). First, consider the case of impulse

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8Specifically, for each set of parameters, we find numerically the optimal time-consistent debt policy (see the algorithm described after Proposition 4 and footnote 9) and verify that it is indeed the targeted ICR policy with policy parameters \( \hat{y} \) and \( y_r \). Then, perturb the parameters of the optimal policy so that \( y_i > y^*_i \) and \( \overline{y}_r < y^*_r \), and verify that \( w'(y_i), w'(y^*_i), w'(\overline{y}_r), \) and \( w'(y^*_r) \) all converge to zero as \( y_i, y^*_i, \overline{y}_r, \) and \( y^*_r \) all converge to \( \hat{y} \). Similarly, we perturb the parameters of the optimal policy so that reflecting issuance/repurchase boundaries \( y_i \) and \( \overline{y}_r \) are distinct, and verify that \( w''(y_i) \) and \( w''(\overline{y}_r) \) converge to zero as \( y_i \) and \( \overline{y}_r \) converge to \( \hat{y} \).
control at boundaries $y_i$ and $\bar{y}_r$. By the smooth-pasting principle,

$$v'(y_i)y_i - v(y_i) = v'(y_i^*)y_i^* - v(y_i^*) = v'(y_r^*)y_r^* - v(y_r^*) = v'(\bar{y}_r^*)\bar{y}_r - v(\bar{y}_r) = 0.$$ 

As we argue in Appendix, this implies that $\partial a_k / \partial \hat{y} = \partial A_k / \partial \hat{y} = 0$ for $\hat{y} \in \{y_i, y_i^*, \bar{y}_r, \bar{y}_r^*\}$, which in turn, implies that $\partial b_k / \partial \hat{y} = \partial B_k / \partial \hat{y} = 0$. The latter implies that $p'(y_i) = p'(y_i^*) = 0$ and $p'(\bar{y}_r) = p'(y_r^*) = 0$. Thus, the issuance/repurchase boundaries can be replaced with reflecting boundaries. This implies that the appropriate boundary conditions at the issuance/repurchase boundaries are

$$v'(y_i)y_i = v(y_i),$$

$$v'(\bar{y}_r)y_r = v(\bar{y}_r),$$

$$p'(y_i) = p'(\bar{y}_r) = 0.$$ 

By the super-contact principle:

$$v''(y_i) = v''(\bar{y}_r) = 0.$$ 

Again, we can show that

$$p''(y_i) = p''(\bar{y}_r) = 0.$$ 

Note that the targeted ICR policies satisfy these conditions. Indeed, in such policies, $\bar{y}_r \to \hat{y}$ and $y_i \to y_i$, and the function $v$ converges to a linear around $\hat{y}$. However, no other policies can satisfy these equations. Intuitively, if there were positive proportional adjustment costs, then the issuance and repurchase boundaries in general would be different. However, with zero proportional adjustment costs, there are no gains from keeping them apart.

**Optimal Time-Consistent Policy** Proposition 3 implies that solution to (5) is in the class $\hat{S}$. By Proposition 2, for targeted ICR policies it is sufficient for all credibility constraints to check that $y_b \leq y_{bm}$, hence, $\hat{\Sigma} = \arg\max_{\Sigma \in \hat{S}} \{ W(\Sigma) : y_b \leq y_{bm} \}$ is the optimal time-consistent targeted ICR debt policy. Therefore,

**Proposition 4.** Suppose that $e(y_b|\hat{\Sigma}) \geq 0$ for some $y \in (y_b, y_{bm}]$. Then, $\hat{\Sigma}$ is the optimal time-consistent policy in $S$. 

Proposition 4 provides a simple way of solving a potentially very complex problem of
Figure 2: Comparison to Commitment Solution

Parameters: $\mu = 2\%$, $r = 5\%$, $\pi = 10\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\eta = 5.6$. The solid blue lines depict values under reflecting issuance boundary at $y_1 = 1$, the dotted yellow line depicts values in the MPE, the dashed green line depicts values under the optimal time-consistent debt policy. The parameters of the optimal time-consistent policy are $\hat{y} = 2.89$, $y_c = 2.31$, $y_b = 0.289$. The parameters of the optimal policy with commitment are $\hat{y} = 1.44$, $y_c = 0.67$, $y_b = 0.396$.

finding the optimal time-consistent debt policy. Specifically, one needs to simply find the optimal targeted ICR policy such that $y_b \leq y_{bm}$, and verify that under this policy the equity value is non-negative for some $y \in (y_b, y_{bm})$. In fact, due to the closed-form solutions in Section 4.1, this problem is reduced to a single variable optimization program.\(^9\)

4.3 Effect of Credibility on Leverage

We next explore how the credibility requirement affects the firm’s leverage choice.

To do so, we first compare the optimal time-consistent policy $\Sigma^*$ to the optimal policy with commitment, which we define as follows. Suppose that the equity holders can commit to a particular issuance/repurchase policy $\Sigma$ that need not satisfy credibility constraints. Then, the optimal policy with commitment is

$$\Sigma_c \equiv \arg \max_{\Sigma \in \mathcal{S}} W(\Sigma).$$

By Proposition 3, the optimal policy with commitment also takes a form of a targeted ICR policy, yet, it will generally differ from the optimal time-consistent policy. In other words, the credibility constraints sometimes bind in the optimal time-consistent policy.

To get an insight into why the optimal policy with commitment violates the credibility constraints, we first use closed-form solutions for $w$ to find the optimal policy subject to only $y_b \leq y_{bm}$ (e.g., via grid search over $x_r, x_{*r}, x_i, x_{*i}$). If this policy is a targeted ICR policy, then Proposition 2 implies that it also satisfies all the rest of credibility constraints, and hence, is the optimal time-consistent policy. We use this algorithm to verify the optimality of the targeted ICR policy in all our specifications.

\(^9\) Note that even without Proposition 3, there is a computationally simple algorithm for finding optimal time-consistent policy. Specifically, we first use closed-form solutions for $w$ to find the optimal policy subject to only $y_b \leq y_{bm}$ (e.g., via grid search over $x_r, x_{*r}, x_i, x_{*i}$). If this policy is a targeted ICR policy, then Proposition 2 implies that it also satisfies all the rest of credibility constraints, and hence, is the optimal time-consistent policy. We use this algorithm to verify the optimality of the targeted ICR policy in all our specifications.
constraints, consider the illustration in Figure 2. The optimal policy with commitment $\Sigma^c$ differs from the optimal time-consistent policy $\Sigma^*$ in several respects. Naturally, the firm value is higher in $\Sigma^c$. Under $\Sigma^*$, the debt price at target ICR is close to the risk-free debt price due to the fact that the equity holders are committed to compensate with repurchase even very large shocks to cash flows (up to 53% drop). However, the repurchase of such a large amount is not credibility as illustrated in the first panel of Figure 2, where the equity value under $\Sigma^c$ is below that in the MPE for low ICRs. The reason is that in these states, the equity holders need to repurchase a substantial amount of debt at a very high price (which is close to the price of risk-free debt). Thus, after significant cash flow drops, the equity holders would prefer to abandon $\Sigma^c$ and switch to the MPE dynamics, which makes $\Sigma^c$ not time-consistent. In order to maintain time-consistency, in the optimal time-consistent policy $\Sigma^*$, the equity holders compensate only moderate shocks to cash flows with repurchase (up to 20%), however, have a wider range of ICRs for which they wait until the cash flows recover sufficiently so that they can get back to target.

Thus, it is repurchases that are particularly costly for the equity holders and the credibility constraints limit the maximal amount of repurchase that the equity holders can credibly promise to make. This is illustrated in Table 1. Since the equity holders do not capture any tax benefits in the MPE, while they get tax benefits from following certain policies in $\mathbb{F}$, higher tax benefits relax the credibility constraints. This leads to a higher maximal repurchase in the optimal time-consistent debt policy (23% versus 20% in the baseline) as well as higher leverage. On the contrary, a higher volatility of Brownian shocks $\sigma$ makes the equity value in the MPE higher, hence, makes the credibility constraints stricter. Thus, the maximal repurchase in the optimal time-consistent policy is lower (12% versus 20% in the baseline), and the leverage is lowered as well (14% versus 21% in the baseline).

It is interesting to note that if we focus on issuance only policies, then the commitment solution coincides with the optimal time-consistent policy, and in particular, the credibility constraints would not affect the leverage dynamics. The policy depicted in Figure 3 is the optimal issuance only policy, and it results in the equity value that is above that in the MPE for all $y \geq y_{bm}$, hence, it is time-consistent. This observation extends the result in Benzoni et al. (2019) to the case of jump-diffusion process for cash flows. The important conceptual point is that it is large repurchases that are particularly costly for equity holders and can lead to the violation of credibility constraints. For this reason, the gap between the commitment and no-commitment solutions arises in our model, but not in the model with only issuance policies or with diffusion process for cash flows.
Table 1: Effect of Credibility Constraints on Leverage

| Base case specification: \( \tau = 10\% \), \( \sigma = 25\% \), \( \lambda = 1/3 \), \( \zeta = -15\% \), \( \mu = 2\% \), \( r = 5\% \). The credit spread is computed at \( \hat{y} \) and is equal to \( 1/\hat{p} - r \). The leverage ratio is computed at \( \hat{y} \) and is equal to \( \hat{p}/(\hat{e} + \hat{p}) \). Maximal repurchase is in percentage of outstanding debt and is equal to \( (1 - y_e / \hat{y}) \).

<table>
<thead>
<tr>
<th>leverage</th>
<th>ICR target</th>
<th>credit spread</th>
<th>repurchase boundary</th>
<th>default boundary</th>
<th>MPE default boundary</th>
<th>maximal repurchase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base case</td>
<td>21%</td>
<td>2.89</td>
<td>45 bps</td>
<td>2.32</td>
<td>0.289</td>
<td>0.289</td>
</tr>
<tr>
<td>( \tau = 25% )</td>
<td>48%</td>
<td>1.17</td>
<td>100 bps</td>
<td>0.90</td>
<td>0.289</td>
<td>0.289</td>
</tr>
<tr>
<td>( \sigma = 35% )</td>
<td>14%</td>
<td>4.08</td>
<td>72 bps</td>
<td>3.60</td>
<td>0.218</td>
<td>0.218</td>
</tr>
</tbody>
</table>

Figure 3: Comparison to Issuance Only Policies

Parameters: \( \mu = 2\% \), \( r = 5\% \), \( \pi = 35\% \), \( \sigma = 25\% \), \( \lambda = 1/3 \), \( \eta = 5.6 \). The solid blue lines depicts values under optimal issuance only policy, the dotted yellow line depicts values in the MPE, the dashed green line depicts values under the optimal time-consistent debt policy. Depicted is the optimal issuance only policy characterized by \( y_i = y_i^* = 1.64 \) and \( y_b = 0.255 \).
5 Empirical Implications

In this section, we describe implications of our model for leverage dynamics, optimal debt maturity, and the effect of volatility of different shocks on leverage.

5.1 Leverage Dynamics

We first describe leverage dynamics under the optimal time-consistent debt policy. In the baseline specification, the firm issues console. The risk-free rate is \( r = 5\% \). The drift of cash-flows under the risk-neutral measure is \( \mu = 2\% \) and the volatility is \( \sigma = 25\% \). Following Graham (2000), we set tax benefits of debt to \( \pi = 10\% \). Downward Poisson jumps occur on average every three years \( (\lambda = 1/3) \) and their average size is \( 15\% \) \( (\zeta = -15\% )\).

Figure 4 illustrates the dynamics under the optimal time-consistent debt policy. There are two qualitatively very different regimes: the “normal” and the “distress” regimes. In the stable regime, the equity holders stick to the ICR target of \( \hat{y} = 2.9 \) and maintain a relatively low leverage ratio of 21\% by compensating all positive or sufficiently small negative shocks to cash flows (up to 20\% downward jumps) through the debt issuance or repurchase, respectively. Because of that, the price of debt is stable and close to the price of risk-free debt (the credit spread is 45 bps).

Large negative shocks to cash flows lead to the distress regime, where the equity holders temporarily abandon the ICR target and do not issue/repurchase debt until either the fundamentals improve or the firm goes bankrupt (see gray regions in Figure 4). As a result, the price of debt drops after the shock and becomes sensitive to cash flow shocks. Despite a lower debt price and a constant debt level, the firm’s leverage ratio jumps up and stays above the target leverage ratio of 21\%, because of the drop in the equity value as the default becomes more likely. If the cash flows recover to the ICR level of 2.3, the firm repurchases a chunk of debt and returns to the stable regime. Otherwise, default occurs when the ICR drops below 0.3.

Interestingly, the two regimes and the ICR target arise endogenously as the equity holders’ optimal credible response to the magnitude of shocks and the size of tax benefits within a rich class of policies that allow for repurchase and issuance regions with discrete as well as incremental debt adjustments. Changes in the underlying environment will affect the ICR target and the relative size of the stable and distress regions.

For example, consider the case of high tax benefits when \( \pi = 40\% \). Naturally, the leverage is much higher compared to the low tax benefits case (initial leverage ratio be-
Figure 4: Leverage dynamics for the baseline specification

Solid lines depict dynamics under the optimal time-consistent policy, dotted lines depict dynamics under DeMarzo and He (2019) policy, dashed lines depict dynamics under Leland (1994) policy. The distress regime of the optimal time-consistent policy is highlighted in gray.
Figure 5: Leverage dynamics for the high benefits specification

Solid lines depict dynamics under the optimal time-consistent policy, dotted lines depict dynamics under DeMarzo and He (2019) policy, dashed lines depict dynamics under Leland (1994) policy. The distress regime of the optimal time-consistent policy is highlighted in gray.
comes 61% compared to 21%), and correspondingly, ICR target is lower. The endogenous reaction to the shocks changes. The equity holders prefer to exit the distress regime faster by repurchasing earlier. In particular, when they are at the ICR target, they are willing to compensate large drops in cash flows with repurchase (up to 24% compared to 20% in the baseline). This explains why in Figure 5 the second distress regime is much shorter compared to the baseline case depicted in Figure 4. Interestingly, the increase in leverage outweighs these larger repurchases by the equity holders, and as a result, debt is more risky and the credit spread at the target ICR is higher (141 bps versus 45 bps in the baseline).

Comparison to DeMarzo and He (2019) and Leland (1994) Our outcome is very different from the alternative benchmark proposed by DeMarzo and He (2019). Both papers consider the environment with costless leverage adjustments. Although somewhat extreme (as firms do face issuance/repurchase costs and often embed contractual commitments into debt contract), this environment provides a natural theoretical benchmark for assessing the value added of debt covenants, state-contingencies, exogenous commitment devices, etc., and hence, the scope of their use in practice.

DeMarzo and He (2019) argue that the MPE is a reasonable outcome in this environment when no exogenous commitment devices are available, such as collateral, covenants, state contingencies, etc. As we described in Section 3.1, in the MPE, the debt holders do not believe that the equity holders can resist temptation to issue new debt in the future. This depresses the price of current debt issuance and leads to a particularly bad outcome in which the ratchet effect dissipates all surplus from the new debt issuances. When compared to this benchmark, various exogenous commitment devices have value, as they alleviate the ratchet effect.

In contrast, we argue that certain debt policies can be sustained endogenously as long as the promise to follow them is credible. This outcome leads to a significant improvement over the outcome in DeMarzo and He (2019), and hence, presents a higher bar for justification of exogenous commitment devices. In particular, if a commitment device only slightly improves on the no-credibility outcome, then in order to justify its existence one needs to argue why the outcome that it sustains cannot be sustained endogenously through the reputational type of dynamics in the present paper.

The leverage dynamics are quite different under the optimal time-consistent policy and in the MPE. In Figure 4, dotted lines correspond to dynamics in the MPE and solid
lines represent dynamics under the optimal time-consistent policy. In the MPE, the firm constantly issues debt at a speed that varies with the level of interest coverage. Hence, the debt level grows steadily over time. Because the equity holders cannot sustain any debt discipline, the price of debt is significantly depressed compared to our optimal time-consistent debt policy. The credit spread is on average 186 bps in the no-credibility outcome compared to the average credit spread of 52 bps in the optimum. Further, the debt price is sensitive to cash flow shocks in the MPE.

Interestingly, lack of credibility need not imply that the debt level is always higher in the MPE compared to the optimal time-consistent policy. Rather, in the optimum, the equity holders tailor better the debt issuance/repurchase to the economic conditions. This can be clearly seen from the evolution of the debt levels and leverage ratios in Figure 4. During most of the first year, the firm accumulates debt faster under the optimal time-consistent policy than in the MPE, because on average the cash flows grow during this period. After the first significant negative shocks to cash flows (around year 2), the leverage ratio in both outcomes jumps up. Yet, after the cash flows improve, the firm repurchases debt and lowers leverage under the optimal time-consistent policy, while the leverage stays high in the MPE. With subsequent distress periods, this divergence in the firm’s leverage continues to increase as seen in Figure 4. The ability to credibly get back to the stable regime under the optimal time-consistent debt policy is what allows the firm to gain from debt issuance when cash flows are high.

We also compare our dynamics to that under the policy proposed in Leland (1994). In Leland (1994), the firm commits not to adjust debt level after the initial issuance. As can be seen from Figure 4, if this is the case, the firm issues a lower debt level compared to the optimal time-consistent policy. Because the firm does not actively manage debt, both interest coverage and debt price are sensitive to cash flow fluctuations. Further, the leverage ratio is lower than in the optimal outcome. This extra precaution is explained by the fact that the firm cannot actively manage debt (in particular, repurchase it after negative shocks).

Finally, note that unless the credibility constraints binds (as is the case in Figure 4), the default threshold in the optimal time-consistent policy is below that in the MPE and the no-issuance policy of Leland (1994). Figure 5 demonstrates that this makes debt riskier in the MPE. Specifically, the firm defaults around year 5 in the MPE, while it

---

10 In DeMarzo and He (2019), the firm prefers to have zero leverage at $t = 0$. In the simulations, we suppose that in both optimal time-consistent outcome and no-credibility outcome, the firm starts with debt level $F_0$ as in the optimal time-consistent policy.
avoids default under the optimal policy.

5.2 Volatility of Shocks and Maturity

We next analyze how different types of shocks affect optimal time-consistent debt policy. We consider changes in the policy boundaries, leverage ratio and credit spread at ICR target as we vary the volatility of Brownian shocks $\sigma$, the size $\zeta$ and intensity $\lambda$ of Poisson downward jumps.\footnote{In our comparative statics, as we vary $\zeta$ or $\lambda$, we adjust $\mu$ so that to hold the drift of the cash flow process $\hat{\mu} = \mu + \frac{1}{\eta}$ constant.} Given that the leverage dynamics is very different in the normal and distress regions, we also consider median leverage ratio and credit spread right after a sufficiently large shock that puts the firm in the distress region but does not bankrupt it.\footnote{Leverage ratio at $y$ equals $\frac{p(y)}{c(y) + p(y)}$ and the credit spread at $y$ equals $\frac{c+\zeta}{p(y)} - \xi - r$. We consider median leverage ratio and credit spread rather than means, because the latter are not guaranteed to exist due to leverage ratio and credit spread going to infinity as $y$ approaches $y_b$.} This environment captures “fallen angels” that are recently downgraded from investment grade to speculative grade.

We report comparative statics for three scenarios: (i) the base case of 10 year bonds and tax benefits $\pi = 10\%$; (ii) the high tax benefits case of 10 year bonds and $\pi = 40\%$; and (iii) the case of console with tax benefits $\pi = 10\%$. As we discuss below, this comparison would allow us to study the effect of the binding credibility constraints and debt maturity on comparative statics.

First, consider the base case in Table 2. Higher intensity or higher expected losses from downward jumps lead to a higher ICR target $\hat{y}/c$, a lower leverage ratio and a higher credit spread at $\hat{y}$. Observe that $\hat{y}/y_b$ also increases as $\zeta$ becomes smaller/$\lambda$ becomes larger. That is, when downward jumps become larger or more frequent, it is optimal for the firm to create a larger safety buffer by lowering the leverage and targeting a higher ICR relative to the default boundary. However, this safety buffer is not sufficient to compensate for the higher risk of default, and so, credit spreads at $\hat{y}$ increase.

Interestingly, when downward jumps are more severe/frequent, lower leverage chosen by the firm also translates into a smaller spike in credit spread and leverage ratio when the firm becomes a “fallen angel.” For example, credit spread goes up on average 43 times after a downgrade for a firm with infrequent jumps ($\lambda = 1/4$) as opposed to only 9 times for a firm with higher jump frequency ($\lambda = 1/2$). Thus, the safety buffer prevents credit spreads from jumping too high in distress. The leverage jumps as well after a downgrade (roughly 2.5-3 times), despite the fact that the firm stops issuing debt. This occurs,
because of the proximity to default, which depreciates significantly equity value.

In the high tax benefits case, the effect of downward jumps is qualitatively the same as in the base case (see Table 3). Quantitatively, higher tax benefits incentivize the firm to increase leverage in the optimal time-consistent policy, which leads to higher credit spreads both at the target ICR and in distress.

The comparative statics is more complicated with respect to the volatility of Brownian shocks. In the base case in Table 2, leverage ratio is decreasing and ICR is increasing with respect to $\sigma$, while leverage ratio is increasing and ICR is decreasing with respect to $\sigma$ in the high tax benefits case in Table 3. This qualitative difference arises because the credibility constraints in the optimal time-consistent policy are slack when $\pi$ is high ($y_b < y_{bm}$ in Table 3), but some constraints are binding when $\pi$ is low ($y_b = y_{bm}$ in Table 2).

To explain the intuition for these comparative statics, consider the Leland model as a benchmark (Leland (1994)). Volatility $\sigma$ has two effects there. On the one hand, the probability that the cash flow process declines to any given bankruptcy threshold and the firm will default increases (i.e., cash flows become riskier). This effect increases expected bankruptcy costs and therefore reduces optimal borrowing. On the other hand, higher volatility reduces the default boundary due to the option value effect. Intuitively, when

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
 & at target & & & & & & \\
 & leverage ratio & credit spread & median leverage ratio & median credit spread & target ICR & repurchase ICR & default ICR \\
\hline
Base case & 27% & 23 bps & 80% & 466 bps & 1.73 & 0.76 & 0.36 \\
\hline
$\zeta = -15\%$ & 39% & 13 bps & 97% & 962 bps & 0.47 & 0.20 & 0.16 \\
\hline
$\zeta = -20\%$ & 27% & 23 bps & 80% & 466 bps & 1.73 & 0.76 & 0.36 \\
\hline
$\zeta = -25\%$ & 17% & 38 bps & 57% & 323 bps & 4.44 & 2.23 & 0.52 \\
\hline
$\lambda = 1/4$ & 31% & 19 bps & 95% & 821 bps & 0.62 & 0.21 & 0.16 \\
\hline
$\lambda = 1/3$ & 27% & 23 bps & 80% & 466 bps & 1.73 & 0.76 & 0.36 \\
\hline
$\lambda = 1/2$ & 18% & 35 bps & 55% & 326 bps & 5.54 & 3.32 & 0.69 \\
\hline
$\sigma = 10\%$ & 30% & 27 bps & 86% & 246 bps & 1.54 & 0.63 & 0.42 \\
\hline
$\sigma = 25\%$ & 27% & 23 bps & 80% & 466 bps & 1.73 & 0.76 & 0.36 \\
\hline
$\sigma = 40\%$ & 22% & 26 bps & 65% & 598 bps & 2.17 & 1.13 & 0.28 \\
\hline
\end{tabular}
\caption{Comparative statics in the base case}
\end{table}

Parameters: $\xi = 1/10$, $\pi = 10\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -20\%$, $\mu = 2\%$, $r = 5\%$.  

Table 2: Comparative statics in the base case
deciding whether to inject cash into a money-losing firm or to announce default, equity holders trade-off saving interest payments (the benefit of default) against the possibility that things improve in the future and the firm recovers (the benefit of waiting). Higher volatility increases the latter and therefore reduces the default boundary. This effect reduces expected bankruptcy costs and increases optimal borrowing. In the Leland model, the first effect dominates when volatility is low, while the second effect dominates when volatility is high (see Table II and Figure 8 in Leland (1994)).

In our model, the second effect dominates when the credibility constraints do not bind. This is the case, e.g., when tax benefits are high (see Table 3). Intuitively, when the firm is at its ICR target, a marginally higher volatility does not matter much for the firm riskiness, because the firm smooths out these shocks by repurchasing and issuing debt. Volatility only matters when the firm gets hit by a big shock that gets it into the financial distress region, which is a relatively small probability event. At the same time, the second effect is as significant as in the Leland model, because the default boundary is determined by the same trade-off as in the Leland model, when the firm is already in the financial distress region. Thus, a marginal increase in $\sigma$ increases the chances of recovering from large negative shocks, but only has a relatively small positive effect on the probability

### Table 3: Comparative statics in the high tax benefits case

<table>
<thead>
<tr>
<th>Parameters</th>
<th>at target</th>
<th>in distress</th>
<th>optimal policy</th>
<th>MPE default</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>leverage</td>
<td>credit spread</td>
<td>median leverage ratio</td>
<td>median credit spread</td>
</tr>
<tr>
<td>High benefits</td>
<td>44%</td>
<td>97 bps</td>
<td>93%</td>
<td>837 bps</td>
</tr>
<tr>
<td>$\zeta = -15%$</td>
<td>52%</td>
<td>65 bps</td>
<td>96%</td>
<td>927 bps</td>
</tr>
<tr>
<td>$\zeta = -20%$</td>
<td>44%</td>
<td>97 bps</td>
<td>93%</td>
<td>837 bps</td>
</tr>
<tr>
<td>$\zeta = -25%$</td>
<td>37%</td>
<td>141 bps</td>
<td>89%</td>
<td>7676 bps</td>
</tr>
<tr>
<td>$\lambda = 1/4$</td>
<td>46%</td>
<td>94 bps</td>
<td>94%</td>
<td>870 bps</td>
</tr>
<tr>
<td>$\lambda = 1/3$</td>
<td>44%</td>
<td>97 bps</td>
<td>93%</td>
<td>837 bps</td>
</tr>
<tr>
<td>$\lambda = 1/2$</td>
<td>40%</td>
<td>128 bps</td>
<td>84%</td>
<td>778 bps</td>
</tr>
<tr>
<td>$\sigma = 10%$</td>
<td>42%</td>
<td>94 bps</td>
<td>89%</td>
<td>313 bps</td>
</tr>
<tr>
<td>$\sigma = 25%$</td>
<td>44%</td>
<td>97 bps</td>
<td>93%</td>
<td>837 bps</td>
</tr>
<tr>
<td>$\sigma = 40%$</td>
<td>46%</td>
<td>110 bps</td>
<td>93%</td>
<td>1257 bps</td>
</tr>
</tbody>
</table>

Parameters: $\xi = 0.1$, $\pi = 40\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -20\%$, $\mu = 2\%$, $r = 5\%$. 

Table 3: Comparative statics in the high tax benefits case
Table 4: Comparative statics in the case of console
Parameters: $\xi = 0$, $\pi = 10\%$, $c = 8\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -20\%$, $\mu = 2\%$, $r = 5\%$.

with which the cash flow process hits any given lower threshold. Consequently, the equity holders expand the turbulent region (ratio $y_r/y_b$ increases with $\sigma$) to increase their chances of escaping from it and borrow more. This explains why the leverage ratio increases in volatility $\sigma$ whenever the credibility constraints do not bind.

The comparative statics is the opposite once the credibility constraints start to bind (see Table 2). An increase in volatility increases the value of equity upon deviation from the debt policy and makes it harder to satisfy the credibility constraints. Thus, in this case the maximal repurchase amount must be sufficiently small so that the the equity holders’ promise to do such a repurchase is credible. In Table 2, as $\sigma$ increases from 10% to 40%, the maximal shock that the equity holders compensate with repurchase goes down from 59% to 48%. This makes debt more risky and the firm borrows less, despite the fact that the equity holders expand the distress region (ratio $y_r/y_b$ increases with $\sigma$) to increase their chances of escaping default while in the distress region.

Finally, we consider the effect of longer maturity. Table 4 reports comparative statics for the case of console debt. This case is qualitatively similar to the base case, because the credibility constraints bind at the optimal time-consistent debt policy. The quantitative difference is that the leverage ratios are lower, while the credit spreads are higher when the firm issues debt of longer maturity.
Figure 6: Firm Value (solid line) and Initial Debt Issued (dashed line) as a Function of Maturity

Parameters: $\mu = 2\%$, $r = 5\%$, $\pi = 10\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -15\%$.

The intuition for this is as follows. Shorter maturity debt serves as a commitment device to reduce debt burden. While this commitment does not affect the firm much in normal times, because it issues new debt to stay at the target ICR, in distress times, this commitment device is valuable as it allow the firm to reduce the likelihood of bankruptcy in states where large repurchase promises are not credible. Thus, the firm can borrow more in debt of shorter maturities. Note that this commitment device also makes debt more risky in distress when the firm issues short-term debt: median credit spread is 251 bps for console compared to 466 bps for 10-year bonds.

5.3 Optimal Maturity

We next analyze the optimal debt maturity. Suppose that the equity holders choose debt maturity $\xi$ and the optimal time-consistent targeted ICR policy $\hat{\Sigma}$ at $t = 0$. Following Leland and Toft (1996), coupon $c$ is set so that the debt is priced at part (i.e., $\hat{p} = 1$). We suppose that the same coupon is applied in computing the MPE of the debt issuance game.

Figure 6 depicts the firm value as a function of maturity in the baseline specification. The optimal maturity is interior and equals (approximately) 5 years. The optimal maturity balanced to forces. On the one hand, principal payments are not tax deductible, and hence, as in Leland and Toft (1996), debt with longer maturity allows the equity holders to better exploit tax benefits (for comparison, the optimal maturity in Leland and Toft (1996) is infinite). On the other hand, shorter maturity allows the equity holders to reduce
Table 5: The Effect of Debt Maturity on Optimal Leverage and Debt Policy

Parameters: $\mu = 2\%$, $r = 5\%$, $\pi = 10\%$, $\sigma = 25\%$, $\lambda = 1/3$, $\zeta = -15\%$

<table>
<thead>
<tr>
<th>maturity</th>
<th>coupon</th>
<th>firm value</th>
<th>optimal leverage</th>
<th>target ICR</th>
<th>default ICR</th>
<th>MPE default ICR</th>
</tr>
</thead>
<tbody>
<tr>
<td>console</td>
<td>5.45%</td>
<td>30.80</td>
<td>21%</td>
<td>2.88</td>
<td>0.289</td>
<td>0.289</td>
</tr>
<tr>
<td>30 years</td>
<td>5.21%</td>
<td>31.22</td>
<td>29%</td>
<td>2.13</td>
<td>0.386</td>
<td>0.386</td>
</tr>
<tr>
<td>10 years</td>
<td>5.12%</td>
<td>31.60</td>
<td>36%</td>
<td>1.72</td>
<td>0.486</td>
<td>0.486</td>
</tr>
<tr>
<td>5 years</td>
<td>5.11%</td>
<td>31.66</td>
<td>37%</td>
<td>1.67</td>
<td>0.548</td>
<td>0.548</td>
</tr>
<tr>
<td>1 year</td>
<td>5.10%</td>
<td>31.62</td>
<td>36%</td>
<td>1.71</td>
<td>0.607</td>
<td>0.627</td>
</tr>
<tr>
<td>6 months</td>
<td>5.10%</td>
<td>31.61</td>
<td>36%</td>
<td>1.72</td>
<td>0.614</td>
<td>0.640</td>
</tr>
</tbody>
</table>

The likelihood of default while in the turbulent region. In the turbulent region, repurchase promises are not credible, and hence, the equity holders have to restrain themselves from active debt management. Shorter debt maturity allows the equity holders to commit to reduce the debt burden in the turbulent region.

Figure 6 also shows that at the optimal maturity, the firm issues close to maximal amount of debt. Intuitively, at longer maturities the debt is too risky and the equity holders decide to reduce leverage. Debt of shorter maturities entails smaller the tax benefits, and hence, it is optimal for the firm to issue less debt. Table 5 demonstrates this relationship between debt maturity and optimal firm leverage. Interestingly, despite the fact that the leverage is reduced for longer maturities, the debt is still more risky, which is reflected in a higher coupon.

Further, Table 5 shows default thresholds for the optimal time-consistent targeted ICR policy and the MPE. The credibility constraints bind for longer maturities, but not for shorter maturities. To see why, recall that the equity value in the MPE is the same as if the firm did not issue any new debt after the initial issuance. The faster the debt matures, the smaller tax benefits from the initial debt issuance captured by the equity holders. Thus, the equity value in the MPE is increasing with debt maturity, and so, the credibility constraints are more stringent for longer maturity debt.

We next consider the effect of volatility of different shocks on the optimal debt maturity. Volatility of Brownian shocks affects significantly only the debt maturity but not other characteristics of the policy. When Brownian component of the cash flow process is more volatile, the firm shortens debt maturity, but does not change significantly the ICR.
### Table 6: Comparative Statics of Optimal Maturity with respect to Shock Volatility

In comparative statics with respect to $\zeta$ and $\lambda$, parameter $\mu$ is adjusted to keep the drift of cash flows $\hat{\mu}$ at the level in the base case. The credit spread is computed at $\hat{y}$ and is equal to $c - r$. The leverage ratio is computed at $\hat{y}$ and is equal to $\hat{p}/(\hat{e} + \hat{p})$.

<table>
<thead>
<tr>
<th>maturity</th>
<th>leverage ratio</th>
<th>credit spread at target</th>
<th>target ICR</th>
<th>repurchase boundary ICR</th>
<th>default boundary ICR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base case $\tau = 10%$</td>
<td>4.9 years</td>
<td>37%</td>
<td>11 bps</td>
<td>1.67</td>
<td>0.70</td>
</tr>
<tr>
<td>$\sigma = 5%$</td>
<td>13.2 years</td>
<td>37%</td>
<td>11 bps</td>
<td>1.67</td>
<td>0.73</td>
</tr>
<tr>
<td>$\sigma = 25%$</td>
<td>4.9 years</td>
<td>37%</td>
<td>11 bps</td>
<td>1.67</td>
<td>0.70</td>
</tr>
<tr>
<td>$\sigma = 40%$</td>
<td>2.3 years</td>
<td>37%</td>
<td>11 bps</td>
<td>1.66</td>
<td>0.69</td>
</tr>
<tr>
<td>$\zeta = -10%$</td>
<td>26.3 years</td>
<td>53%</td>
<td>7 bps</td>
<td>0.45</td>
<td>0.26</td>
</tr>
<tr>
<td>$\zeta = -15%$</td>
<td>4.9 years</td>
<td>37%</td>
<td>11 bps</td>
<td>1.67</td>
<td>0.70</td>
</tr>
<tr>
<td>$\zeta = -20%$</td>
<td>2.4 years</td>
<td>26%</td>
<td>15 bps</td>
<td>3.79</td>
<td>1.20</td>
</tr>
<tr>
<td>$\lambda = 1/4$</td>
<td>11.4 years</td>
<td>40%</td>
<td>11 bps</td>
<td>0.86</td>
<td>0.39</td>
</tr>
<tr>
<td>$\lambda = 1/3$</td>
<td>4.9 years</td>
<td>37%</td>
<td>11 bps</td>
<td>1.67</td>
<td>0.70</td>
</tr>
<tr>
<td>$\lambda = 1/2$</td>
<td>2.2 years</td>
<td>34%</td>
<td>10 bps</td>
<td>3.44</td>
<td>1.31</td>
</tr>
</tbody>
</table>

(a) Low Tax Benefits $\tau = 10\%$

<table>
<thead>
<tr>
<th>maturity</th>
<th>leverage ratio</th>
<th>credit spread at target</th>
<th>target ICR</th>
<th>repurchase boundary ICR</th>
<th>default boundary ICR</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Benefits $\tau = 25%$</td>
<td>22 year</td>
<td>50%</td>
<td>53 bps</td>
<td>1.21</td>
<td>0.74</td>
</tr>
<tr>
<td>$\sigma = 5%$</td>
<td>22 year</td>
<td>50%</td>
<td>53 bps</td>
<td>1.21</td>
<td>0.74</td>
</tr>
<tr>
<td>$\sigma = 25%$</td>
<td>8.6 years</td>
<td>48%</td>
<td>44 bps</td>
<td>1.26</td>
<td>0.73</td>
</tr>
<tr>
<td>$\zeta = -10%$</td>
<td>console</td>
<td>63%</td>
<td>26 bps</td>
<td>0.29</td>
<td>0.17</td>
</tr>
<tr>
<td>$\zeta = -15%$</td>
<td>22 year</td>
<td>50%</td>
<td>53 bps</td>
<td>1.21</td>
<td>0.74</td>
</tr>
<tr>
<td>$\zeta = -20%$</td>
<td>console</td>
<td>36%</td>
<td>181 bps</td>
<td>2.33</td>
<td>1.92</td>
</tr>
<tr>
<td>$\lambda = 1/5$</td>
<td>console</td>
<td>55%</td>
<td>43 bps</td>
<td>0.18</td>
<td>0.11</td>
</tr>
<tr>
<td>$\lambda = 1/3$</td>
<td>22 year</td>
<td>50%</td>
<td>53 bps</td>
<td>1.21</td>
<td>0.74</td>
</tr>
<tr>
<td>$\lambda = 1$</td>
<td>1.6 years</td>
<td>37%</td>
<td>33 bps</td>
<td>8.57</td>
<td>3.73</td>
</tr>
</tbody>
</table>

(b) High Tax Benefits $\tau = 25\%$
target or the leverage ratio. As a result, the credit spread remains virtually unchanged. These effects hold for both high and low tax benefits (Table 6a and 6b).

At the same time, Poisson shocks affect both debt maturity and the leverage ratio. We first consider the baseline case of small tax benefits in Table 6a. When downward jumps are more frequent or more severe, maturity of debt shortens, the leverage is reduced, and the ICR target increases. As a result, credit spreads are not affected significantly by the volatility of shocks.

The effects of the frequency of Poisson shocks is amplified when the tax benefits are larger and the firm borrows more (see Table 6b). Interestingly, the effect of the size of downward shocks is non-monotonic. When the size of downward shocks is moderate, larger shock size shorten the maturity and reduce the leverage ratio (shift from $\zeta = -10\%$ to $\zeta = -15\%$). However, when downward shocks are sufficiently large, it is optimal for the firm to issue console and significantly reduce leverage and increase ICR target. Nevertheless, the credit spread increases significantly (more than three times as $\zeta$ drops from $-15\%$ to $-20\%$).

6 Conclusion

This paper introduces two innovations into the standard trade-off theory of capital structure. We allow for downward jumps in cash flows and consider the optimal time-consistent policy in the environment where the equity holders can costlessly issue/repurchase debt. The combination of these two new assumptions allows us to obtain a realistic leverage dynamics. In these dynamics, the firm can be in one of two regimes – stable or distress, – which are qualitatively very different. In the stable regime, the firm manages its leverage ratio to a target. The credit spread is small and the debt price does not respond to shocks to fundamentals. In contrast, in the distress regime, which occur after particularly large negative cash flow shocks, the firm does not issue/repurchase debt. Credit spreads are high with the debt price being sensitive to cash flows shocks. These dynamics captures the difference between investment- and speculative-grade bonds as well as how a firm can transition between the two regimes.

We show that credibility constraints often bind and taking them into account affects the comparative statics. While more severe/frequent jumps tend to reduce leverage, the volatility of Brownian shocks increases leverage when credibility constraints do not bind, but it decreases leverage when they are binding. The optimal maturity is generally interior and it balances the commitment value of short-term debt and the higher tax benefits of
long-term debt.

References


DeMarzo, P. and He, Z.: 2019, Leverage dynamics without commitment.


### A Appendix

#### A.1 Omitted Proofs

**Proof of Proposition 1.** The argument for (25) is provided in the text, and we are left to prove that $E_m$ is strictly convex in $F$. Consider $y_{bm} < y' < y''$. For any sequence of shocks $dZ_t$, $dN_t$, and $dS_t$, the process $y_t$ that starts at $y''$ is always higher than the process that starts at $y'$. Thus, when the process $y_t$ starts at $y''$, $\tau_{bm}$ occurs later than when it starts at $y'$, which implies that

$$
E \left[ \int_t^{\tau_{bm} \wedge \tau_m} e^{-r(s-t)} ds + 1\{\tau_m \leq \tau_{bm}\} \right| y_t = y', \Sigma^0] < E \left[ \int_t^{\tau_{bm} \wedge \tau_m} e^{-r(s-t)} ds + 1\{\tau_m \leq \tau_{bm}\} \right| y_t = y'', \Sigma^0].
$$

Given (27), $p_m(y') < p_m(y'')$. Thus, $p_m'(y) > 0$ for $y > y_{bm}$. Finally,

$$
\frac{\partial^2}{\partial F^2} E_m(Y, F) = \frac{\partial}{\partial F} \left\{ e_m(y) - e_m'(y)y \right\} = e_m''(y) \frac{y^2}{F} = p_m'(y) \frac{y}{F} > 0,
$$
where we used $p'_m(y) = ye''_m(y)$ in the last equality. This completes the proof of strict convexity of $E_m$ in $F$. \hfill \Box

**Proof of Proposition 2.** Fix some targeted ICR policy. It follows immediately from (25) that $y_b \leq y_{bm}$ and $e(\bar{y}) \geq e_m(\bar{y})$ are necessary conditions for the incentive constraints to be satisfied. To show that they are also sufficient, we first show that if $y_b \leq y_{bm}$, then $e(y) \geq e_m(y)$ for all $y \in (y_{bm}, y_r]$. We will show the contrapositive that if $e(\bar{y}) < e_m(\bar{y})$ for some $\bar{y} \in (y_{bm}, y_r]$, then $y_b > y_{bm}$. Suppose to contradiction that $e(\bar{y}) < e_m(\bar{y})$, but $y_b \leq y_{bm}$. Let the stopping time $\bar{\tau}$ be the first time when the state $y_t$ reaches $\bar{y}$ when the equity holders do not issue or repurchase new debt for $y_t \in (y_b, \bar{y})$. For all $y \in (y_b, \bar{y})$,

$$
e(y) = \mathbb{E} \left[ \int_{\bar{\tau}}^{\tau_b \wedge \bar{\tau}} e^{-r(s-t)}(1 - \pi)(y_s - c)ds - \xi ds + 1\{\bar{\tau} < \tau_b\}e^{-r(\bar{\tau}-t)}e(\bar{y}) \mid y_t = y \right]$$

$$< \mathbb{E} \left[ \int_{\bar{\tau}}^{\tau_b \wedge \bar{\tau}} e^{-r(s-t)}(1 - \pi)(y_s - c)ds - \xi ds + 1\{\bar{\tau} < \tau_b\}e^{-r(\bar{\tau}-t)}e_m(\bar{y}) \mid y_t = y \right]$$

$$\leq \mathbb{E} \left[ \int_{\bar{\tau}}^{\tau_b \wedge \bar{\tau}} e^{-r(s-t)}(1 - \pi)(y_s - c)ds - \xi ds + 1\{\bar{\tau} < \tau_{bm}\}e^{-r(\bar{\tau}-t)}e_m(\bar{y}) \mid y_t = y \right]$$

$$= e_m(y).$$

The first equality is by the fact that the equity holders do not issue any debt until the state reaches $\bar{y}$ when they follow the targeted ICR policy. The first inequality is by the fact that $e(\bar{y}) < e_m(\bar{y})$ and the event $\bar{\tau} < \tau_b$ has a positive probability starting from any state $y \in (y_b, \bar{y})$. The second inequality is by the fact that if the equity holders do not issue any debt after the initial issuance, then the equity value at state $\bar{y}$ equals $e_m(\bar{y})$ and the stopping time $\tau_{bm}$ is the optimal default time for the equity holders, and in particular, is weakly preferred to the stopping time $\tau_b$. The last equality is by the fact that $e_m$ gives the value of equity when the equity holders do not issue any debt for $t > 0$. Therefore, $e(y) < e_m(y)$ for all $y \in (y_b, \bar{y})$. Since $e_m(y) = 0$ for all $y \in (y_b, y_{bm}]$, $e(y) < 0$ for all $y \in (y_b, y_{bm}]$, which contradicts the fact that $e(y_b) = 0$. Thus, it must be that $y_b > y_{bm}$, which is the desired conclusion.

We are left to show that the incentive-compatibility constraints (25) are satisfied for all $y \in (y_r, \bar{y}]$. Since for all $y \in (y_r, \bar{y}]$, $e(y)$ is a convex combination of $e(y_r)$ and $e(\bar{y})$, which are above $e_m(y_r)$ and $e_m(\bar{y})$, respectively. Since function $e_m$ is convex (by Proposition 1), $e(y) \geq e_m(y)$ for $y \in (y_r, \bar{y}]$, which concludes the proof. \hfill \Box

**Proof of Proposition 3.** It is convenient to define parameters of the debt policy relative to the default boundary: $x_i \equiv y_i/y_0$, $x_i^* \equiv y_i^*/y_0$, $x_r \equiv y_r/y_0$, $x_r^* \equiv y_r^*/y_0$, $x_r \equiv y_r/y_0$. Let

$$\tilde{w}(x) = \frac{ce}{(r + \xi)x} + \frac{1}{x} \sum_{k=1}^{3} A_k x^{-\gamma_k}.$$ 

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Note that $w(y) = \phi + \frac{1}{y_b} \tilde{w}(y/y_b)$ for $y \in [y_r, y_i]$ and the solution to program (33) can be obtained by solving

$$\max_{x \in X} \{ \tilde{w}(x) : y_b = y_{b0} \},$$

where $x = (x_r, x_i, x^*_r, x^*_i, x_r^* r)$ and $X$ is the set of all admissible $x$s. Using the fact that $a_k = c_k - b_k/r$ and $A_k = C_k - B_k/r$ together with conditions (15), (17) – (19), (21), and (22) on function $e$ and conditions (9) – (13) on function $p$ (see explicit formulas for those conditions in Online Appendix B.2), we can determine coefficients $a_k$s and $A_k$s from the following system of equations:

$$\phi y_{b0} + \frac{c\pi}{r + \xi} + \sum_{k=1}^{3} a_k = 0,$$

$$\frac{\phi \eta}{\eta + 1} y_{b0} + \frac{c\pi}{r + \xi} + \sum_{k=1}^{3} a_k \frac{\eta}{\eta - \gamma_k} = 0,$$

$$\frac{c\pi}{(r + \xi)x_r} + \frac{1}{x_r} \sum_{k=1}^{3} a_k x_r^{-\gamma_k} = \frac{c\pi}{(r + \xi)x_r} + \frac{1}{x_r} \sum_{k=1}^{3} A_k x_r^{-\gamma_k},$$

$$\frac{c\pi}{(r + \xi)x_i} + \frac{1}{x_i} \sum_{k=1}^{3} A_k x_i^{-\gamma_k} = \frac{c\pi}{(r + \xi)x_i} + \frac{1}{x_i} \sum_{k=1}^{3} A_k x_i^{-\gamma_k},$$

$$\sum_{k=1}^{3} \frac{\eta(1 + \gamma_k)}{\eta - \gamma_k} \left[ a_k x_r^{\eta - \gamma_k} - A_k x_r^{\eta - \gamma_k} \right] = \frac{c\pi}{r + \xi} (x_r^\eta - x_r^\eta).$$

Coefficients $b_k$s and $B_k$s are determined from

$$\sum_{k=1}^{3} b_k = -1, \quad \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} = 0,$$

$$\sum_{k=1}^{3} b_k x_r^{-\gamma_k} = \sum_{k=1}^{3} B_k x_r^{-\gamma_k},$$

$$\sum_{k=1}^{3} B_k (x_r^{-\gamma_k} - x_r^{\eta - \gamma_k}) = 0,$$

$$\sum_{k=1}^{3} B_k (x_i^{-\gamma_k} - x_i^{\eta - \gamma_k}) = 0,$$

$$\sum_{k=1}^{3} b_k \frac{\gamma_k}{\eta - \gamma_k} x_r^{\eta - \gamma_k} - \sum_{k=1}^{3} B_k \frac{\gamma_k}{\eta - \gamma_k} x_r^{\eta - \gamma_k} = 0.$$
The remaining condition is the smooth-pasting condition at the default boundary $e'(y_{b0}) = 0$, which becomes
\[
\phi y_{b0} - \sum_{k=1}^{3} \left( a_k - b_k \frac{c + \xi}{r + \xi} \right) \gamma_k = 0.
\]
(46)

Suppose that boundaries $y_i$ and $y_r$ are non-reflecting. We will show that then, they can be replaced by reflecting boundaries without any changes in the firm value. By the smooth-pasting principle, in the optimal $x$,
\[
\bar{w}'(x_i) = \bar{w}'(x_i^*) = \bar{w}'(\pi_r) = \bar{w}'(x_r^*) = 0.
\]
(47)

Differentiating equations (34) – (39) with respect to $x_i$ and taking into account that $y_{b0}$ is held fixed:
\[
\begin{align*}
\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} &= 0, \\
\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} \frac{\eta}{\eta - \gamma_k} &= 0, \\
\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} x_r^{\gamma_k-1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} \tilde{r}^{\gamma_k-1} &= 0, \\
\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} \left( x_r^{\gamma_k-1} - \tilde{r}^{\gamma_k-1} \right) &= 0, \\
\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} \left( x_i^{\gamma_k-1} - x_i^{\gamma_k-1} \right) &= - \frac{c\pi}{(r + \xi)x_i^2} - \sum_{k=1}^{3} A_k (\gamma_k + 1)x_i^{\gamma_k-2}, \\
\sum_{k=1}^{3} \frac{\partial B_k}{\partial x_i} \left[ \frac{\partial a_k}{\partial x_i} x_i^{\eta-\gamma_k} - \frac{\partial A_k}{\partial x_i} \tilde{r}^{\eta-\gamma_k} \right] &= 0.
\end{align*}
\]

Note that $w'(x_i) = - \frac{c\pi}{(r + \xi)x_i^2} - \sum_{k=1}^{3} A_k (\gamma_k + 1)x_i^{\gamma_k-2}$. Thus, using (47), $\frac{\partial a_k}{\partial x_i} = \frac{\partial A_k}{\partial x_i} = 0, k = 1, 2, 3$. Differentiating equations (40) – (43), (45), and (46) with respect to $x_i$, we get that $\frac{\partial b_k}{\partial x_i} = \frac{\partial B_k}{\partial x_i} = 0, k = 1, 2, 3$. Differentiating equation (44) with respect to $x_i$, we get that
\[
\begin{align*}
\sum_{k=1}^{3} \frac{\partial B_k}{\partial x_i} \left( x_i^{\gamma_k-1} - x_i^{\gamma_k-1} \right) - \sum_{k=1}^{3} \gamma_k B_k x_i^{\gamma_k-1} &= 0.
\end{align*}
\]

Thus,
\[
\begin{align*}
p'(y_i) &= - \frac{c + \xi}{(r + \xi)y_{b0}} \sum_{k=1}^{3} \gamma_k B_k (y_i/y_{b0})^{\gamma_k-1} = 0.
\end{align*}
\]

By the analogous argument, we get that $p'(y_i^*) = p'(y_r^*) = 0$.

The argument is slightly different for $\pi_r$. Differentiating equations (34) – (38), (40) – (41),
and (46) with respect to $\pi_r$:

\[
\begin{align*}
\sum_{k=1}^{3} \frac{\partial a_k}{\partial \pi_r} &= 0, \\
\sum_{k=1}^{3} \frac{\partial a_k}{\partial \pi_r} \frac{\eta}{\eta - \gamma_k} &= 0, \\
\sum_{k=1}^{3} \frac{\partial a_k}{\partial \pi_r} x_r^{-\gamma_k - 1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial \pi_r} \pi_r^{-\gamma_k - 1} &= \bar{w}'(\pi_r), \\
\sum_{k=1}^{3} \frac{\partial A_k}{\partial \pi_r} (x_r^{s-\gamma_k - 1} - \pi_r^{-\gamma_k}) &= \bar{w}'(\pi_r), \\
\sum_{k=1}^{3} \frac{\partial A_k}{\partial \pi_r} (x_i^{s-\gamma_k - 1} - x_i^{-\gamma_k - 1}) &= 0, \\
\sum_{k=1}^{3} \frac{\partial b_k}{\partial \pi_r} &= -1, \\
1 + \sum_{k=1}^{3} \frac{\partial b_k}{\partial \pi_r} \frac{\eta}{\eta - \gamma_k} &= 0, \\
\sum_{k=1}^{3} \left( \frac{\partial a_k}{\partial \pi_r} \frac{c + \xi \partial b_k}{r + \xi \partial \pi_r} \right) \gamma_k &= 0.
\end{align*}
\]

Using (47), $\frac{\partial a_k}{\partial \pi_r} = \frac{\partial b_k}{\partial \pi_r} = \frac{\partial A_k}{\partial \pi_r} = 0$, $k = 1, 2, 3$. Differentiating (42) – (45) with respect to $\pi_r$ and using $\frac{\partial b_k}{\partial \pi_r} = 0$, $k = 1, 2, 3$, we get

\[
\begin{align*}
\sum_{k=1}^{3} \frac{\partial B_k}{\partial \pi_r} x_r^{s-\gamma_k} &= \sum_{k=1}^{3} \frac{\partial b_k}{\partial \pi_r} x_r^{-\gamma_k} = 0, \\
\sum_{k=1}^{3} \frac{\partial B_k}{\partial \pi_r} \pi_r^{-\gamma_k} + p'(\bar{y}_r) \bar{y}_b \frac{r + \xi}{c + \xi} &= \sum_{k=1}^{3} \frac{\partial b_k}{\partial \pi_r} x_r^{-\gamma_k} = 0, \\
\sum_{k=1}^{3} \frac{\partial B_k}{\partial \pi_r} (x_i^{s-\gamma_k} - x_i^{* - \gamma_k}) &= 0, \\
\sum_{k=1}^{3} \frac{\partial B_k}{\partial \pi_r} \frac{\gamma_k}{\eta - \gamma_k} - \sum_{k=1}^{3} B_k \gamma_k \pi_r^{\eta - \gamma_k} - \sum_{k=1}^{3} \frac{\partial b_k}{\partial \pi_r} \frac{\gamma_k}{\eta - \gamma_k} x_r^{\eta - \gamma_k} &= 0.
\end{align*}
\]

Note that $p'(\bar{y}_r) = - \frac{c + \xi}{(r + \xi)\bar{y}_0} \sum_{k=1}^{3} B_k \gamma_k \pi_r^{\gamma_k - 1}$. Thus, inverting this system, we get that $\frac{\partial B_k}{\partial \pi_r} = 0$, $k = 1, 2, 3$, and $p'(\bar{y}_r) = 0$. Therefore,

\[
p'(y_i) = p'(y_i^*) = p'(y_r^*) = p'(\bar{y}_r) = 0.
\]
By \( w'(y_i) = p'(y_i) = 0 \), function \( w \) would not change if we set reflecting issuance boundary at \( y_i \). Thus, we can focus on reflecting issuance boundaries with \( y_i = y_i^* \). Similarly, by \( w'(\overline{y}_r) = p'(\overline{y}_r) = 0 \), function \( w \) would not change if we set reflecting repurchase boundary at \( \overline{y}_r \).

For reflecting repurchase/issuance boundaries at \( \overline{y}_r \) and \( y_i \), conditions (37) and (38) on \( A_k \)s are replaced by

\[
\frac{c\pi}{r + \xi} + \sum_{k=1}^{3} A_k (\gamma_k + 1) \overline{x}_r^{-\gamma_k} = 0, \tag{48}
\]

\[
\frac{c\pi}{r + \xi} + \sum_{k=1}^{3} A_k (\gamma_k + 1) x_i^{-\gamma_k} = 0, \tag{49}
\]

and conditions (43) and (44) on \( B_k \)s are replaced by

\[
\sum_{k=1}^{3} \gamma_k B_k \overline{x}_r^{-\gamma_k} = 0, \tag{50}
\]

\[
\sum_{k=1}^{3} \gamma_k B_k x_i^{-\gamma_k} = 0. \tag{51}
\]

By the super-contact principle, in the optimal \( x \),

\[
\tilde{w}''(x_i) = \tilde{w}''(\overline{x}_r) = 0. \tag{52}
\]

Note that

\[
\tilde{w}''(x_i) = \frac{2c\pi}{(r + \xi)x_i^3} + \sum_{k=1}^{3} (\gamma_k + 1)(\gamma_k + 2) A_k x_i^{-\gamma_k - 3}
\]

\[
= \frac{2}{x_i^3} \left( \frac{c\pi}{r + \xi} + \sum_{k=1}^{3} (\gamma_k + 1) A_k x_i^{-\gamma_k} \right) + \sum_{k=1}^{3} (\gamma_k + 1) \gamma_k A_k x_i^{-\gamma_k - 3}
\]

\[
= \sum_{k=1}^{3} (\gamma_k + 1) \gamma_k A_k x_i^{-\gamma_k - 3},
\]

where we used (49) in the last line. Differentiating equations (34) – (36), (39), and (48) – (49)
with respect to $x_i$:

$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} = 0,$$

$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} \frac{\eta}{\eta - \gamma_k} = 0,$$

$$\sum_{k=1}^{3} \frac{\partial a_k}{\partial x_i} x_i^{-\gamma_k - 1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} x_i^{-\gamma_k - 1} = 0,$$

$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} (\gamma_k + 1) x_i^{-\gamma_k} = 0,$$

$$\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_i} (\gamma_k + 1) x_i^{-\gamma_k} = \sum_{k=1}^{3} A_k (\gamma_k + 1) x_i^{-\gamma_k - 1} = \tilde{w}''(x_i) x_i^2,$$

$$\sum_{k=1}^{3} \eta (1 + \gamma_k) \left[ \frac{\partial a_k}{\partial x_i} x_i^{-\gamma_k} - \frac{\partial A_k}{\partial x_i} x_i^{-\gamma_k} \right] = 0.$$

Thus, by the super-contact principle, $\frac{\partial a_k}{\partial x_i} = \frac{\partial A_k}{\partial x_i} = 0$, $k = 1, 2, 3$. Differentiating equations (40) – (42), (50), (45), and (46) with respect to $x_i$, we get that $\frac{\partial b_k}{\partial x_i} = \frac{\partial B_k}{\partial x_i} = 0$, $k = 1, 2, 3$. Note that

$$p''(y_i) = \frac{c + \xi}{(r + \xi) y_0^2} \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) B_k x_i^{-\gamma_k - 2} = \frac{c + \xi}{(r + \xi) y_0^2} \sum_{k=1}^{3} \gamma_k^2 B_k x_i^{-\gamma_k - 2},$$

where we used (51) to get the last equality. Differentiating equation (51), we get

$$\sum_{k=1}^{3} \gamma_k \frac{\partial B_k}{\partial x_i} x_i^{-\gamma_k} - \sum_{k=1}^{3} \gamma_k^2 B_k x_i^{-\gamma_k - 1} = 0.$$

Therefore, $p''(y_i) = 0$. 

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Next, differentiating equations (34) – (36), (39), and (48) – (49) with respect to \( \pi_r \):

\[
\sum_{k=1}^{3} \frac{\partial a_k}{\partial \pi_r} = 0, \\
\sum_{k=1}^{3} \frac{\partial a_k}{\partial \pi_r} \eta - \gamma_k = 0, \\
\sum_{k=1}^{3} \frac{\partial a_k}{\partial \pi_r} x_i^{\gamma_k - 1} - \sum_{k=1}^{3} \frac{\partial A_k}{\partial \pi_r} x_i^{\gamma_k - 1} = - \sum_{k=1}^{3} A_k (\gamma_k + 1) x_i^{\gamma_k - 2}, \\
\sum_{k=1}^{3} \eta (1 + \gamma_k) \left[ \frac{\partial a_k}{\partial \pi_r} x_i^{\gamma_k - 1} - \frac{\partial A_k}{\partial \pi_r} x_i^{\gamma_k - 1} \right] = \frac{c + \xi}{r + \xi} x_i^{\gamma_k - 1} + \sum_{k=1}^{3} \eta (1 + \gamma_k) A_k x_i^{\gamma_k - 1}, \\
\sum_{k=1}^{3} \frac{\partial A_k}{\partial \pi_r} (\gamma_k + 1) x_i^{\gamma_k - 1} = 0, \\
\sum_{k=1}^{3} \frac{\partial A_k}{\partial \pi_r} (\gamma_k + 1) x_i^{\gamma_k - 1} - \sum_{k=1}^{3} A_k \gamma_k (\gamma_k + 1) x_i^{\gamma_k - 1} = 0.
\]

Thus, using the super-contact principle and (48), we get that the right-hand side of these equations are zero. Hence, \( \frac{\partial a_k}{\partial \pi_r} = \frac{\partial A_k}{\partial \pi_r} = 0, k = 1, 2, 3 \). Differentiating equations (40) – (42), (51), (45), and (46) with respect to \( \pi_r \), we get that \( \frac{\partial b_k}{\partial x_i} = \frac{\partial B_k}{\partial x_i} = 0, k = 1, 2, 3 \). Note that

\[
p''(\bar{y}_r) = \frac{c + \xi}{(r + \xi)y_6} \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) B_k \pi_r^{\gamma_k - 2} = \frac{c + \xi}{(r + \xi)y_6} \sum_{k=1}^{3} \gamma_k^2 B_k \pi_r^{\gamma_k - 2},
\]

where we used (50) to get the last equality. Differentiating equation (50), we get

\[
\sum_{k=1}^{3} \gamma_k \frac{\partial B_k}{\partial \pi_r} \pi_r^{\gamma_k - 2} - \sum_{k=1}^{3} \gamma_k^2 B_k \pi_r^{\gamma_k - 1} = 0.
\]

Therefore, \( p''(\bar{y}_r) = 0 \).

To sum up, both \( x_i \) and \( \pi_r \) satisfy the following equations in \( X \):

\[
\sum_{k=1}^{3} \gamma_k \frac{\partial B_k}{\partial \pi_r} \pi_r^{\gamma_k - 2} - \sum_{k=1}^{3} \gamma_k^2 B_k \pi_r^{\gamma_k - 1} = 0.
\]
\[ w'(X) = 0 : \sum_{k=1}^{3} A_k (\gamma_k + 1) X^{-\gamma_k} = -\frac{c\pi}{r + \xi}, \]
\[ w''(X) = 0 : \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) A_k X^{-\gamma_k} = 0, \]
\[ p'(X\bar{y}_b) = 0 : \sum_{k=1}^{3} \gamma_k B_k X^{-\gamma_k} = 0, \]
\[ p''(X\bar{y}_b) = 0 : \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) B_k X^{-\gamma_k} = 0. \]

Note that \((X^{-\gamma_1}, X^{-\gamma_2}, X^{-\gamma_3})\) can be determined from either first three equations or first two equations and the last. Since both \(x_i\) and \(x_r\) satisfy these equations, it must be that both systems do not have full rank. Thus, \(\frac{(\gamma_k+1)A_k}{B_k}\) is a constant across \(k\), and \(\frac{A_k}{B_k}\) is a constant across \(k\). But this means that \(\gamma_k\) should be constant across \(k\), which is a contradiction to the fact that \(\gamma_k\)s are distinct solutions to the characteristic equation. Thus, it is necessary that \(\bar{y}_r = y_i\), which is the targeted ICR policy. \(\square\)

### A.2 Derivation of Equation (31)

To determine \(\mathbb{E}[\hat{p}dF_t]\) in equation (30), we consider separately the cases with and without Poisson jumps. If no jumps occur in the interval \([t, t + dt]\) (i.e., \(dN_t = 0\)), then the equity holders issue/repurchase debt to compensate for all Brownian shocks and reissue maturing debt so that

\[ dy_t = dY_t/Y_t - dF_t/F_t- = dY_t/Y_t - (d\Gamma_t - \xi F_t-)/F_t- = 0. \]

Hence, in this case

\[ \mathbb{E}[d\Gamma_t|dN_t = 0] = \mathbb{E}[F_{t-}dY_t/Y_t|dN_t = 0] + \xi F_{t-} dt = (\hat{\mu} + \xi) F_{t-} dt, \]

and

\[ \mathbb{E}[dF_t|dN_t = 0] = \mathbb{E}[F_{t-}dY_t/Y_t|dN_t = 0] = \hat{\mu} F_{t-} dt. \]

The continuation value in this case is equal to

\[ \mathbb{E}[E(\hat{y}_F_t, F_t)|dN_t = 0] = \mathbb{E}[E(\hat{y}_F_{t-} + dF_t)|dN_t = 0] \]
\[ = E(\hat{y}_F_{t-}, F_{t-}) + \mathbb{E}[e(\hat{y})dF_t|dN_t = 0] \]
\[ = E(\hat{y}_F_{t-}, F_{t-}) + \hat{\mu} F_{t-} dt. \]
If there is a Poisson jump $dY_t/Y_t = e^{-\tilde{S}_t} - 1$ so that $\hat{y} e^{-\tilde{S}_t} > y_r$, then the equity holders compensate this jump so that the state returns to $\hat{y}$. In order to do so, they repurchase $F_{t-} (1 - e^{-\tilde{s}})$ units of debt (i.e., $dF_t = F_{t-} (e^{-\tilde{s}} - 1) < 0$). Thus, if $\tilde{S}_t = \tilde{s} < \ln(\hat{y}/y_r)$, then

$$E \left[ dF_t \mid dN_t = 1, \tilde{S}_t = \tilde{s} \right] = F_{t-} \left( e^{-\tilde{s}} - 1 \right),$$

and the continuation value is

$$E \left[ (\hat{y}/F_t) dN_t = 1, \tilde{S}_t = \tilde{s} \right] = E \left[ e(\hat{y}) (F_{t-} + dF_t) dN_t = 0, \tilde{S}_t = \tilde{s} \right]$$

$$= E(\hat{y} F_{t-}) e^{-\tilde{s}}.$$

Therefore, we can rewrite equation (30) as

$$E(\hat{y} F_{t-}, F_{t-}) = (1 - \pi)(\hat{y} F_{t-} - c F_{t-}) dt - \xi F_{t-} dt + \hat{p} (\hat{\mu} + \xi) F_{t-} dt + \lambda dt \int_{0}^{\ln(\hat{y}/y_r)} \hat{p} F_{t-} (e^{-\tilde{s}} - 1) \eta e^{-\eta \tilde{s}} d\tilde{s}$$

$$+ (1 - r dt - \lambda dt) \left( E(\hat{y} F_{t-}, F_{t-}) + \hat{\epsilon} \hat{\mu} F_{t-} dt \right)$$

$$+ (1 - r dt) \left\{ \lambda dt \int_{0}^{\ln(\hat{y}/y_r)} E(\hat{y} F_{t-}, F_{t-}) e^{-\tilde{s}} \eta e^{-\eta \tilde{s}} d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} E(e^{-\tilde{s}} \hat{y} F_{t-}, F_{t-}) \eta e^{-\eta \tilde{s}} d\tilde{s} \right\}. $$

Normalizing by $F_{t-}$, we get

$$\hat{\epsilon} = (1 - \pi)(\hat{y} - c) dt - \xi + \hat{p} (\hat{\mu} + \xi) dt + \lambda dt \int_{0}^{\ln(\hat{y}/y_r)} \hat{p} (e^{-\tilde{s}} - 1) \eta e^{-\eta \tilde{s}} d\tilde{s}$$

$$+ (1 - r dt - \lambda dt) (\hat{\epsilon} + \hat{\epsilon} \hat{\mu} dt)$$

$$+ (1 - r dt) \left\{ \lambda dt \int_{0}^{\ln(\hat{y}/y_r)} \hat{\epsilon} e^{-\tilde{s}} \eta e^{-\eta \tilde{s}} d\tilde{s} + \lambda dt \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} e^{-\tilde{s}} \hat{y} \eta e^{-\eta \tilde{s}} d\tilde{s} \right\}. $$

Hence, we get the HJB equation (31).
B Online Appendix (Not for Publication)

This appendix presents auxiliary calculations for the results provided in the main text.

B.1 Auxiliary Derivations for Section 3

Derivation of Condition (12) Condition (12) can be written more explicitly by computing the expectation:

\[
1 + b_1 \frac{\eta}{\eta - \gamma_1} + b_2 \frac{\eta}{\eta - \gamma_2} + b_3 \frac{\eta}{\eta - \gamma_3} = 0. \tag{53}
\]

By \( p(e^{-\bar{s}}y) = 0 \) for all \( \bar{s} > \ln(y/y_b) \), the HJB equation (6) for \( y \in (y_b, y_r) \) can be written as

\[
(r + \lambda + \xi)p(y) = c + \xi + (\hat{\mu} + \xi)yp'(y) + \frac{1}{2}\sigma^2yp''(y) + \lambda \int_{\ln(y/y_b)}^{\ln(y/y_b)} \{p(e^{-\bar{s}}) \eta e^{-\eta \bar{s}}\} d\bar{s}.
\]

Let us compute the integral using the conjectured \( p \) in (7):

\[
\int_{0}^{\ln(y/y_b)} \{p(e^{-\bar{s}}) \eta e^{-\eta \bar{s}}\} d\bar{s} = \int_{0}^{\ln(y/y_b)} \left\{ \frac{c + \xi}{r + \xi} \left(1 + \sum_{k=1}^{3} b_k e^{\gamma_k \bar{s}} (y/y_b)^{-\gamma_k}\right) \eta e^{-\eta \bar{s}}\right\} d\bar{s}
\]

\[
= \frac{c + \xi}{r + \xi} (1 - (y/y_b)^{-\eta}) + \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k} \eta \int_{0}^{\ln(y/y_b)} e^{-(\eta - \gamma_k) \bar{s}} d\bar{s}
\]

\[
= \frac{c + \xi}{r + \xi} (1 - (y/y_b)^{-\eta}) + \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k} \eta \left(1 - (y/y_b)^{-(\eta - \gamma_k)}\right)
\]

\[
= \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} (y/y_b)^{-\gamma_k} - (y/y_b)^{-\eta} \frac{c + \xi}{r + \xi} \left(1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k}\right).
\]

Thus, we can write the HJB equation (6) as

\[
(r + \lambda + \xi) \left(1 + \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k}\right) = c + \xi - (\hat{\mu} + \xi) \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} \gamma_k b_k (y/y_b)^{-\gamma_k}
\]

\[
+ \frac{1}{2}\sigma^2 \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} \gamma_k (1 + \gamma_k) b_k (y/y_b)^{-\gamma_k}
\]

\[
+ \lambda \frac{c + \xi}{r + \xi} + \lambda \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} (y/y_b)^{-\gamma_k}
\]

\[
- \lambda (y/y_b)^{-\eta} \frac{c + \xi}{r + \xi} \left(1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k}\right).
\]
Cancelling the terms at the constant,

\[
\lambda(y/y_b)^{-\eta} \left( 1 + \frac{3}{k=1} b_k \frac{\eta}{\eta - \gamma_k} \right) = \frac{3}{k=1} b_k (y/y_b)^{-\gamma_k} \left\{ -(\mu + \xi) \gamma_k + \frac{\sigma^2}{2} \gamma_k (1 + \gamma_k) + \lambda \frac{\eta}{\eta - \gamma_k} - (r + \lambda + \xi) \right\}.
\]

Thus, we get that \( \gamma_k \)'s must solve the characteristic equation (8). Further, matching terms at \((y/y_b)^{-\eta}\), we get that coefficients \( b_k \)'s satisfy condition (12), which is the desired result.

Condition (12) can be interpreted as follows. It requires that even if conjecture \( p(y) = \frac{c + \xi}{r + \xi} \left( 1 + \sum_{k=1}^3 b_k (y/y_b)^{-\gamma_k} \right) \) was applied to \( y \) below \( y_b \), it would not change debt pricing in the range \([y_b, y_r]\). Indeed, this conjecture only describes debt pricing on \([y_b, y_r]\), and \( p(y) = 0 \) for \( y < y_b \). Because of the memoryless property of the exponential distribution of downward jumps, in order to derive the debt price in the region \( \gamma(y_b, y_r) \), it is sufficient to ensure that \( \mathbb{E}[p(Sy_b)] = 0 \) rather than that \( p(y) = 0 \) for all \( y < y_b \). Condition (12) is equivalent to the requirement that \( \mathbb{E}[p(Sy_b)] = 0 \) even if conjecture \( p(y) = \frac{c + \xi}{r + \xi} \left( 1 + \sum_{k=1}^3 b_k (y/y_b)^{-\gamma_k} \right) \) is extended to \( y \) below \( y_b \).

**Derivation of Condition (13)** Condition (13) can be written more explicitly by computing the expectation:

\[
\sum_{k=1}^3 b_k \frac{\gamma_k}{\eta - \gamma_k} \left( \frac{y_r}{y_b} \right)^{\eta - \gamma_k} = \sum_{k=1}^3 B_k \frac{\gamma_k}{\eta - \gamma_k} \left( \frac{\bar{y}_r}{y_b} \right)^{\eta - \gamma_k}.
\]

Let us derive it. For \( y \in (y_b, y_r) \), the HJB equation (6) becomes

\[
(r + \lambda + \xi)p(y) = c + \xi + (\hat{\mu} + \xi)y'p'(y) + \frac{1}{2} \sigma^2 y^2 p''(y) + \lambda \int_{\ln(y/y_b)}^{\ln(y/y_r)} \left\{ p \left( e^{-\bar{s}} y \right) \eta e^{-\bar{s}} \right\} d\bar{s},
\]

because \( p(e^{-\bar{s}} y) = 0 \) for all \( \bar{s} > \ln(y/y_b) \). Using \( p(y) = p^*_\bar{s} \) for \( y \in [y_r, \bar{y}_r] \), we get that \( p(y) \) in the region \( y \in (\bar{y}_r, y) \) satisfies

\[
(r + \lambda + \xi)p(y) = c + \xi + (\hat{\mu} + \xi)y'p'(y) + \frac{1}{2} \sigma^2 y^2 p''(y) + \lambda \int_{\ln(y/y_r)}^{\ln(y/y_r)} \left\{ p^*_\bar{s} \eta e^{-\bar{s}} \right\} d\bar{s}
\]

\[
+ \lambda \int_{\ln(y/y_r)}^{\ln(y/\bar{y}_r)} \left\{ p^*_\bar{s} \eta e^{-\bar{s}} \right\} d\bar{s} + \lambda \int_{0}^{\ln(y/\bar{y})} \left\{ p \left( e^{-\bar{s}} y \right) \eta e^{-\bar{s}} \right\} d\bar{s}.
\]

Using the conjecture (7), we can compute each integral more explicitly as follows. To simplify the expressions, we use notation \( x = y/y_b, x_r = y_r/y_b, x = \bar{y}/y_b, \hat{x} = \bar{y}/y_b, x^* = y^*/y_b, \hat{x} = \hat{y}/y_b \). The first integral in (55) equals

\[51\]
\[
\int_{\ln(y/y_r)}^{\ln(y/y_b)} \frac{c + \xi}{r + \xi} \left( 1 + \sum_{k=1}^{3} b_k e^{\gamma_k \xi} (y/y_b)^{-\gamma_k} \right) \eta e^{-\eta \xi} d\xi
\]

\[
= \frac{c + \xi}{r + \xi} \left( (\frac{y}{y_r})^{-\eta} - (\frac{y}{y_b})^{-\eta} \right) + \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k (y/y_b)^{-\gamma_k} \frac{\eta}{\eta - \gamma_k} \left( (\frac{y}{y_r})^{-(\eta - \gamma_k)} - (\frac{y}{y_b})^{-(\eta - \gamma_k)} \right)
\]

\[
= \frac{c + \xi}{r + \xi} x^{-\eta} (x_r^{-\eta} - 1) + \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k x^{-\eta} \frac{\eta}{\eta - \gamma_k} (x_r^{-\eta - \gamma_k} - 1).
\]

Analogously, the last integral equals

\[
\int_{0}^{\ln(y/y_r)} \frac{c + \xi}{r + \xi} \left( 1 + \sum_{k=1}^{3} B_k e^{\gamma_k \xi} (y/y_b)^{-\gamma_k} \right) \eta e^{-\eta \xi} d\xi
\]

\[
= \frac{c + \xi}{r + \xi} \left( 1 - (\frac{y}{y_r})^{-\eta} \right) + \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} B_k (y/y_b)^{-\gamma_k} \frac{\eta}{\eta - \gamma_k} \left( 1 - (\frac{y}{y_r})^{-(\eta - \gamma_k)} \right)
\]

\[
= \frac{c + \xi}{r + \xi} (1 - x^{-\eta \bar{\eta}}) + \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} B_k \frac{\eta}{\eta - \gamma_k} (x^{-\gamma_k} - x^{-\eta \bar{\eta} - \gamma_k}).
\]

The second integral equals

\[
\int_{\ln(y/y_r)}^{\ln(y/y_b)} p^*_r \eta e^{-\eta \xi} d\xi = p^*_r \left( (\frac{y}{y_r})^{-\eta} - (\frac{y}{y_b})^{-\eta} \right)
\]

\[
= p^*_r x^{-\eta} (x_r^{-\eta} - x_b^{-\eta}).
\]

Therefore, the HJB equation becomes

\[
(r + \lambda + \xi) \frac{c + \xi}{r + \xi} \left( 1 + \sum_{k=1}^{3} B_k x^{-\gamma_k} \right)
= c + \xi - (\mu + \xi) \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} \gamma_k B_k x^{-\gamma_k} + \frac{\sigma^2}{2} \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) B_k x^{-\gamma_k}
\]

\[
+ \lambda \frac{c + \xi}{r + \xi} x^{-\eta} (x_r^{-\eta} - 1) + \lambda \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k x^{-\eta} \frac{\eta}{\eta - \gamma_k} (x_r^{-\eta - \gamma_k} - 1) + \lambda p^*_r x^{-\eta} (\bar{\eta}^{-\eta} - x_r^{-\eta})
\]

\[
+ \lambda \frac{c + \xi}{r + \xi} (1 - x^{-\eta \bar{\eta}}) + \lambda \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} B_k \frac{\eta}{\eta - \gamma_k} (x^{-\gamma_k} - x^{-\eta \bar{\eta} - \gamma_k}).
\]

Given that \(\gamma_k\)s solve the characteristic equation (8),

\[
x_r^{-\eta} - 1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} (x_r^{-\eta - \gamma_k} - 1) + \frac{r + \xi}{c + \xi} p^*_r (\bar{\eta}^{-\eta} - x_r^{-\eta}) - \bar{\eta}^{-\eta} - \sum_{k=1}^{3} B_k \frac{\eta}{\eta - \gamma_k} \bar{\eta}^{-\eta - \gamma_k} = 0.
\]
Using (12),
\[ x_t^\eta + \sum_{k=1}^3 b_k \frac{\eta}{\eta - \gamma_k} x_t^\eta - \gamma_k + \frac{r + \xi}{c + \xi} p^*_r (x_t^\eta - x_t^\eta) - \sum_{k=1}^3 B_k \frac{\eta}{\eta - \gamma_k} x_t^\eta - \gamma_k = 0, \]
or equivalently,
\[ \sum_{k=1}^3 b_k \frac{\eta}{\eta - \gamma_k} x_t^\eta - \gamma_k + \left( \frac{r + \xi}{c + \xi} p^*_r - 1 \right) (x_t^\eta - x_t^\eta) - \sum_{k=1}^3 B_k \frac{\eta}{\eta - \gamma_k} x_t^\eta - \gamma_k = 0. \]

Using \( p_r^* = \frac{c+\xi}{r+\xi} \left( 1 + \sum_{k=1}^3 b_k x_r^\gamma \right) = \frac{c+\xi}{r+\xi} \left( 1 + \sum_{k=1}^3 B_k x_r^\gamma \right), \)
\[ \sum_{k=1}^3 b_k \frac{\eta}{\eta - \gamma_k} x_r^\eta - \gamma_k + \sum_{k=1}^3 B_k x_r^\gamma - \sum_{k=1}^3 b_k x_r^\gamma - \sum_{k=1}^3 B_k \frac{\eta}{\eta - \gamma_k} x_r^\gamma - \gamma_k = 0, \]
or
\[ \sum_{k=1}^3 b_k \frac{\gamma_k}{\eta - \gamma_k} x_r^\gamma - \gamma_k = \sum_{k=1}^3 B_k \frac{\gamma_k}{\eta - \gamma_k} x_r^\gamma, \]
which is the desired condition (13).

The interpretation of condition (13) is as follows. It requires that even if conjecture \( p(y) = \frac{c+\xi}{r+\xi} \left( 1 + \sum_{k=1}^3 B_k (y/y_b)^{-\gamma_k} \right) \) was applied to \( y \) below \( \eta_r \), it would not change debt pricing in the range \([\eta_r, y_i]\). Indeed, because of the memoryless property of the exponential distribution of downward jumps, the debt price in the region \([\eta_r, y_i]\) depends on the price of debt below \( \eta_r \) only through \( \mathbb{E}[p(S\eta_r)] \). Condition (13) is equivalent to the requirement that
\[ \mathbb{E}[p(S\eta_r)] = \left( 1 - \left( \frac{\eta_r}{y_r} \right)^{-\eta} \right) p^*_r + \left( \frac{\eta_r}{y_r} \right)^{-\eta} \mathbb{E}[p(Sy_r)] \]
still holds even if we use conjecture \( p(y) = \frac{c+\xi}{r+\xi} \left( 1 + \sum_{k=1}^3 B_k (y/y_b)^{-\gamma_k} \right) \) in the left-hand side to compute \( \mathbb{E}[p(S\eta_r)] \).

**Derivation of Condition (21)**  Condition (21) can be written more explicitly by computing the expectation:
\[ \frac{\phi_\eta}{\eta + 1} y_b - \rho + \sum_{k=1}^3 c_k \frac{\eta}{\eta - \gamma_k} = 0. \]

Let us derive it. In region \( y \in [y_b, y_r] \), the evolution of \( e \) is given by
\[ (r + \lambda + \xi)e(y) = (1 - \pi)(y - c) - \xi + (\hat{\mu} + \xi)ye'(y) + \frac{1}{2}\sigma^2 ye''(y) + \lambda \int_0^{\ln(y/y_b)} e^{-\hat{\gamma}s} \eta e^{-\eta^2 s} ds. \]
Using the conjectured form of $e$, the integral becomes

$$
\int_0^{\ln(y/y_b)} \left( \phi e^{-\delta y} - \rho + \sum_{k=1}^{3} c_k \left( e^{-\delta y/y_b} \right)^{-\gamma_k} \right) \eta e^{-\eta \delta} d\delta
$$

$$
= \frac{\phi \eta}{\eta + 1} \left( 1 - \left( \frac{y}{y_b} \right)^{-\eta} \right) - \rho \left( 1 - \left( \frac{y}{y_b} \right)^{-(\eta + 1)} \right) + \sum_{k=1}^{3} \frac{c_k \left( y/y_b \right)^{-\gamma_k \eta}}{\eta - \gamma_k} \left( 1 - \left( \frac{y}{y_b} \right)^{-(\eta - \gamma_k)} \right)
$$

$$
= \frac{\phi \eta xy_b}{\eta + 1} \left( 1 - x^{-(\eta + 1)} \right) - \rho \left( 1 - x^{-\eta} \right) + \sum_{k=1}^{3} \frac{c_k x^{-\gamma_k \eta}}{\eta - \gamma_k} \left( 1 - x^{-(\eta - \gamma_k)} \right).
$$

Thus, we can re-write the HJB equation as

$$
(r + \lambda + \xi) \left( \phi y - \rho + \sum_{k=1}^{3} c_k x^{-\gamma_k} \right) = (1 - \pi)(y - c) - \xi + (\hat{\mu} + \xi) \left( \phi y - \sum_{k=1}^{3} \gamma_k c_k x^{-\gamma_k} \right)
$$

$$
+ \frac{1}{2} \sigma^2 \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) c_k x^{-\gamma_k}
$$

$$
+ \frac{\lambda \phi \eta xy_b}{\eta + 1} \left( 1 - x^{-(\eta + 1)} \right) - \lambda \rho \left( 1 - x^{-\eta} \right) + \sum_{k=1}^{3} \frac{c_k x^{-\gamma_k \lambda \eta}}{\eta - \gamma_k} \left( 1 - x^{-(\eta - \gamma_k)} \right).
$$

Since $\gamma_k$s satisfy the characteristic equation (8), terms at $c_k x^{-\gamma_k}$ disappear. Further, given the definition of $\phi$ and $\rho$, the terms at constant and at $y$ disappear as well. Thus, we get that equation (21) must hold.

The interpretation of equation (21) is as follows. It requires that even if the conjecture $e(y) = \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}$ were applied beyond the range $[y_b, y_r]$, this would not change the equity value on $[y_b, y_r]$. By the memoryless property of the exponential distribution of downward jumps, this would be the case if $\mathbb{E}[e(Sy_r)]$ did not change. Condition (22) requires that $\mathbb{E}[e(Sy_r)] = 0$ holds even if we use conjecture $e(y) = \phi y - \rho + \sum_{k=1}^{3} c_k (y/y_b)^{-\gamma_k}$ to compute $\mathbb{E}[e(Sy_r)]$ (rather than $e(y) = 0$ for $y < y_b$).

**Derivation of Condition (22)** Condition (22) can be written more explicitly by computing the expectation:

$$
\sum_{k=1}^{3} \frac{\eta(1 + \gamma_k)}{\eta - \gamma_k} \left[ \left( c_k + b_k \frac{c + \xi}{r + \xi} \right) \left( \frac{y_r}{y_b} \right)^{\eta - \gamma_k} - \left( C_k + B_k \frac{c + \xi}{r + \xi} \right) \left( \frac{y_r}{y_b} \right)^{\eta - \gamma_k} \right] = \frac{c \pi}{r + \xi} \left[ \left( \frac{y_r}{y_b} \right)^{\eta} - \left( \frac{y_b}{y_r} \right)^{\eta} \right].
$$
Let us derive it. In region $y \in [\overline{y}_r, \underline{y}_r]$, the evolution of $e$ is given by

$$(r + \lambda + \xi)e(y) = (1 - \pi)(y - c) - \xi + (\hat{\mu} + \xi)\phi e' + \frac{1}{2} \sigma^2 y^2 e''(y) + \lambda \int_{\overline{y}_r}^{\underline{y}_r} e^{-\delta s} \eta e^{-\eta s} \ ds + \lambda \int_{\overline{y}_r}^{\underline{y}_r} e^{-\delta s} \eta e^{-\eta s} \ ds.$$ 

We use the conjectures in (20) to compute each integral. The first integral becomes

$$\int_{\overline{y}_r}^{\underline{y}_r} \left( \phi e^{-\delta s} - \rho + \sum_{k=1}^{3} c_k \left( e^{-\delta s} \right)^{-\gamma_k} \right) \eta e^{-\eta s} \ ds$$

$$= \frac{\phi y \eta}{\eta + 1} \left( \left( \frac{y}{\overline{y}_r} \right)^{-(\eta + 1)} - \left( \frac{y}{\underline{y}_r} \right)^{-(\eta + 1)} \right) - \rho \left( \left( \frac{y}{\overline{y}_r} \right)^{\eta} - \left( \frac{y}{\underline{y}_r} \right)^{\eta} \right)$$

$$+ \sum_{k=1}^{3} c_k \left( \frac{y}{\eta - \gamma_k} \right)^{\eta - \gamma_k} \left( \left( \frac{y}{\overline{y}_r} \right)^{-(\eta - \gamma_k)} - \left( \frac{y}{\underline{y}_r} \right)^{-(\eta - \gamma_k)} \right)$$

$$= \frac{\phi x^{-\eta} y \eta}{\eta + 1} \left( x_r^{\eta + 1} - 1 \right) - \rho x^{-\eta} (x_r^{\eta} - 1) + \sum_{k=1}^{3} \frac{c_k x^{-\eta} \eta}{\eta - \gamma_k} (x_r^{\eta - \gamma_k} - 1)$$

$$= x^{-\eta} \left[ \phi y \eta \left( x_r^{\eta + 1} - 1 \right) - \rho (x_r^{\eta} - 1) + \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} (x_r^{\eta - \gamma_k} - 1) \right].$$

Denote $e_r^* = e(y_r^*)$. The second integral becomes

$$\int_{\overline{y}_r}^{\underline{y}_r} \left( \frac{p_r^* + e_r^* y e^{-\delta s} - p_r^*}{y_r^* \eta} \right) \eta e^{-\eta s} \ ds$$

$$= \frac{p_r^* + e_r^*}{y_r^*} y \int_{\overline{y}_r}^{\underline{y}_r} \eta e^{-\eta s} \ ds - p_r^* \int_{\overline{y}_r}^{\underline{y}_r} \eta e^{-\eta s} \ ds$$

$$= \frac{p_r^* + e_r^*}{y_r^*} (y_r^{\eta + 1} - y_r) - p_r^* \left( \left( \frac{y}{y_r} \right)^{\eta} - \left( \frac{y}{y_r} \right)^{\eta} \right)$$

$$= y^{-\eta} \left[ \frac{p_r^* + e_r^*}{y_r^*} \left( y_r^{\eta + 1} - y_r \right) - p_r^* \left( y_r^{\eta} - y_r^\eta \right) \right]$$

$$= x^{-\eta} \left[ \frac{p_r^* + e_r^*}{x_r^*} \eta \left( x_r^{\eta + 1} - x_r^{\eta} \right) - p_r^* \left( x_r^{\eta} - x_r^{\eta} \right) \right].$$
The third integral becomes

$$
\int_0^{\ln(y/\eta_r)} \left( \phi e^{-\hat{s}} y - \rho + \sum_{k=1}^{3} C_k \left( e^{-\hat{s}} y/\eta_r \right)^{-\gamma_k} \right) \eta e^{-\eta \hat{s}} d\hat{s}
$$

$$
= \frac{\phi y \eta}{\eta + 1} \left( 1 - \left( \frac{y}{\eta_r} \right)^{-\gamma_r} \right) - \rho \left( 1 - \left( \frac{y}{\eta_r} \right)^{-\gamma_r} \right) + \sum_{k=1}^{3} C_k \left( \frac{y}{\eta_r} \right)^{-\gamma_k} \frac{\eta}{\eta - \gamma_k} \left( 1 - \left( \frac{y}{\eta_r} \right)^{-\gamma_k} \right)
$$

Thus,

$$(r + \lambda + \xi) \left( \phi y \eta x - \rho + \sum_{k=1}^{3} C_k x^{-\gamma_k} \right) = (1 - \pi)(y_b x - c) - \xi + (\hat{\mu} + \xi) \left( \phi y \eta x - \sum_{k=1}^{3} \gamma_k C_k x^{-\gamma_k} \right)
$$

$$
+ \frac{1}{2} \sigma^2 \sum_{k=1}^{3} \gamma_k (\gamma_k + 1) C_k x^{-\gamma_k}
$$

$$
+ \lambda x^{-\eta \gamma} \left[ \frac{\phi y \eta}{\eta + 1} \left( x_r^{\eta+1} - 1 \right) - \rho \left( x_r^{\eta} - 1 \right) + \sum_{k=1}^{3} C_k \eta \left( x_r^{\eta} - \gamma_k \right) \right]
$$

$$
+ \lambda x^{-\eta \gamma} \left[ \frac{p_r^* + e_r^*}{x_r^*} \eta + 1 \left( x_r^{\eta+1} - x_r^{\eta} \right) - p_r^* \left( x_r^{\eta} - x_r^{\eta} \right) \right]
$$

$$
+ \lambda x^{-\eta \gamma} \left[ \rho x_r^{\eta \gamma} - \frac{\phi y \eta x_r^{\eta+1}}{\eta + 1} - \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} x_r^{\eta} x_r^{\eta - \gamma_k} \right].
$$

Matching the coefficients at $\lambda x^{-\eta}$,

$${\frac{\phi y \eta}{\eta + 1} \left( x_r^{\eta+1} - 1 \right) - \rho \left( x_r^{\eta} - 1 \right) + \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} \left( x_r^{\eta} - \gamma_k \right) - 1}$$

$$
+ \frac{p_r^* + e_r^*}{x_r^*} \frac{\eta}{\eta + 1} \left( x_r^{\eta+1} - x_r^{\eta} \right) - p_r^* \left( x_r^{\eta} - x_r^{\eta} \right)
$$

$$
+ \rho x_r^{\eta \gamma} - \frac{\phi y \eta x_r^{\eta+1}}{\eta + 1} - \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} x_r^{\eta} x_r^{\eta - \gamma_k} = 0.
$$

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Given equation (21),

\[
\phi r \frac{y}{\eta + 1} x_r^{\eta + 1} - \rho x_r^\eta + \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k}
\]

\[
+ \frac{p_r^* + e_r^*}{x_r^\eta} \frac{\eta}{\eta + 1} (x_r^{\eta + 1} - x_r^{\eta + 1}) - p_r^* (x_r^\eta - x_r^\eta)
\]

\[
+ \rho x_r^\eta - \frac{\phi \eta y}{\eta + 1} x_r^{\eta + 1} - \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k} = 0.
\]

Denote \(e_r \equiv e(y_r)\) and \(e_r \equiv e(y_r)\). Using \(x_r^\eta - \rho x_r^\eta + \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k} = 0\),

\[
\frac{\phi \eta y}{\eta + 1} x_r^{\eta + 1} - \rho x_r^\eta + \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k}
\]

\[
+ (p_r^* + e_r^*) \frac{\eta}{\eta + 1} x_r^\eta - (p_r^* + e_r^*) \frac{\eta}{\eta + 1} x_r^\eta - p_r^* (x_r^\eta - x_r^\eta)
\]

\[
+ \rho x_r^\eta - \frac{\phi \eta y}{\eta + 1} x_r^{\eta + 1} - \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k} = 0,
\]

or

\[
\frac{\phi \eta y}{\eta + 1} x_r^{\eta + 1} - \rho x_r^\eta + \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k}
\]

\[
- p_r^* \frac{1}{\eta + 1} x_r^\eta + e_r^* \frac{\eta}{\eta + 1} x_r^\eta + p_r^* \frac{1}{\eta + 1} x_r^\eta - e_r \frac{\eta}{\eta + 1} x_r^\eta
\]

\[
+ \rho x_r^\eta - \frac{\phi \eta y}{\eta + 1} x_r^{\eta + 1} - \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k} = 0.
\]

Regrouping the terms,

\[
x_r^\eta \left( \phi \eta y x_r^\eta \frac{1}{\eta + 1} - \rho + \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k} \right)
\]

\[
- x_r^\eta \left( \phi \eta y x_r^\eta \frac{1}{\eta + 1} - \rho + \sum_{k=1}^{3} \frac{C_k \eta}{\eta - \gamma_k} x_r^{\eta - \gamma_k} \right)
\]

\[
+ p_r^* \frac{1}{\eta + 1} x_r^\eta - p_r^* \frac{1}{\eta + 1} x_r^\eta + e_r^* \frac{\eta}{\eta + 1} x_r^\eta - e_r \frac{\eta}{\eta + 1} x_r^\eta = 0,
\]

or
or

\[
x_r(x_\eta) \left( \frac{\phi y \eta}{\eta + 1} - \rho + \sum_{k=1}^{3} \frac{c_k \eta}{\eta + \gamma} x_r^{-\gamma} - e_r \frac{\eta}{\eta + 1} + p_r^* \frac{\eta}{\eta + 1} \right)
= x_r(x_\eta) \left( \frac{\phi y \eta}{\eta + 1} - \rho + \sum_{k=1}^{3} \frac{C_k \eta}{\eta + \gamma} x_r^{-\gamma} - \phi y \eta \frac{1}{\eta + 1} + p_r^* \frac{1}{\eta + 1} \right),
\]

or

\[
x_r(x_\eta) \left( \frac{\phi y \eta}{\eta + 1} - \rho + \sum_{k=1}^{3} \frac{c_k \eta}{\eta + \gamma} x_r^{-\gamma} \right)
= x_r(x_\eta) \left( \frac{\phi y \eta}{\eta + 1} - \rho + \sum_{k=1}^{3} \frac{C_k \eta}{\eta + \gamma} x_r^{-\gamma} \right),
\]

or

\[
\sum_{k=1}^{3} \eta(1 + \gamma) \frac{c_k x_\eta^{-\gamma} - C_k x_\eta^{-\gamma}}{\eta - \gamma} + (p_r^* - \rho) x_\eta = \sum_{k=1}^{3} \eta(1 + \gamma) \frac{C_k x_\eta^{-\gamma}}{\eta - \gamma} + (p_r^* - \rho) x_\eta.
\]

Using \(p_r^* = \frac{c + \xi}{r + \xi}(1 + \sum_{k=1}^{3} b_k x_\eta^{-\gamma}) = \frac{c + \xi}{r + \xi}(1 + \sum_{k=1}^{3} B_k x_\eta^{-\gamma})\), we re-write this equation as

\[
\sum_{k=1}^{3} \eta(1 + \gamma) \frac{c_k x_\eta^{-\gamma} - C_k x_\eta^{-\gamma}}{\eta - \gamma} + \frac{c + \xi}{r + \xi} b_k x_\eta^{-\gamma} - \frac{c + \xi}{r + \xi} B_k x_\eta^{-\gamma} = \frac{c + \xi}{r + \xi}(x_\eta - x_\eta),
\]

or given equation (13),

\[
\sum_{k=1}^{3} \eta(1 + \gamma) \left( c_k x_\eta^{-\gamma} - C_k x_\eta^{-\gamma} + \frac{c + \xi}{r + \xi} B_k x_\eta^{-\gamma} \left( 1 + \frac{\gamma k(\eta + 1)}{\eta - \gamma} \right) \right)
= \frac{c + \xi}{r + \xi}(x_\eta - x_\eta),
\]

which is the desired condition (22).

Condition (22) is interpreted as follows. It requires that even if the conjecture \(e(y) = \phi y - \rho + \sum_{k=1}^{3} C_k (y/y_\eta)^{-\gamma} \) were applied beyond the range \([\bar{y}_r, y_\eta]\), this would not change the
equity value on \([\overline{y}_r, y_i]\). By the memoryless property of the exponential distribution of downward jumps, this would be the case if \(\mathbb{E}[e(S\overline{y}_r)]\) did not change. Condition (22) requires that

\[
\mathbb{E}[e(S\overline{y}_r)] = \left(1 - \left(\frac{\overline{y}_r}{y_r}\right)^{-\eta}\right)\mathbb{E}\left[\frac{e_y^* + p_r^*}{y_r} S\overline{y}_r - p_r^* \right| S \in [y_r/\overline{y}_r, 1]\bigg] + \left(\frac{\overline{y}_r}{y_r}\right)^{-\eta} \mathbb{E}[e(Sy_r)]
\]

holds even if we use conjecture \(e(y) = \phi y - \rho + \sum_{k=1}^{3} C_k (y/y_b)^{-\gamma_k}\) in the left-hand side to compute \(\mathbb{E}[e(S\overline{y}_r)]\).

### B.2 Closed-Form Expressions

#### B.2.1 Reflecting Boundaries

If the issuance boundary \(y_i\) is a reflecting boundary so that \(y_i = y_i^*\), then conditions (64) and (17) are replaced by appropriate limits of them. The former is replaced by

\[
p'(y_i) = 0,
\]

which is obtained as the limit of the condition \((p(y_i) - p(y_i^*)) / (y_i - y_i^*) = 0\) as \(y_i^* \to y_i\):

\[
\lim_{y_i^* \to y_i} \frac{\sum_{k=1}^{3} B_k \left(y_i/y_b\right)^{-\gamma_k} - (y_i^*/y_b)^{-\gamma_k}}{y_i - y_i^*} = \lim_{y_i^* \to y_i} \frac{\sum_{k=1}^{3} B_k y_b^{-\gamma_k} y_i^{-\gamma_k - 1} \left[1 - (y_i^*/y_i)^{-\gamma_k}\right]}{1 - y_i^*/y_i}
\]

\[
= \lim_{y_i^* \to y_i} \sum_{k=1}^{3} -\gamma_k y_i^{-\gamma_k - 1} B_k y_b^{-\gamma_k}
\]

\[
= \lim_{y_i^* \to y_i} p'(y_i).
\]

Condition (17) is replaced by

\[
e'(y_i) y_i = p(y_i) + e(y_i),
\]

which is obtained by taking the limit of the condition (17).

If the repurchase boundary \(\overline{y}_r\) is a reflecting boundary so that \(\overline{y}_r = y_r^*\), then by the analogy with the reflecting issuance boundary, conditions (63) and (73) are replaced by appropriate limits of them:

\[
p'(\overline{y}_r) = 0,
\]

\[
e'(\overline{y}_r) \overline{y}_r = p(\overline{y}_r) + e(\overline{y}_r).
\]
B.2.2 Conditions Determining the Leverage Dynamics

**Debt Price**  The debt price is determined by conditions:

\[ p(y_b) = 0 : \sum_{k=1}^{3} b_k = -1, \quad (60) \]

\[ \text{eq. (12)} : 1 + \sum_{k=1}^{3} b_k \frac{\eta}{\eta - \gamma_k} = 0, \quad (61) \]

\[ p(y_r) = p(y_r^*) : \sum_{k=1}^{3} b_k x_r^{-\gamma_k} = \sum_{k=1}^{3} B_k x_r^{-\gamma_k}, \quad (62) \]

\[ p(y_r) = p(y_r^*) : \sum_{k=1}^{3} B_k \left( x_r^{-\gamma_k} - x_r^{* -\gamma_k} \right) = 0, \quad (63) \]

\[ p(y_i) = p(y_i^*) : \sum_{k=1}^{3} B_k \left( x_i^{-\gamma_k} - x_i^{* -\gamma_k} \right) = 0, \quad (64) \]

\[ \text{eq. (13)} : \frac{\lambda(y/y_b)}{\eta} \gamma_k \sum_{k=1}^{3} b_k x_r^{-\gamma_k} - \frac{\lambda(x_r)}{\eta} \gamma_k \sum_{k=1}^{3} B_k x_r^{-\gamma_k} = 0. \quad (65) \]

Conditions (12) and (13) are obtained by plugging the conjecture (7) into the HJB equation (6) and matching the terms at \( \lambda(y/y_b)^{-\eta} \). Conditions (60) – (65) give six equations on six parameters \( (b_k, B_k)_{k=1,2,3} \), and allow us to derive the debt price without computing the equity value.

If \( y_i \) is a reflecting issuance boundary, then as we showed above, condition (64) is replaced by (56), or more explicitly:

\[ p'(y_i) = 0 : \sum_{k=1}^{3} \gamma_k B_k x_i^{-\gamma_k} = 0. \quad (66) \]

If \( y_r \) is a reflecting repurchase boundary, then as we showed above, condition (63) is replaced by (58), or more explicitly:

\[ p'(y_r) = 0 : \sum_{k=1}^{3} \gamma_k B_k x_r^{-\gamma_k} = 0. \quad (67) \]

**Equity Value**  The equity value is determined by conditions:
\[ e(y_b) = 0 : \phi y_b - \rho + \sum_{k=1}^{3} c_k = 0, \]  
\[ e'(y_b) = 0 : \phi y_b - \sum_{k=1}^{3} c_k \gamma_k = 0, \]  
\[ \text{eq. (21)} : \frac{\phi}{\eta + 1} y_b - \rho + \sum_{k=1}^{3} c_k \frac{\eta}{\eta - \gamma_k} = 0, \]  
\[ \frac{e(y_i) + p_i^*}{y_i} = \frac{e(y_i^*) + p_i^*}{y_i^*} = \frac{c\pi}{(r + \xi) i} + \frac{1}{x_i} \sum_{k=1}^{3} \left( C_k + B_k \frac{c + \xi}{r + \xi} \right) x_i^{-1 \gamma_k} = \frac{c\pi}{(r + \xi) i} + \frac{1}{x_i} \sum_{k=1}^{3} \left( C_k + B_k \frac{c + \xi}{r + \xi} \right) x_i^{1 - \gamma_k}, \]  
\[ \frac{e(y_r) + p_r^*}{y_r} = \frac{e(y_r^*) + p_r^*}{y_r^*} = \frac{c\pi}{(r + \xi) r} + \frac{1}{x_r} \sum_{k=1}^{3} \left( C_k + B_k \frac{c + \xi}{r + \xi} \right) x_r^{-1 \gamma_k} = \frac{c\pi}{(r + \xi) r} + \frac{1}{x_r} \sum_{k=1}^{3} \left( C_k + B_k \frac{c + \xi}{r + \xi} \right) x_r^{1 - \gamma_k}, \]  
\[ \text{eq. (22)} : \sum_{k=1}^{3} \left\{ \frac{\eta (1 + \gamma_k)}{\eta - \gamma_k} \left[ \left( C_k + B_k \frac{c + \xi}{r + \xi} \right) x_r^{\eta - \gamma_k} - \left( C_k + B_k \frac{c + \xi}{r + \xi} \right) x_r^{\eta \gamma_k} \right] \right\} = \frac{c\pi}{r + \xi} (x_r^{\eta} - x_r^{\eta \gamma_k}). \]  

Conditions (21) and (22) are obtained by plugging the conjecture (20) into the HJB equation (14) and matching the terms at \( \lambda(y/y_b)^{-\eta} \). Conditions (68) – (74) give seven equations on six parameters \((c_k, C_k)_{k=1,2,3}\) and default threshold \(y_b\), and allow us to derive the equity value.

If \(y_i\) is a reflecting issuance boundary, then as we showed above, condition (71) is replaced by (57), or more explicitly:

\[ \frac{c\pi}{r + \xi} + \sum_{k=1}^{3} \left( C_k (\gamma_k + 1) + B_k \frac{c + \xi}{r + \xi} \right) x_i^{-1 \gamma_k} = 0. \]  

If \(y_r\) is a reflecting repurchase boundary, then as we showed above, condition (73) is replaced by (59), or more explicitly:

\[ \frac{c\pi}{r + \xi} + \sum_{k=1}^{3} \left( C_k (\gamma_k + 1) + B_k \frac{c + \xi}{r + \xi} \right) x_r^{-1 \gamma_k} = 0. \]
Enterprise Value per Unit of Debt

Using the conjectured form of function $v$ in (32), the conditions on $v$ can be written explicitly as

$$
\phi y_b + \frac{c \pi}{r + \xi} + \sum_{k=1}^{3} a_k = 0, \quad (77)
$$

$$
\frac{\phi \eta}{\eta + 1} y_b + \frac{c \pi}{r + \xi} + \sum_{k=1}^{3} \frac{a_k}{\eta - \gamma_k} = 0, \quad (78)
$$

$$
\frac{c \pi}{(r + \xi)x_r} + \sum_{k=1}^{3} a_k x_r^{-\gamma_k - 1} = \frac{c \pi}{(r + \xi)x_r^*} + \frac{1}{x_r^*} \sum_{k=1}^{3} A_k x_r^{* - \gamma_k}, \quad (79)
$$

$$
\frac{c \pi}{(r + \xi)\pi_r} + \frac{1}{x_r^*} \sum_{k=1}^{3} A_k \pi_r^{-\gamma_k} = \frac{c \pi}{(r + \xi)\pi_r^*} + \frac{1}{x_r^*} \sum_{k=1}^{3} A_k \pi_r^{* - \gamma_k}, \quad (80)
$$

$$
\frac{c \pi}{(r + \xi)x_i} + \frac{1}{x_i^*} \sum_{k=1}^{3} A_k x_i^{-\gamma_k} = \frac{c \pi}{(r + \xi)x_i^*} + \frac{1}{x_i^*} \sum_{k=1}^{3} A_k x_i^{* - \gamma_k}, \quad (81)
$$

$$
\sum_{k=1}^{3} \frac{\eta (1 + \gamma_k)}{\eta - \gamma_k} \left[ a_k x_i^{\eta - \gamma_k} - A_k \pi_r^{\eta - \gamma_k} \right] = \frac{c \pi}{r + \xi} (\pi_r^\eta - x_i^\eta). \quad (82)
$$

If $y_i$ is the reflecting issuance boundary, then the condition $v(y_i)/y_i = v(y_i^*)/y_i^*$, is replaced by

$$
v'(y_i) y_i = v(y_i), \quad (83)
$$

or more explicitly,

$$
\frac{c \pi}{r + \xi} + \sum_{k=1}^{3} A_k (\gamma_k + 1)x_i^{-\gamma_k} = 0. \quad (84)
$$

If $\pi_r$ is the reflecting repurchase boundary, then the condition $v(\pi_r)/\pi_r = v(\pi_r^*)/\pi_r^*$ is replaced by

$$
v'(\pi_r) \pi_r = v(\pi_r), \quad (85)
$$

or more explicitly,

$$
\frac{c \pi}{r + \xi} + \sum_{k=1}^{3} A_k (\gamma_k + 1)\pi_r^{-\gamma_k} = 0. \quad (86)
$$

B.2.3 Conditions Determining Leverage Dynamics under Targeted ICR Policies

Debt Price

The coefficients $b_i$s satisfy the following conditions:

$$p(y_b) = 0 : b_1 + b_2 + b_3 = -1, \quad (87)$$

$$p(y_r) = \hat{\phi} : \frac{c + \xi}{r + \xi} (1 + b_1 x_r^{-\gamma_1} + b_2 x_r^{-\gamma_2} + b_3 x_r^{-\gamma_3}) = \hat{\phi}, \quad (88)$$

$$\text{eq. (12)} : b_1 \frac{\eta}{\eta - \gamma_1} + b_2 \frac{\eta}{\eta - \gamma_2} + b_3 \frac{\eta}{\eta - \gamma_3} = -1. \quad (89)$$
Further, we show that this HJB equation together with (88) imply

$$\sum_{k=1}^{3} b_k x_r^{-\gamma_k} \left( r + \xi - \frac{\lambda \gamma_k}{\eta - \gamma_k} \left( \frac{\hat{x}}{x_r} \right)^{-\eta} \right) = 0. \quad (90)$$

Using the conjectured solution, we can re-write the HJB equation (28) as follows

$$(r + \xi + \lambda \hat{y}) \hat{p} = c + \xi + \lambda \hat{p} \left( 1 - \left( \frac{\hat{y}}{y_r} \right)^{-\eta} \right) + \lambda \frac{c + \xi}{r + \xi} \int_{\ln\left(\frac{\hat{y}}{y_r}\right)}^{\ln\left(\frac{\hat{y}}{y_b}\right)} \left( 1 + \sum_{k=1}^{3} b_k (e^{-\hat{y}/y_b})^{-\gamma_k} \right) \eta e^{-\eta \xi} d\xi,$$

or equivalently,

$$\left( r + \xi + \lambda \left( \frac{\hat{y}}{y_r} \right)^{-\eta} \right) \hat{p} = c + \xi + \lambda \frac{c + \xi}{r + \xi} \left( \left( \frac{\hat{y}}{y_r} \right)^{-\eta} - \left( \frac{\hat{y}}{y_b} \right)^{-\eta} \right) + \lambda \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k \left( \frac{\hat{y}}{y_r} \right)^{-\gamma_k \eta} \left( \left( \frac{\hat{y}}{y_r} \right)^{-\eta \gamma_k} - \left( \frac{\hat{y}}{y_b} \right)^{-\eta \gamma_k} \right).$$

Using the notation $\hat{x} = \hat{y}/y_b$ and $x_r = y_r/y_b$,

$$\left( r + \xi + \lambda \left( \frac{\hat{x}}{x_r} \right)^{-\eta} \right) \hat{p} = c + \xi + \lambda \frac{c + \xi}{r + \xi} \left( \left( \frac{\hat{x}}{x_r} \right)^{-\eta} - \hat{x}^{-\eta} \right) + \lambda \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k \hat{x}^{-\gamma_k \eta} \left( \left( \frac{\hat{x}}{x_r} \right)^{-\eta \gamma_k} - \hat{x}^{-\eta \gamma_k} \right).$$

Plugging in $\hat{p}$, we get

$$c + \xi \left( r + \xi + \lambda \left( \frac{\hat{x}}{x_r} \right)^{-\eta} \right) \left( 1 + \sum_{k=1}^{3} b_k x_r^{-\gamma_k} \right) = c + \xi + \lambda \frac{c + \xi}{r + \xi} \left( \left( \frac{\hat{x}}{x_r} \right)^{-\eta} - \hat{x}^{-\eta} \right) + \lambda \frac{c + \xi}{r + \xi} \sum_{k=1}^{3} b_k \hat{x}^{-\gamma_k \eta} \left( \left( \frac{\hat{x}}{x_r} \right)^{-\eta \gamma_k} - \hat{x}^{-\eta \gamma_k} \right).$$

Simplifying,

$$\left( r + \xi \right) \left( 1 + \sum_{k=1}^{3} b_k x_r^{-\gamma_k} \right) + \lambda \left( \frac{\hat{x}}{x_r} \right)^{-\eta} \left( 1 + \sum_{k=1}^{3} b_k x_r^{-\gamma_k} \right) = r + \xi + \lambda \left( \left( \frac{\hat{x}}{x_r} \right)^{-\eta} - \hat{x}^{-\eta} \right) + \lambda \sum_{k=1}^{3} b_k \hat{x}^{-\gamma_k \eta} \left( \left( \frac{\hat{x}}{x_r} \right)^{-\eta \gamma_k} - \hat{x}^{-\eta \gamma_k} \right).$$

or

$$\left( r + \xi \right) \sum_{k=1}^{3} b_k x_r^{-\gamma_k} + \lambda \left( \frac{\hat{x}}{x_r} \right)^{-\eta} \sum_{k=1}^{3} b_k x_r^{-\gamma_k} = -\lambda \hat{x}^{-\eta} + \lambda \sum_{k=1}^{3} b_k \hat{x}^{-\gamma_k \eta} \left( \left( \frac{\hat{x}}{x_r} \right)^{-\eta \gamma_k} - \hat{x}^{-\eta \gamma_k} \right),$$

or

$$\left( r + \xi \right) \sum_{k=1}^{3} b_k x_r^{-\gamma_k} + \lambda \sum_{k=1}^{3} b_k \hat{x}^{-\gamma_k \eta} x_r^{-\gamma_k} = -\lambda \hat{x}^{-\eta} + \lambda \sum_{k=1}^{3} b_k \eta \left( \hat{x}^{-\eta \gamma_k} x_r^{-\gamma_k} - \hat{x}^{-\eta} \right).$$
Using (89),
\[(r + \xi) \sum_{k=1}^{3} \frac{b_k x_r^{-\gamma_k}}{\eta - \gamma_k} = \lambda \sum_{k=1}^{3} \frac{b_k \gamma_k}{\eta} (\hat{x}^{-\eta} x_r^{-\gamma_k}),\]
or
\[\sum_{k=1}^{3} b_k x_r^{-\gamma_k} \left( r + \xi - \frac{\lambda \gamma_k}{\eta - \gamma_k} \left( \frac{\hat{x}}{x_r} \right)^{-\eta} \right) = 0,\]
which is the desired condition (90).

**Equity Value** The equity value satisfies the following conditions:

\[e(y_b) = 0 : \quad -\rho + \phi y_b + c_1 + c_2 + c_3 = 0, \quad (91)\]
\[e'(y_b) = 0 : \quad \phi y_b - c_1 \gamma_1 - c_2 \gamma_2 - c_3 \gamma_3 = 0, \quad (92)\]
\[\text{eq. (21)} : \quad \frac{\phi}{\eta} + \frac{1}{y_b} - \rho + \sum_{k=1}^{3} \frac{c_k}{\eta} - \frac{\gamma_k}{\eta} = 0. \quad (93)\]

We can re-write condition \(\frac{e(y_r)+\hat{p}}{y_r} = \frac{\hat{e}+\hat{p}}{\hat{x}}\) more explicitly as
\[
\frac{\phi y_b x_r - \rho + \sum_{i=1}^{3} c_i x_r^{-\gamma_i} + \hat{p}}{x_r} = \frac{\hat{e} + \hat{p}}{\hat{x}}. \quad (94)
\]

Finally, we show that using our conjecture and (93), we can rewrite the HJB equation (31) as
\[
\left( r - \hat{\mu} + \frac{\lambda}{\eta + 1} + \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^{-(\eta+1)} \right) \hat{e} = (1 - \pi)(\hat{x} y_b - c) - \xi
\]
\[+ \hat{p} \left[ \hat{\mu} + \xi - \frac{\lambda}{\eta + 1} - \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^{-(\eta+1)} + \lambda (\hat{x}/x_r)^{-\eta} \right]
\[+ \lambda (\hat{x}/x_r)^{-\eta} \left( \frac{\phi}{\eta} + \frac{1}{y_b} - \rho + \sum_{k=1}^{3} \frac{c_k}{\eta} - \frac{\gamma_k}{\eta} \right), \quad (95)
\]

We can re-write the HJB equation (31) as
\[
\left( r - \hat{\mu} + \frac{\lambda}{\eta + 1} + \frac{\lambda \eta}{\eta + 1} (\hat{y}/y_r)^{-(\eta+1)} \right) \hat{e} = (1 - \pi)(\hat{y} - c) - \xi + \hat{p} (\hat{\mu} + \xi)
\]
\[+ \lambda \left[ \hat{\mu} + \xi - \frac{\lambda}{\eta + 1} \left( (\hat{y}/y_r)^{-(\eta+1)} \right) - \lambda (\hat{y}/y_r)^{-\eta} \right]
\[+ \lambda \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_r)} e^{\left( e - \hat{\gamma} \right) \eta} = \eta s \, ds. \]
Using the conjectured form of $e$:

$$
\left( r - \hat{\mu} + \frac{\lambda}{\eta + 1} + \frac{\lambda \eta}{\eta + 1} (\hat{y}/y_r)^{-(\eta+1)} \right) \hat{e} = (1 - \pi)(\hat{y} - c) - \xi + \hat{p} (\hat{\mu} + \xi)
$$

$$
+ \lambda \frac{\hat{p} \eta}{\eta + 1} \left( 1 - (\hat{y}/y_r)^{-(\eta+1)} \right) - \lambda \hat{p} (1 - (\hat{y}/y_r)^{-\eta})
$$

$$
+ \lambda \int_{\ln(\hat{y}/y_r)}^{\ln(\hat{y}/y_b)} \left( \phi \hat{y} e^{-\hat{s}} - \rho + \sum_{k=1}^{3} c_k (e^{-\hat{s} \hat{y}/y_b})^{-\gamma_k} \right) e^{-\eta \hat{s}} d\hat{s}.
$$

or

$$
\left( r - \hat{\mu} + \frac{\lambda}{\eta + 1} + \frac{\lambda \eta}{\eta + 1} (\hat{y}/y_r)^{-(\eta+1)} \right) \hat{e} = (1 - \pi)(\hat{y} - c) - \xi + \hat{p} (\hat{\mu} + \xi)
$$

$$
+ \lambda \frac{\hat{p} \eta}{\eta + 1} \left( 1 - (\hat{y}/y_r)^{-(\eta+1)} \right) - \lambda \hat{p} (1 - (\hat{y}/y_r)^{-\eta})
$$

$$
+ \lambda \frac{\hat{p} \eta}{\eta + 1} \left( (\hat{y}/y_r)^{-(\eta+1)} - (\hat{y}/y_b)^{-(\eta+1)} \right)
$$

$$
- \lambda \rho (\hat{y}/y_r)^{-\eta} - (\hat{y}/y_b)^{-\eta}
$$

$$
+ \lambda \sum_{k=1}^{3} c_k (\hat{y}/y_b)^{-\gamma_k} \eta \left( (\hat{y}/y_r)^{-(\eta-\gamma_k)} - (\hat{y}/y_b)^{-(\eta-\gamma_k)} \right),
$$

or using $\hat{x} = \hat{y}/y_b$ and $x_r = y_r/y_b$,

$$
\left( r - \hat{\mu} + \frac{\lambda}{\eta + 1} + \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^{-(\eta+1)} \right) \hat{e} = (1 - \pi)(\hat{x} y_b - c) - \xi
$$

$$
+ \hat{p} \left[ \hat{\mu} + \xi - \frac{\lambda}{\eta + 1} - \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^{-(\eta+1)} + \lambda (\hat{x}/x_r)^{-\eta} \right]
$$

$$
+ \lambda \frac{\hat{p} \eta}{\eta + 1} \left( (\hat{x}/x_r)^{-(\eta+1)} - \hat{x}^{-\eta} \right)
$$

$$
- \lambda \rho (\hat{x}/x_r)^{-\eta} - \hat{x}^{-\eta}
$$

$$
+ \lambda \sum_{k=1}^{3} c_k \hat{x}^{-\gamma_k} \eta \left( (\hat{x}/x_r)^{-(\eta-\gamma_k)} - \hat{x}^{-\eta-\gamma_k} \right),
$$

65
or

\[
(r - \hat{\mu} + \frac{\lambda}{\eta + 1} + \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^-(\eta+1)) \hat{e} = (1 - \pi)(\hat{x}y_b - c) - \xi
\]

\[
+ \hat{p} \left[ \hat{\mu} + \xi - \frac{\lambda}{\eta + 1} - \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^-(\eta+1) + \lambda(\hat{x}/x_r)^-\eta \right]
\]

\[
+ \frac{\lambda \phi \hat{x} y_b}{\eta + 1} (\hat{x}/x_r)^-(\eta+1) - \lambda \rho (\hat{x}/x_r)^-\eta
\]

\[
+ \lambda \sum_{k=1}^{3} \frac{c_k \hat{x}^{-\gamma_k \eta} (\hat{x}/x_r)^-(\eta-\gamma_k)}{\eta - \gamma_k}
\]

\[
+ \lambda \hat{x}^{-\eta} \left( \rho - \frac{\phi y_b \eta}{\eta + 1} - \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} \right).
\]

Using (93),

\[
(r - \hat{\mu} + \frac{\lambda}{\eta + 1} + \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^-(\eta+1)) \hat{e} = (1 - \pi)(\hat{x}y_b - c) - \xi
\]

\[
+ \hat{p} \left[ \hat{\mu} + \xi - \frac{\lambda}{\eta + 1} - \frac{\lambda \eta}{\eta + 1} (\hat{x}/x_r)^-(\eta+1) + \lambda(\hat{x}/x_r)^-\eta \right]
\]

\[
+ \lambda (\hat{x}/x_r)^-\eta \left( \frac{\phi y_b \eta}{\eta + 1} y_b - \rho + \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} x_r^{-\gamma_k} \right),
\]

or

\[
\left( r - \hat{\mu} + \frac{\lambda}{\eta + 1} \right) (\hat{x}/x_r)^\eta + \frac{\lambda \eta x_r}{(\eta + 1)x} \right) \hat{e} = ((1 - \pi)(\hat{x}y_b - c) - \xi) (\hat{x}/x_r)^\eta
\]

\[
+ \hat{p} \left[ \left( \hat{\mu} + \xi - \frac{\lambda}{\eta + 1} \right) (\hat{x}/x_r)^\eta - \frac{\lambda \eta x_r}{(\eta + 1)x} + \lambda \right]
\]

\[
+ \lambda \left( \frac{\phi y_b \eta}{\eta + 1} y_b - \rho + \sum_{k=1}^{3} \frac{c_k \eta}{\eta - \gamma_k} x_r^{-\gamma_k} \right),
\]

which is the desired equation (95).