Intertemporal Preference with Loss Aversion:
Consumption and Risk-Attitude

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Abstract

We study the consumption and portfolio selection problem of an agent who faces consumption irreversibility: there is disutility from changing consumption levels. The derived preference exhibits intertemporal loss aversion toward consumption changes with the previous consumption level being the reference point. The optimization problem involves the non-monotonic and non-concave utility function. By combining a duality method and the super-contact principle, we derive the closed-form solution. We show that the consumption policy involves an inaction interval for the consumption-wealth ratio, which can explain the four stylized facts about consumption at once. The optimal portfolio choice exhibits a U-shape in the inaction interval, which sheds light on the empirical debate on the relationship between a household’s financial wealth and the share invested in risky assets.

JEL Classification Codes: C61, D11, D15, E21, G11

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1 Introduction

In this paper we study a model of intertemporal preference with loss aversion. We develop the model by introducing disutility (utility adjustment cost) from changing consumption levels. Disutility implies an irreversibility of consumption decisions in the following sense: if an agent increases or decreases consumption now, she cannot reverse the decision in the future without incurring utility costs.\footnote{The utility adjustment costs can be interpreted as mental costs from changing consumption. Thus, our model of irreversible consumption, in a formal sense, is similar to that of irreversible investment (see e.g., Abel and Eberly (1996), Dixit and Pindyck (1998)).} By solving a continuous-time consumption and investment problem in closed form, we show that the irreversibility can explain the four stylized facts in consumption at once: (i) excess smoothness, (ii) excess sensitivity, (iii) the disappearance of (i) and (ii) for large shocks, and (iv) asymmetric sensitivities to wealth (income) shocks. We also show that the optimal portfolio choice can explain the empirical puzzle on the relationship between the financial wealth and the risk share. Moreover, our model induces risk attitude with time-varying risk aversion.

Consumption irreversibility is closely related to loss aversion. The utility function can be expressed as two equivalent forms: one with consumption irreversibility and the other with intertemporal loss aversion (ILA). The equivalence is established by rewriting the agent’s utility function with respect to consumption levels as that with respect to the changes in consumption. It is necessary to use the first form for the solution analysis. The second is conceptually more useful when we discuss implications. It is particularly seen from the second form that the preference displays loss aversion toward a consumption change with the previous consumption level being the reference point. Note that the utility function representing the preference exhibits a minimal departure from the canonical time-separable utility function. Our model is perhaps the simplest among those displaying loss aversion toward a consumption gamble (Bowman et al. (1999), Kösezi and Rabin (2006, 2007, 2009), and Pagel (2017)), since the previous level of consumption is conceptually the simplest possible reference point.\footnote{Kösezi and Rabin (2006, 2007, 2009) propose models in which the reference point is stochastic and based on expectations. They also acknowledge that past consumption is a natural reference point. For example, Kösezi and Rabin (2009) write, “past consumption is in many circumstances a major determinant of expected future consumption.”}

Technically the irreversibility or existence of adjustment costs makes the utility function non-monotonic and non-concave in consumption; the op-
Optimization problem involving such an objective function is non-trivial. To tackle this difficulty, we first transform the original problem to a dual problem by using a martingale approach. Then, the dual problem is eventually reduced to a solvable two-sided singular control problem: it can be properly handled by the super-contact principle often used for analyzing a firm’s investment decisions (e.g., Abel and Eberly (1996) and Dixit and Pindyck (1998)). We provide optimal policies in closed form with a full verification of their optimality. The optimal policies are determined by three preference parameters for subjective discounting, risk aversion, and ILA. In particular, we investigate how ILA affects the optimal policies and disentangle the effects of risk aversion and those of ILA.

The optimal consumption policy involves inaction, due to adjustment costs (Figure 1). More precisely, it is characterized by two numbers, say $\underline{c}$ and $\bar{c}$ (with $\underline{c} < \bar{c}$). When the ratio $c_{t-1}/X_t$ of the previous consumption level to wealth, which we call the consumption-wealth ratio, is inside $[\underline{c}, \bar{c}]$, called the inaction interval, consumption is not adjusted. When the ratio is outside the interval, consumption is adjusted immediately so that the ratio is restored to the nearest boundary of the interval. The inaction interval becomes wider as ILA increases, i.e., the agent adjusts consumption less frequently as ILA increases.

The consumption policy described in Figure 1 explains the following four consumption puzzles documented in the empirical literature at once: the excess smoothness of consumption (Deaton, 1987), its excess sensitivity (Flavin, 1981), the disappearance of both smoothness and sensitivity with large wealth or income shocks (magnitude hypothesis, Jappelli and Pista-function to be monotone increasing and concave.

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See Cox and Huang (1989) and Karatzas et al. (1987).
ferri (2010)), and the asymmetric sensitivities to wealth (income) changes (Shea (1995) and Bowman et al. (1999)). First, the agent does not increase consumption immediately in response to a permanent good shock when the magnitude of the shock is moderate enough so that the consumption-wealth ratio slightly decreases (since wealth increases) but does not go out of the inaction interval. Second, however, while such a good shock does not increase consumption now, it increases the probability of increasing consumption in subsequent periods. Therefore, consumption in subsequent periods turns out to be excessively sensitive to changes in income or wealth. Third, both excessive sensitivity and excess smoothness disappear for a large shock since the consumption-wealth ratio immediately reaches a boundary of the inaction interval for such a large shock. Fourth, notice $\bar{c} > \bar{c}$ in Figure 1. This means that consumption exhibits asymmetric sensitivities to wealth (or income) changes: the sensitivity of a consumption decrease in response to a decrease in wealth is higher than that of a consumption increase in response to an increase in wealth. Note that this asymmetry can be explained by loss aversion models (Bowman et al., 1999) and the aforementioned three consumption puzzles can be explained by consumption commitment models (Chetty and Szeidl, 2016). However, to the best of our knowledge, our model is the first that can explain all four stylized facts at once.

The optimal portfolio policy features a U-shaped relationship between financial wealth and the share of risky assets in wealth, called the \textit{risky share}. If ILA increases, the U-shaped relationship becomes more pronounced: the minimum risky share decreases with loss aversion while its maximum is determined by risk aversion and unchanged by loss aversion (Figure 5(a)).

The U-shaped relationship provides a potential explanation to the debate in the literature on the relationship between the household’s financial wealth and the risky share. Calvet et al. (2009) and Calvet and Sodini (2014) show evidence that the risky share tends to increase with financial wealth, which is consistent with the habit or commitment or decreasing relative risk aversion (DRRA) models. However, Brunnermeier and Nagel (2008) and Chiappori and Paiella (2011) find a neutral or slightly negative relationship, providing evidence consistent with the standard CRRA model. We show that both the upward and downward changes of the risky share can happen in response to increases in wealth if the household’s preference exhibits intertemporal loss aversion. The U-shaped relationship implies that the risky share is decreasing in wealth on the left side of the inaction interval and increasing in wealth on its right side (see Figure 3). If good (bad) shocks arrive more frequently, the wealth process tends to stay longer in the increasing (decreasing) region of the interval, which makes the relation-
ship appear to be positive (negative) during the data period. We confirm this intuition by simulation. We generate populations with different ILA and simulate three types of sample paths: (a) bullish, (b) bearish, and (c) no trend (neither bullish nor bearish). Then, we regress the change of the risky share on the change of financial wealth within each sample in the same manner as Brunnermeier and Nagel (2008). We find that for case (a) there is a significant positive impact of the wealth increase on the risky share, while for case (b) there is a significant negative impact. We find, however, no relationship for case (c). This result sheds light on the discrepancy in the empirical literature.

Attitudes toward risk are characterized by the revealed coefficient of relative risk aversion (RCRRA), i.e., the level of relative risk aversion inferred by an outsider who observes the risky investment of the agent but assumes the agent is not loss averse. The RCRRA has an inverted U-shape relationship with the consumption-wealth ratio and wealth, due to its inverse relationship with the risky share. The RCRRA is higher than the actual coefficient of relative risk aversion (CRRA) of the underlying felicity function inside the inaction interval and is equal to the CRRA at its boundary points. The inverted U-shape becomes wider and higher as ILA increases. That is, the maximum value of RCRRA increases with ILA, while its minimum value is fixed as the CRRA at each boundary (Figure 5(b)). An individual with high ILA and low risk aversion can appear to be highly risk averse to a small shock (i.e., extreme risk aversion), but appear to be aggressive to a large shock or consecutive shocks that make the consumption-wealth ratio stay near a boundary (i.e., excessive risk-taking) in time. Such an individual’s consumption-wealth ratio moves in the middle of the inaction interval so her RCRRA is high during the times when the moderate good and bad shocks alternate.

**Related literature:** There exists a vast literature on loss aversion including reference dependence and thus we do not attempt to summarize it here (see, e.g., Kahneman and Thaler (2000), Barberis and Thaler (2003), Barberis and Huang (2008), and O’Donoghue and Sprenger (2018) for surveys). The key difference between our model and existing models is that the reference point in our model is the previous consumption level. For example, Benartzi and Thaler (1995) and Barberis et al. (2001) model loss aversion toward wealth changes. In Yogo (2008) a geometric average of past consumption is used as a reference point. In addition, note that an important aspect of the prospect theory is probability distortion, which our model does not have. See Ai and Bansal (2018) on a characterization of preferences exhibiting probability distortion, based on recursive formulation.

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sion. Our model is indeed a continuous-time generalization of their model for the case in which the weight is given to the most recent consumption in calculation of the reference point.

Pagel (2017) studies life-cycle consumption/savings decisions when the agent has the expectations-based loss aversion proposed by Kösgezi and Rabin (2006, 2007, 2009) and shows that the expectations-based loss aversion generates excess smoothness and sensitivity of consumption. Thanks to its anchoring with past consumption, our model exhibits stricter rigidity in optimal consumption, as implied by the inaction interval. Pagel (2017) does not consider the portfolio selection problem as we do in our model, by which our model additionally explains the empirical puzzle of the relationship between financial wealth and the risky share. Furthermore, our model admits closed-form expressions for optimal consumption policies, whereas multi-period expectations-based loss aversion models rarely admit closed-form expressions.

Our paper is originally motivated by the classical work of Duesenberry (1949) in that our model builds on the assumption of the partial irreversibility of consumption decisions. His critique is closely related to (external and internal) habit formation and thus has significantly contributed to the development of the habit literature. Part of our results resembles those from the habit models. For example, both our model and habit models generate time-varying risk aversion. Habit models, however, neither imply the higher sensitivity of consumption to wealth (income) changes when consumption is adjusted downward than when consumption is adjusted upward, nor explain the magnitude hypothesis, i.e., the hypothesis that both the excess smoothness and the excess sensitivity disappear for large shocks.

Our model is also related to dynamic consumption and investment models with fixed consumption adjustment costs or with consumption commitments (Grossman and Laroque (1990), Flavin and Nakagawa (2008), and

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6 The expectations-based loss aversion models exhibit the first-order precautionary motive, implying that consumption tends to be low. In our model the rigidity implies that overconsumption happens as frequent as underconsumption.

7 Closed-form expressions allow us to derive clear analytic results and have a potential to be applied for asset management in the context in which the user needs stable cash flows from investment. Also, our framework is easily extended to the case with multi-goods. See our companion paper Choi et al. (2020a) for these analyses.

8 "This critique is based on a demonstration that two fundamental assumptions of aggregate demand theory are invalid. These assumptions are (i) that every individual’s consumption behaviour is independent of that of every other individual, and (ii) that consumption relations are reversible in time (Duesenberry, 1949)."

9 See, e.g., Abel (1990), Constantinides (1990), Detemple and Zapatero (1991), and Campbell and Cochrane (1999).
Chetty and Szeidl (2007, 2016)). These models and our model share common features: consumption adjustments are infrequent, consumption responses to moderate wealth shocks exhibit excess smoothness and excess sensitivity, and these properties disappear when consumption responds to large wealth shocks. These models, however, do not exhibit asymmetric sensitivities of consumption for downward adjustments and upward adjustments, as our model does. They assume monetary adjustment costs which are subtracted from wealth, whereas we assume mental adjustment costs which are subtracted from the utility value.\footnote{More precisely, when there are the monetary adjustment costs, the optimal policy requires a third number located in the middle of the inaction interval to which the ratio $c_{t-}/X_t$ is adjusted if the ratio reaches the boundary points. Thus, the asymmetric consumption sensitivity does not appear in consumption commitment models.} Our model predicts a U-shaped relationship between financial wealth and the risky share, while the habit and consumption commitment models generate a monotone increasing relationship (Brunnermeier and Nagel, 2008).

Finally Dybvig (1995), Riedel (2009) and Jeon et al. (2018) study models in which consumption decisions are completely irreversible, i.e., consumption is not allowed to decline over time (other than a predetermined depreciation), and Shrikhande (1997) and Watson and Scott (2015) study models with mental adjustment costs.\footnote{Shrikhande (1997) did not solve the problem that he outlined in the setup, but instead tackled the different problem with monetary costs in the main body of the paper.} All these models are special cases of our model. We provide further discussions on these models as well as habit and commitment models in the main body of the paper.

The rest of the paper is organized as follows. Section 2 explains the discrete-time utility setup with consumption irreversibility and its equivalent form with IRA, and provide their continuous-time versions. Section 3 describes the agent’s problem and provides the solution analysis. We investigate the model implications for the consumption policy in Section 4 and for the portfolio choice and the risk attitude in Section 5. Section 6 concludes the paper. All the proofs are provided in the Appendix.

\section{Preference}

\subsection{Intertemporal Loss Aversion}

We first set up the utility function with consumption irreversibility modeled by mental utility costs from changing the level of consumption.\footnote{The preference can also be derived from a natural extension of the canonical time-separable utility function. These utility functions derived in different ways are fundamentally equivalent.} Later we...
rewrite and reinterpret the preference in terms of loss aversion.

Consider an agent living in a one-good economy until \( t = T \) that can be finite or infinite. For an exogenously given consumption level \( c_0 \), the agent has the following utility function:

\[
U = \mathbb{E} \left[ \sum_{t=0}^{T} \hat{\delta}^t \left( u(c_t) - \alpha_t(\Delta u(c_t))^+ - \beta_t(\Delta u(c_t))^- \right) \right],
\]

where \( \alpha_t + \beta_t > 0 \). Here \( \mathbb{E} \) denotes expectation, \( \hat{\delta} \) is the subjective discount factor \( (0 < \hat{\delta} < 1) \), \( c_t \) is consumption at time \( t \), \( u \) is strictly increasing concave felicity function, \( \alpha_t \) and \( \beta_t \) are possibly dependent on time and state of the world, \( \Delta u(c_t) \equiv u(c_t) - u(c_{t-1}) \) for \( t > 0 \) and \( \Delta u(c_0) \equiv u(c_0) - u(c_{0-}) \), and superscript \( + \) (\( - \)) means the positive (negative) part of a real-valued function, i.e., \( f^+ = \max(f, 0) \) (\( f^- = \max(-f, 0) \)).

The first part of the utility function (1) is a standard expected utility with felicity function \( u \), the second part \( \alpha_t(\Delta u(c_t))^+ \) is the utility cost of adjusting consumption upward, and the third part \( \beta_t(\Delta u(c_t))^- \) is the utility cost for of adjusting consumption downward. Parameter \( \alpha_t \) (\( \beta_t \)) is a proportional cost of an upward (downward) consumption adjustment. This formulation means that there are mental utility costs from changing consumption decision: the consumption decision is partially irreversible.

Conceptually the preference is closely related to intertemporal loss aversion. To see this, we rewrite the period-\( t \) component of the utility as follows.

\[
u(c_t) - \alpha_t(\Delta u(c_t))^+ - \beta_t(\Delta u(c_t))^- = \begin{cases} 
  u(c_{t-1}) + (1 - \alpha_t)(\Delta u(c_t))^+ & \text{if } \Delta c_t > 0, \\
  u(c_{t-1}) - (1 + \beta_t)(\Delta u(c_t))^- & \text{if } \Delta c_t < 0.
\end{cases}
\]

Equation (2) shows that the agent calculates a gain or a loss with reference to the previous level of consumption. The loss has a higher weight than the gain if \( \alpha_t > -\beta_t \) (see Assumption 1), implying that the period utility function exhibits loss aversion.

Note that in a one-period multi-good economy Brunnermeier (2004) shows that learning to re-optimize in accordance with a change in income results in adjustment costs. In this sense, our model can be regarded as a reduced-form model of an intertemporal preference with learning costs. Considering the period-\( t \) utility of the agent, i.e., \( u(c_t) - \alpha_t(\Delta u(c_t))^+ - \beta_t(\Delta u(c_t))^- \), our model extends the following models: Dybvig (1995) and Riedel (2009) consider models with \( \alpha_t = 0 \) and \( \beta_t = \infty \), Shrikhande (1997) considers a model where \( \alpha_t = -1, \beta_t > 1 \) and Watson and Scott (2015) consider a model with \( \alpha_t = 0 \) and \( \beta_t > 0 \). The period utility function (2) is an extension of a special case of the model proposed by Bowman et al. (1999); their model with \( \alpha = 1 \) corresponds to our model with \( \alpha_t < 0 \).
Note that consumption at \( t \) is the result of its adjustments from the initial level of consumption \( c_0 \), i.e.,

\[
u(c_t) = u(c_0) + \Delta u(c_0) + \cdots + \Delta u(c_t) = u(c_0) + \sum_{s=0}^{t} \Delta u(c_s).
\]

Thus, we can rewrite the first part of the utility function as follows: defining \( \delta \equiv 1 - \hat{\delta} \)

\[
E \left[ \sum_{t=0}^{T} \delta^t u(c_t) \right] = E \left[ \sum_{t=0}^{T} \delta^t \left( u(c_0) + \sum_{s=0}^{t} \Delta u(c_s) \right) \right] = \frac{1 - \delta^{T+1}}{\delta} u(c_0) + E \left[ \sum_{t=0}^{T} \delta^t \frac{1 - \delta^{T-t+1}}{\delta} \Delta u(c_t) \right]. \tag{3}
\]

Equation (3) is the key when transforming the utility function (1) of the form with consumption irreversibility into the form with ILA that will appear later in (5). Equation (3) rewrites the utility function in terms of absolute consumption levels (the left-hand side) as that in terms of changes in consumption levels (the right-hand side). The first term in the right-hand side is the utility value if the initial level \( c_0 \) is maintained until \( t = T \). The second term is the expected present value of changes in utility due to changes in consumption, which has a natural interpretation that each change has a permanent effect on the agent’s utility value.

Combining the permanent effect of a change in consumption and the adjustment costs, we can calculate the present value of utility gains and losses. The present value for a positive change is equal to \( \frac{1 - \delta^{T-t+1}}{\delta} - \alpha_t \sigma (\Delta u(c_t))^+ \) and the present value of utility loss for a negative change (in absolute value) is equal to \( \frac{1 - \delta^{T-t+1}}{\delta} + \beta_t \sigma (\Delta u(c_t))^− \).

The agent does not increase consumption at \( t \) unless \( \frac{1 - \delta^{T-t+1}}{\delta} - \alpha_t > 0 \), since the total utility gain would not be positive otherwise. This observation leads to the following assumption. Here, note that \( \alpha_t < 0 \) is interpreted as utility gains from increasing consumption.

**Assumption 1.**

\[-\beta_t < \alpha_t < \frac{1 - \delta^{T-t+1}}{\delta}, \quad \beta_t \geq 0 \quad \text{for every} \quad t \geq 0.\]

Now we are ready to define intertemporal loss aversion.

**Definition 2.1.** We define the coefficient of intertemporal loss aversion (ILA) \( L_t \) at \( t \) to be the ratio of the marginal utility loss to the marginal utility gain, i.e.,

\[
L_t \equiv \frac{(1 - \delta^{T-t+1} + \beta_t) \sigma \Delta u(c_t)}{(1 - \delta^{T-t+1} - \alpha_t) \sigma \Delta u(c_t)} = \frac{1 - \delta^{T-t+1} + \delta \beta_t}{1 - \delta^{T-t+1} - \delta \alpha_t}. \tag{4}
\]
Finally we rewrite the utility function (1) by using (3) and ILA as follows:

\[ U = \frac{1 - \dot{\delta}^{T+1}}{\delta} u(c_0) + \frac{1}{\delta} \mathbb{E} \left[ \sum_{t=0}^{T} D(t) \left( (\Delta u(c_t))^+ - L_t(\Delta u(c_t))^-) \right) \right], \quad (5) \]

where \( D(t) \equiv \dot{\delta}^t (1 - \dot{\delta}^{T-t+1} - \delta \alpha_t) \). The agent’s utility in (5) consists of three parts: (1) the present value (PV) of the utility value of the previously given level of consumption, (2) the PV of the increases in the utility value due to increases in consumption, and (3) the negative of the PV of the decreases in the utility value due to decreases in consumption multiplied by the coefficient of ILA.

Before we move on to continuous-time case, note that the utility function (5) is not necessarily increasing in consumption.\(^{13}\) This is not surprising, since loss aversion often violates the first-order stochastic dominance as shown by Kösegezi and Rabin (2007) and Masatlioglu and Raymond (2016). Masatlioglu and Raymond also show that the preference proposed by Kösegezi and Rabin (2007) exhibits global risk aversion only when it satisfies the first-order stochastic dominance for the case where the consumption utility is non-linear (Proposition 6, Masatlioglu and Raymond (2016)). We establish a similar result in Appendix A: a sufficient and necessary condition for the utility function (5) to be monotone increasing and concave.

### 2.2 Continuous-Time Preference

We now consider a continuous-time limit of the utility functions. From now, for explicit solutions, we consider the infinite horizon. Heuristically, as the length of the time interval goes to zero, the limit is

\[ U = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) - \alpha_t (du(c_t))^+ - \beta_t (du(c_t))^- \right) dt \right], \quad (6) \]

where \( \delta > 0 \) is the subjective discount rate, and \( c_t \) is the rate of consumption (that we will simply call *consumption*), and \( \alpha_t (du(c_t))^+ \) means the utility cost (or gain if \( \alpha_t < 0 \)) of an upward consumption adjustment and \( \beta_t (du(c_t))^− \) is the utility cost of a downward consumption adjustment with \( \alpha_t + \beta_t > 0, \beta_t \geq 0 \). The terms \((du(c_t))^+\) and \((du(c_t))^−\) are, however, not precisely defined. We provide a rigorous formal definition as follows:

\[ U = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) - \alpha_t du_t^+ - \beta_t du_t^- \right) dt \right], \quad (7) \]

\(^{13}\)For example, let \( C^0 \) be the consumption profile such that \( c_t = 0 \) for every \( t \geq -1 \) and \( C^1 \) be the consumption profile such that \( c_t = 1 \) for every \( t \geq -1 \) except \( c_0 = 2 \) and suppose that \( L_1 > 1 \). Then \( U(C^0) > U(C^1) \) if \( D(1) \) is sufficiently close to 1.

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where \( u_t^+ \) and \( u_t^- \) denote positive and negative variations of \( u(c_t) \) over \([0-, t]\), which we will call positive and negative variation processes.\(^{14}\) Note that the utility loss is infinite if the agent adjusts consumption too often within a finite time interval. Definition (7) is rigorous and it precludes any consumption policies which result in a process \( c_t \) with an infinite total variation over a finite time interval as suboptimal.

Similarly to the discrete-time case, we define the coefficient of ILA. An important case of the model is the case when \((\alpha_t, \beta_t) = (\alpha, \beta)\), for all \( t \geq 0 \), where \( \alpha, \beta \) are non-negative constants satisfying Assumption 1, i.e., \(-\beta < \alpha < 1/\delta\). In this case, ILA is constant for all \( t \geq 0 \):

\[
L_t = L \equiv \frac{1 + \delta \beta}{1 - \delta \alpha}, \text{ where } -\beta < \alpha < 1/\delta.
\] (8)

Note that \( L = 1 \) if and only if \( \alpha = \beta = 0 \). Thus, the agent with \( L = 1 \) has no loss aversion. ILA is an increasing function of the adjustment costs and the subject discount rate. Increases in adjustment costs raise the weight of utility losses relative to that of utility gains and thus raise the ILA coefficient. An individual with a higher subjective discount rate discounts future utility more heavily and current utility loss has a higher weight in her utility, and thus she has higher intertemporal loss aversion.\(^{15}\)

Given the same \( c_0^- \), the utility functions with two different pairs of parameters \((\alpha, \beta)\) and \((\alpha', \beta')\) are affine transformations of the other if the two have the same ILA. In other words, they induce the same ordinal preference. The following proposition rewrites the utility function (7) in terms of ILA:

**Proposition 2.1.** The utility function (7) can be rewritten as follows:

\[
U = \frac{u(c_0^-)}{\delta} + \frac{1 - \delta \alpha}{\delta} \mathbb{E} \left[ \int_0^{\infty} e^{-\delta t} \left( du_t^+ - Ldu_t^- \right) \right].
\] (9)

We will use the utility function (7) of the form with consumption irreversibility for our solution analysis, not (9) of the form with ILA, since the former allows us to apply a martingale approach and the super-contact principle to the analysis. Form (9) will be used later when we interpret optimal policies with respect to ILA and discuss the implications.

\(^{14}\)The positive (negative) variation over \([T_1, T_2]\) of a process \( X_t \) is defined as \( \sup_{[t_0, t_1, \ldots, t_N]} \sum (X_{t_{i+1}} - X_{t_i})^+ \) (\( \sup_{[t_0, t_1, \ldots, t_N]} \sum (X_{t_{i+1}} - X_{t_i})^- \)) where \( [t_0, t_1, \ldots, t_N] \) is an arbitrary partition of \([T_1, T_2]\). The total variation is defined as the sum of the positive and negative variations. The positive variation, negative variation, and total variation are all infinite if any one of them is infinite (Theorem 2.6, Wheeden (2015)).

\(^{15}\)Note that Researchers measure loss aversion in a laboratory context without taking the intertemporal effects of consumption changes into consideration (Kahneman et al. (1990), Tversky and Kahneman (1991), Abdellaoui et al. (2007), Abdellaoui et al. (2008)). Our result suggests that the subjective time discounting factor can be important when measuring the loss aversion coefficient in an intertemporal context.
3 Model and Solution Analysis

3.1 Problem

Financial Market: We consider a standard continuous-time financial market as in Grossman and Laroque (1990) and Flavin and Nakagawa (2008). Namely, there exist two assets: a risk-free asset with a constant interest rate \( r \) and a market portfolio \( S_t \) that we call the risk asset. It satisfies

\[
dS_t/S_t = \mu dt + \sigma dB_t,
\]

for some constants \( \mu, \sigma, \mu > r \), and \( B_t \) is a 1-dimensional standard Brownian Motion defined on a standard probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \{\mathcal{F}_t\}_{t \geq 0} \) is an augmented filtration generated by \( B_t \).

As in Grossman and Laroque (1990) and in Flavin and Nakagawa (2008) we abstract from labor income and assume that the agent’s wealth is kept in the form of financial assets. The assumption is equivalent to an alternative assumption that the agent receives a stream of labor income whose full capitalized value can be utilized for investments in financial assets without any restriction.

Optimization Problem: The agent’s wealth process \( (X_t)_{t \geq 0} \) evolves according to the following dynamics:

\[
dx_t = \left[ rX_t + \pi_t(\mu - r) - c_t \right] dt + \sigma\pi_t dB_t, \quad X_0 = X > 0, \quad (10)
\]

where \( \pi_t \) is the dollar amount invested in the risky asset at time \( t \). Note that by our assumption the adjustment costs are subtracted from the utility value, not from wealth. Hence, the wealth process is the same as in Merton (1969, 1971) and the financial market has no frictions and is complete.

We provide technical conditions for admissible policies in Appendix B. Let II denote the set of admissible policies satisfying the technical conditions.

---

16This is a simplified version of \( n \)-risky assets, where the value of the \( i \)-th risky asset (including accumulated dividends) \( \hat{z}_{i,t} \) follows \( d\hat{z}_{i,t} = \hat{z}_{i,t}(\mu_i dt + dw_{i,t}) \), where \( w_{1,t}, \ldots, w_{n,t} \) are arithmetic Brownian motions without drift and have positive definite instantaneous covariance matrix \( \Sigma \). Assuming \( \mu_1, \ldots, \mu_n \) and \( \Sigma \) are constant, a general equilibrium consideration leads naturally to the capital asset pricing model (CAPM) and a two-fund separation theorem as in Grossman and Laroque (1990) and Flavin and Nakagawa (2008). The agent’s portfolio selection can be described as an allocation between the riskless asset and the market portfolio, which consists of all risky assets in the economy. In this case, \( B_t \) is constructed from \((w_{1,t}, \ldots, w_{n,t})\).

17See Koo (1998) for calculation of capitalized value of a stream of income in the presence or absence of frictions.

18Our problem is thus different from those of Shrikhande (1997) and Cuoco and Liu (2000), where the adjustment costs are subtracted from wealth and the financial market has frictions.
Problem 1 (Primal Problem (Dynamic Version)).

Given $c_0 = c > 0$ and $X_0 = X > 0$, we consider the following optimization problem of the agent:

$$V(X, c) = \max_{(c_t, \pi_t) \in \Pi} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right], \quad (11)$$

subject to the wealth process (10).

Finally we assume that the felicity function takes the following form:

$$u(c) = \begin{cases} 
\frac{c^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1, \\
\log c, & \gamma = 1.
\end{cases} \quad (12)$$

That is, the agent has constant relative risk aversion $\gamma$. We also assume that the agent has constant ILA $L$ with $\alpha$ and $\beta$ as in (8). The following assumption guarantees the existence of a solution to the problem without adjustment costs when the felicity function is (12) (Merton (1969, 1971)).

Assumption 2.

$$K \equiv r + \frac{\delta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \theta^2 > 0 \quad \text{with} \quad \theta \equiv \frac{\mu - r}{\sigma}. \quad (13)$$

3.2 Solution Analysis

Problem 1 is dynamic in character, since it is subject to the dynamic wealth constraint (10). We transform the problem to a static problem (Subsection 3.2.1). Again we transform the static problem into a dual problem. Mathematically the dual problem in our case is a two-sided singular control problem. We will establish a duality relationship between the value function of the dual problem and that of the primal problem.

3.2.1 Dual Problem

We define the stochastic discount factor or state price density by

$$\xi_t \equiv e^{-rt} Z_t, \quad \text{and} \quad Z_t \equiv e^{-\frac{1}{2} \theta^2 t - \theta B_t}, \quad t \geq 0.$$  

Then, we have the following budget constraint, implying the present value of the consumption stream is no more than the initial wealth $X_0 = X$.

$$\mathbb{E} \left[ \int_0^\infty \xi_t c_t dt \right] \leq X. \quad (13)$$

Problem 2 (Primal Problem (Static Version)).

Given $c_0 = c > 0$ and $X_0 = X > 0$, we consider the following optimization problem of the agent:

$$\max_{(c_t, \pi_t) \in \Pi} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right] \quad (14)$$
subject to budget constraint (13).

We will show that Problem 1 is equivalent to Problem 2 and call both of them the primal problem. Thus, the optimized value in Problem 2 is the same as $V(X,c)$.

We now consider the Lagrangian of Problem 2:

$$L = E \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right] + y \left( X - E \left[ \int_0^\infty \xi_t c_t dt \right] \right)$$

$$= E \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t) - ye^{\delta t} \xi_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right] + yX,$$

where $y_t$ is the process for the marginal utility (shadow price) of wealth defined by

$$y_t = ye^{\delta t} \xi_t, \quad t \geq 0,$$

and its initial value $y > 0$ is the Lagrange multiplier for the budget constraint (13). Maximization of Lagrangian (15) involves the following problem, which we will call the dual problem.\(^{19}\)

**Problem 3** (Dual problem).

$$J(y,c) = \max_{c_t \in \Pi(c)} E \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t) - y_t c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right]$$

$$= \max_{c_t \in \Pi(c)} E \left[ \int_0^\infty e^{-\delta t} \left( h(y_t,c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right],$$

where $h(y,c) \equiv u(c) - yc$ and $\Pi(c)$ is the class of all admissible consumption policies $(c_t)$ such that there exists $(\pi_t)$ with $(c_t, \pi_t) \in \Pi$.

Note that the state variables for the dual problem (Problem 3) are the marginal value of wealth $y_t$ and the given (previous) level of consumption $c_{t-}$, so that the dual value function can be written as a function of $(y,c)$.

### 3.2.2 Solution to the Dual Problem and Duality

The advantage of the dual formulation is that the dual problem only involves the choice of consumption, not the portfolio choice. Since irreversibility generates action/inaction choices, the optimal consumption policy in our case naturally involves inaction; the agent decides whether to adjust or not to adjust consumption. Thus, if we properly apply the super-contact principle, we can obtain the explicit solution to the dual problem. Specifically, the

\(^{19}\)Usually the dual problem of a maximization problem involves minimization. Here the problem still involves a maximization. We, however, call it the dual problem, since the important variable is the marginal utility of wealth $y_t$, the dual to wealth, in the problem.
dual problem in our case is reduced to a two-sided singular control problem. Its rigorous treatment is given in Appendix D and the Supplemental Material. We provide an intuitive explanation below.

Suppose that the agent adjusts consumption by \( dc \) over an infinitesimal time period \( [t, t + dc] \), then the benefit can be calculated in utility terms:

\[
J(y_t, c_{t-} + dc) - J(y_t, c_{t-}) \approx J_c(y_t, c_{t-})dc,
\]

where \( J \) is the dual value function and \( J_c = \partial J/\partial c \). On the other hand,

\[
\text{Adjustment Cost} = \begin{cases} \alpha du^+_t = \alpha u'(c_{t-})dc & \text{if } dc > 0, \\ \beta du^-_t = -\beta u'(c_{t-})dc & \text{if } dc < 0. \end{cases}
\]

Thus, the adjustment is optimal only when the benefit is greater than or equal to the cost, i.e., \( J_c(y_t, c_{t-}) \geq \alpha u'(c_{t-}) \) if \( dc > 0 \) and \( -J_c(y_t, c_{t-}) \geq \beta u'(c_{t-}) \) if \( dc < 0 \). Note that \( J_c(y_t, c_{t-}) \), the marginal valuation of consumption implied by the dual value function, is different from the marginal utility of consumption \( u'(c_{t-}) \); as indicated in equation (17) the former measures the benefit of adjusting consumption, taking into consideration the total effects of the decision on the future utility values, and the latter measures the benefit of marginal increase in consumption over an infinitesimal time period \( [t, t + dt] \) and is proportional to the cost of adjustment. \( J_c(y_t, c_{t-}) \) can become negative; it is negative when the previous consumption level has become relatively high due to infrequent adjustments. Hence, inaction is optimal, i.e., the agent does not adjust consumption if \( -\beta u'(c_{t-}) < J_c(y_t, c_{t-}) < \alpha u'(c_{t-}) \). When no action is optimal, the dual value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

\[
\delta J(y, c) = \frac{\partial^2 J(y, c)}{2 \partial y^2} + (\delta - r)y \frac{\partial J(y, c)}{\partial y} + u(c) - yc.
\]

The left-hand side of the HJB equation (18) is the rate of return in utility terms required by the agent and the right-hand side is the expected rate of return as the sum of the expected change in the utility value \( J \) and the instantaneous utility flow \( u(c) - yc \) for the dual problem.

The smooth pasting condition implies that we have

\[
J_c(y_t, c_{t-}) = \alpha u'(c_{t-}) \quad (19)
\]

when the agent increases consumption, and

\[
J_c(y_t, c_{t-}) = -\beta u'(c_{t-}) \quad (20)
\]

when the agent reduces consumption. The agent adjusts consumption upward only when her marginal utility of consumption reaches the high value.
\( \alpha u'(c_{t-}) \) and adjusts downward when her marginal utility of consumption reaches the low value \(-\beta u'(c_{t-})\). Hence, the agent’s optimal decision can be described by three regions in the state space: the inaction region (NR region), the increasing region (IR region), and the decreasing region (DR region) of the state space \( D = \{(y, c) | y > 0, c > 0\} \). If the given initial level of consumption \( c_{0-} \) and the marginal utility of wealth \( y_0 \) is such that \( y_0/u'(c_{0-}) \) lies in the IR or DR regions, then consumption is immediately adjusted to the nearest boundary of the inaction region (NR region). If \( y_0/u'(c_{0-}) \) is inside the NR region, then consumption is not adjusted, i.e., consumption is set equal to \( c_{0-} \), until the marginal utility ratio \( y_t/u'(c_{0-}) \) goes outside the region; consumption is adjusted downward if and only if the marginal utility of wealth \( y_t \) goes above \( u'(c_{0})z_{\beta} \) and is adjusted upward if and only if it goes below \( u'(c_{0})z_{\alpha} \). See Figure 2 for the illustration.

![Diagram](image-url)

Figure 2: DR, NR, and IR regions and consumption adjustment

The following proposition recovers the value function of the primal problem (Problem 2) from the dual value function by establishing a duality relationship.

**Proposition 3.1.** The value function \( V \) is concave and satisfies

\[
V(X, c) = \min_{y > 0} [J(y, c) + yX].
\]  

(21)

Namely, the value function is the concave conjugate of the dual value function if we use a standard term in convex analysis. The proposition also implies that if the Lagrange multiplier \( y \) is properly chosen, the optimal policy for the dual problem is indeed optimal for the primal problem. The proposition shows that the value function is concave, even if the utility function is generally non-concave. A well-known result from the convex
analysis (Proposition 1 and Theorem 1 in Section 8.6 of Luenberger (1969)) implies

\[ J(y, c) = \max_{X > 0} [V(X, c) - yX], \quad (22) \]

i.e., the dual value function is the convex conjugate of the value function.

### 3.3 Risk Aversion and ILA

Before we investigate optimal policies in detail, we state a direct consequence of our solution analysis in the previous subsection combined with Proposition 2.1. Note again that it is the utility function (7) of the form with consumption irreversibility that we used when we analyzed the problem. The following proposition confirms the link between (7) and (9).

**Proposition 3.2.** Given \( x \) and \( c_0 \), \( \gamma \), and \( \delta \), the optimal consumption policy and the optimal portfolio policy are the same for every pair of \((\alpha, \beta)\) if loss aversion \( L \) is the same.

Proposition 3.2 implies that the three preference parameters \( \delta \), \( \gamma \) and \( L \) determine optimal consumption and investment decisions. The definition of ILA does not involve \( \gamma \) (see (4) and (8)). We will disentangle the effects of risk aversion and ILA when investigating how ILA affects optimal polices.

### 4 Optimal Consumption Policy

By using duality in (22), the optimal consumption policy in the dual problem is translated into the optimal policy in the primal problem, which is explicitly described in Proposition 4.1. Figure 1 also illustrates the dynamics of the consumption-wealth ratio process \( c_t/X_t \).

**Proposition 4.1.** (a) There exist \( \underline{c} \) and \( \bar{c} \) such that the consumption is not adjusted if \( c_t/X_t \) is in \([\underline{c}, \bar{c}]\). Consumption is adjusted upward (downward, resp.) to the nearest boundary if it is below \( \underline{c} \) (above \( \bar{c} \), resp.).

(b) There exists a stationary distribution for \( c_t/X_t \).

Inaction and incremental adjustment of consumption are consistent with the excess smoothness of consumption (Deaton, 1987) and its excess sensitivity to current income (Flavin, 1981). Consumption is unchanged in the inaction interval and is adjusted only when the consumption-wealth ratio hits one of the boundary points. A household with nontrivial ILA (i.e., \( L > 1 \)) hardly changes consumption in response to small permanent income shocks if the consumption-wealth ratio does not reach \( \bar{c} \) or \( \underline{c} \) after the shock,
which implies excess smoothness. Regarding the excess sensitivity, suppose
there is a small or medium good shock. After the shock, consumption may
not immediately respond. However, consumption is more likely to increase
later since the probability of increasing in the next period is higher. In this
sense, the consumption adjustment reflects accumulated effects and thus
appears to be excessive to the anticipated change in income (or wealth) in
our model. However, both excess sensitivity and excess smoothness disap-
pear for large wealth or income shocks in our model (magnitude hypothesis,
Jappelli and Pistaferri (2010)). The reason is that the consumption-wealth
ratio immediately reaches \( \bar{c} \) or \( \underline{c} \) after consecutive large shocks.

The existence of a stationary distribution for the consumption-wealth
ratio is consistent with the stationarity hypothesis of the log consumption-
wealth ratio in the empirical literature (Lettau and Ludvigson (2001, 2010)).
Its explicit form is given in the proof of Proposition 4.1.

The following proposition is a consequence of Proposition 4.1(a).

**Proposition 4.2.** When the agent adjusts consumption, the sensitivity of
consumption to wealth changes, taking the following form:

\[
\Delta c_t = \begin{cases} 
\bar{c}(\Delta X_t)^+ & \text{when consumption is adjusted upward}, \\
-\underline{c}(\Delta X_t)^- & \text{when consumption is adjusted downward}.
\end{cases}
\]

That is, the sensitivity of a consumption increase to an increase in wealth
(income) is equal to \( \bar{c} \), and the sensitivity of a consumption decrease to a
decrease in wealth is equal to \( \bar{c} \). Since \( \bar{c} < \underline{c} \) the sensitivity for a downward
adjustment is higher than that for an upward adjustment, consistent with
empirical evidence documented by Shea (1995) and Bowman et al. (1999).

5 Risky Share and Risk Attitude

5.1 Risky Share and RCRRA

Applying Itô’s lemma to the optimal wealth process, we derive the optimal
portfolio in terms of the marginal utility of wealth \( y_t \).

**Proposition 5.1.** Suppose that \( c_{t-} \) and \( X_t \) at time \( t \) are given such that
\( c_{t-}/X_t \in [\underline{c}, \bar{c}] \). Then, the optimal portfolio \( \pi_t^* \) is as follows:

\[
\pi_t^* = \frac{\theta}{\sigma c_{t-}} \left( C_1(m_1 - 1) \left( \frac{y_t^*}{(c_{t-})^{-\gamma}} \right)^{m_1-1} + C_2(m_2 - 1) \left( \frac{y_t^*}{(c_{t-})^{-\gamma}} \right)^{m_2-1} \right),
\]

where \( y_t^* \) is the unique solution to \( X_t = -J_y(y_t^*, c_{t-}) \) and \( m_1, m_2, C_1, \) and
\( C_2 \) are given in the proof.
In the absence of adjustment costs (the case \( \alpha = \beta = 0 \) or \( L = 1 \)), the optimal portfolio is given by \( \pi_t^* = \frac{\mu - r}{\gamma \sigma^2} X_t \) (Merton, 1969). Based on the classical result, for given \( c_{t-} \) we can rewrite the optimal portfolio and the risky share, i.e., the share of the risky asset investment in wealth, in terms of \( X_t \) by using the duality. In other words, if \( X_t \in [x c_{t-}, \bar{x} c_{t-}] \), we have

\[
\pi_t^* = \frac{\mu - r}{\gamma \sigma^2} X_t - F(X_t, c_{t-}) \quad \text{and} \quad \frac{\pi_t^*}{X_t} = \frac{\mu - r}{\gamma \sigma^2} - \frac{F(X_t, c_{t-})}{X_t},
\]

where \( F(\cdot, \cdot) \) is a non-negative function. In this case, we can call the first term a myopic demand for the risky asset and the remaining term a hedging demand, which hedges against changes in the consumption wealth ratio. Intuitively, the second term in (24) is the demand that crowds out the myopic demand, in order to maintain the current consumption level since frequent changes in consumption incur high utility costs. In this sense, the second term itself is positive. The risky share \( \pi_t^*/X_t \) exhibits a U-shaped relationship with wealth (or the wealth to consumption ratio). The ratio of the hedging demand to wealth, \( F(X_t, c_{t-})/X_t \), takes the largest value inside the inaction region at which the total risky share takes the minimum value (see Figure 3). This result is summarized in Theorem 5.1.

**Theorem 5.1.** The risky share is a function of the wealth-to-consumption ratio \( X_t/c_{t-} \) over the interval \([x, \bar{x}]\) and satisfies the following properties:

(a) The risky share is smaller than \( \frac{\mu - r}{\gamma \sigma^2} \) inside the inaction interval \((x, \bar{x})\).

(b) The risky share is equal to \( \frac{\mu - r}{\gamma \sigma^2} \) only if \( X_t/c_{t-} = x \) or \( X_t/c_{t-} = \bar{x} \).

(c) There exists \( \hat{x} \in (x, \bar{x}) \) such that the risky share attains its minimum value. The risky share is decreasing in \([x, \hat{x}]\) and increasing in \([\hat{x}, \bar{x}]\).

Theorem 5.1 implies that in terms of risk-taking the agent with \( L > 1 \) behaves in the same way as a non-loss averse agent (\( L = 1 \)) only when the wealth-to-consumption (or consumption-wealth) ratio reaches a boundary of the inaction interval. When the ratio hits a boundary of the inaction interval, the risky share is equal to the myopic demand. In addition, the risky share is increasing and decreasing with wealth depending on the position of the wealth-to-consumption ratio within the inaction interval (see Theorem 5.1(c) and Figure 3).

By using the optimal portfolio we calculate an effective measure of risk aversion. Motivated by the classical portfolio selection result, we define the revealed coefficient of relative risk aversion (RCRRA) by

\[
\text{RCRRA}(t) \equiv \frac{\mu - r}{\sigma^2} \frac{X_t}{\pi_t^*}. \quad (25)
\]
The RCRRA is the coefficient of relative risk aversion inferred from the agent’s portfolio allocation at time $t$ by an outsider who regards the agent as non-loss averse, i.e., $L = 1$ for the agent. The RCRRA changes over time in accordance with the consumption-wealth ratio.

Note that the RCRRA has the inverse relationship with the optimal risky share. That is, the RCRRA is generally greater than $\gamma$ (Theorem 5.1(a)). Theorem 5.1(b) says that the RCRRA is equal to $\gamma$ only at the boundaries of the inaction interval, an envelope theorem type result; the agent acts as if not facing the costs when she adjusts consumption.

5.2 Risky Share and Wealth Change

Here we investigate how the risky share changes in response to increases in wealth. Models with habit, commitment, or DRRA (decreasing relative risk aversion) predict that the risky share increases with wealth while standard models with CRRA preferences predict no change in the risky share. The empirical literature is inconclusive. Calvet et al. (2009) and Calvet and Sodini (2014) provide results consistent with habit or commitment or DRRA models. However, Chiappori and Paiella (2011) and Brunnermeier and Nagel (2008) provide evidence consistent with the standard CRRA model and even show a slightly negative relationship between the risky share and wealth. We shed light on the debate by showing that the choice of sample paths generated from one underlying data-generating process (e.g., time-series of the stock market data) can produce the positive, neutral, and negative relationships between the risky share and wealth.

Figure 4 shows four types of sample paths we consider. Panel (a) shows
a path, corresponding to a typical bull market, and Panel (b) shows a path corresponding to a typical bear market. Panels (c) and (d) exhibit no trend (neither bullish nor bearish) during the sample periods and hence we call them markets with no trend. In addition, we generate a population of agents by using a joint lognormal density for \((\alpha, \beta)\), where the mean values of \(\alpha\) and \(\beta\) satisfy (4) with \(\delta = 0.015\) and \(L = 2\) or \(L = 2.33\), i.e., the mean value of ILA of the population is 2 or 2.33 with different variances (see the description in Table 1). We simulate time series of their wealth and risky shares and conduct regression analysis similar to that by Brunnermeier and Nagel (2008). We use the following equation for the regression:

\[
\Delta_k \log \frac{\pi_t}{X_t} = \rho \Delta_k \log X_t,
\]

where \(\Delta_k\) denotes a \(k\)-period (year) first-difference operator, i.e., \(\Delta_k g_t \equiv g_t - g_{t-k}\) for a state variable \(g_t\). We give the details of simulation and regression analysis in Appendix K.

The regression results summarized in Table 1 show a positive effect of wealth increase on the risky share when the market is in an upward trend (Panel (a)) and a negative effect when the market is in a downward trend (Panel (b)). There is no wealth effect on the risky share or a very small effect (if any) when the market has no trend in the sample periods (Panels (c) and (d)).

The intuition behind these results comes from the fact that the risky share shows a U-shaped relationship with wealth within the inaction interval (recall Figure 3). There exist two regions inside the interval: the decreasing region (D-region), in which the risky share decreases with wealth, and the increasing region (I-region), in which the risky share increases with wealth. If the market is in an upward trend as in Panel (a), the wealth process is more likely to stay in the I-region of the inaction interval. However, if the market is in a downward trend as in Panel (b), the wealth is more likely to stay in the D-region of the inaction interval.
Table 1: The regression coefficients for markets (a), (b), (c), and (d) in Figure 4. We generate the distribution of households with \( \alpha \) and \( \beta \) using the log-normal distributions with mean \( m_\alpha, m_\beta \) and variance \( v_\alpha, v_\beta \), respectively. The values in the parentheses are p-values greater than 1%. *** means a p-value smaller than 1%. The parameter values are from the benchmark set, i.e., \( (\mu = 0.0784, r = 0.0086, \sigma = 0.2016) \) and \( (\gamma = 3.5, \delta = 0.015) \).

<table>
<thead>
<tr>
<th>((m_\alpha, v_\alpha))</th>
<th>((m_\beta, v_\beta))</th>
<th>Bull market</th>
<th>Bear market</th>
<th>No trend 1</th>
<th>No trend 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha = 0)</td>
<td>((66, 5^2))</td>
<td>0.1845***</td>
<td>-0.1647***</td>
<td>-0.0287 (0.08)</td>
<td>0.0458 (0.03)</td>
</tr>
<tr>
<td>(\alpha = 0)</td>
<td>((66, 15^2))</td>
<td>0.1698***</td>
<td>-0.1611***</td>
<td>-0.0053 (0.07)</td>
<td>0.0177 (0.36)</td>
</tr>
<tr>
<td>(\alpha = 0)</td>
<td>((66, 30^2))</td>
<td>0.1780***</td>
<td>-0.1849***</td>
<td>-0.0112 (0.23)</td>
<td>-0.0040 (0.84)</td>
</tr>
<tr>
<td>(\alpha = 0)</td>
<td>((66, 50^2))</td>
<td>0.1508***</td>
<td>-0.1667***</td>
<td>-0.0113 (0.51)</td>
<td>0.0093 (0.61)</td>
</tr>
<tr>
<td>((10, 10^2))</td>
<td>((66, 5^2))</td>
<td>0.1485***</td>
<td>-0.1607***</td>
<td>-0.0552***</td>
<td>0.0643***</td>
</tr>
<tr>
<td>((10, 10^2))</td>
<td>((66, 15^2))</td>
<td>0.1556***</td>
<td>-0.1795***</td>
<td>-0.0205 (0.24)</td>
<td>0.0536 (0.02)</td>
</tr>
<tr>
<td>((10, 10^2))</td>
<td>((66, 30^2))</td>
<td>0.1636***</td>
<td>-0.1755***</td>
<td>-0.0003 (0.98)</td>
<td>0.0508 (0.02)</td>
</tr>
<tr>
<td>((10, 10^2))</td>
<td>((66, 50^2))</td>
<td>0.1732***</td>
<td>-0.1761***</td>
<td>-0.0441 (0.01)</td>
<td>0.0251 (0.25)</td>
</tr>
<tr>
<td>((10, 20^2))</td>
<td>((66, 50^2))</td>
<td>0.1721***</td>
<td>-0.1798***</td>
<td>0.0085 (0.62)</td>
<td>0.0371 (0.09)</td>
</tr>
</tbody>
</table>

5.3 Effects of ILA

Recall that the agent’s RCRRA near the boundaries of the inaction interval is close to risk aversion \( \gamma \), which means that risk aversion determines the agent’s behavior when the agent is about to adjust consumption. Then, what is the effect of ILA? The following proposition provides the effect of ILA on the frequency of consumption adjustment.

**Proposition 5.2** (Loss Aversion: Consumption). Let \( \bar{c} \) and \( c \) be the two boundaries for consumption adjustment. Then, \( \bar{c} \) increases with \( L \) and \( c \) decreases with \( L \).

Proposition 5.2 implies that the inaction interval increases with ILA. In other words, an increase in ILA delays consumption adjustments over time. This property generally has a significant impact on the pattern of aggregate consumption including its time series properties, e.g., mean, volatility, and autocorrelation (see Choi et al. (2020a)).

**Proposition 5.3** (Loss Aversion: RCRRA). Inside the inaction interval (except at the boundary points), the RCRRA increases with ILA. Moreover, as \( L \) goes to infinity, the minimum value of the risky share goes to zero and the maximum RCRRA goes to infinity.

If \( L = 1 \), the inaction region disappears and the RCRRA is always equal to \( \gamma \). If \( L > 1 \), the RCRRA displays the inverted U-shaped relationship.
Moreover, as $L$ increases, the inaction interval becomes wider and the inverted U-shaped relationship is more pronounced (Figure 5(b)).

Figure 5: The risky share and the RCRRA for different values of ILA. Other parameter values are $\delta = 0.015, \mu = 0.0784, \sigma = 0.2016, r = 0.0015$, and $\gamma = 3.5$.

Figure 6: A simulation path of $X_t$ and the RCRRA I: $\delta = 0.015, r = 0.0086, \mu = 0.0784, \sigma = 0.2016, X_0 = 61.81$, $c = 1$, and $L = 10$. The maximum RCRRA is 7.7329.

Figure 6 shows a typical simulated path of RCRRA. While the underlying relative risk aversion coefficient is $\gamma = 3.5$, the RCRRA can become larger than twice the size of $\gamma$. Panel (a) plots the wealth process; the upper and lower dotted lines in Panel (a) describe the dynamics of the boundaries of the inaction interval for wealth. As seen in Panel (b), the RCRRA tends to be low during the time periods when there are consecutive good shocks or consecutive bad shocks in one direction since they make the consumption-wealth ratio closer to the boundary. The RCRRA is low when changes in wealth are large, for example, in the time interval between $t = 20$ and $t = 30$ in Panel (b). However, the RCRRA tends to be high during the time periods when there is little gain or loss in wealth due to moderately alternating good and bad shocks (e.g., during the time period before $t = 10$ or after $t = 40$).
In particular, the RCRRA takes the maximum value whenever the wealth process hits the middle red dotted lines in Panel (a). In sum, the agent exhibits high risk aversion when the market shows no trend (neither bullish nor bearish), while the agent looks more aggressive (showing her actual risk aversion) when there are large consecutive shocks in one direction.

On one hand, our model is related to habit models in terms of generating the time-varying risk aversion. The mechanism behind changes in implied risk aversion in our model, however, is different from that in habit models. In our model RCRRA depends on whether the consumption-wealth ratio is close to a boundary of the inaction interval and in habit models it depends on whether consumption is close to the current level of habit stock. Thus, implied risk aversion increases as wealth declines in habit models, whereas it can decrease when wealth declines in our model.

On the other hand, Grossman and Laroque (1990) and Chetty and Szeidl (2007) generate low risk aversion to large shocks to wealth but high risk aversion to small shocks in the case where there are durable consumption goods or consumption commitments. Our model generates a similar result if risk aversion is small and ILA is large. Note that Grossman and Laroque (1990) and Chetty and Szeidl (2007) show the relationship by using numerical solutions. We explicitly derive the theoretical relationship between the RCRRA and the consumption-wealth ratio in closed form.

6 Concluding Remarks

We have studied a model of intertemporal loss aversion derived from consumption irreversibility or mental adjustment costs from changing consumption decisions. The optimization problem is nontrivial since the preference is neither monotonic nor concave in consumption. By transforming the original problem to a dual problem and then applying the super-contact principle, we have obtained a closed form solution and investigated the implications. The optimal policies of the agent with intertemporal loss aversion can explain various empirical puzzles in consumption, portfolio selection, and risk attitude.

We can reinterpret our model as that of a durable good. There is a literature on durable goods based on local substitution between consumption at nearby time points (Hindy and Huang (1992, 1993), Hindy et al. (1997), and Cuoco and Liu (2000)). Schroder and Skiadas (2002) show that there exists an isomorphism between the irreversible consumption model of Dybvig (1995) and the model by Hindy and Huang (1993). Since our model includes the irreversible consumption model as a special case, it also includes Hindy
and Huang’s model by extending the isomorphism appropriately. Furthermore, we can extend the current model to the case with multi-consumption goods. We investigate the details of those two extensions in our companion paper (Choi et al., 2020a).

Finally, one might be curious about a model of competitive equilibrium. In our other companion paper (Choi et al., 2020b) we analyze the equilibrium asset prices and credit market implications when there are two types of heterogeneous agents: one group of agents is both risk- and loss-averse and the other group of agents is risk-averse, but not loss-averse.

References


Appendix

A Monotonicity and Concavity

Proposition A.1.

(1) The utility function (5) is monotone increasing if and only if $L_{t+1} \leq \frac{D_t}{D_{t+1}}$ for every $t \geq 0$.

(2) Suppose that $u$ is non-linear, then the utility function (5) is concave if and only if it is monotone increasing.
Proof. For (1), suppose that the utility function \( U \) is increasing in consumption. Let us consider that \( C^1 = (c^1_s)_{s=0}^{T} \) and \( C^2 = (c^2_s)_{s=0}^{T} \), where \( c^1_s = c^2_s \) for every \( s \geq -1 \) except that \( c^2_s > c^1_s \) for some \( t \). For \( j = 1, 2 \),

\[ i) \ c^1_{t-1} \leq c^1_t \leq c^1_{t+1}: \]

\[ U(C^2) - U(C^1) = \frac{1}{\delta} (D(t) - D(t + 1))(u(c^2_t) - u(c^1_t)). \]

\[ ii) \ c^1_{t-1} \leq c^1_t \ \text{and} \ c^1_t \geq c^1_{t+1}: \]

\[ U(C^2) - U(C^1) = \frac{1}{\delta} (D(t) - L_{t+1}D(t + 1))(u(c^2_t) - u(c^1_t)). \]

\[ iii) \ c^1_{t-1} \geq c^1_t \ \text{and} \ c^1_t \leq c^1_{t+1}: \]

\[ U(C^2) - U(C^1) = \frac{1}{\delta} (D(t)L_t - D(t + 1))(u(c^2_t) - u(c^1_t)). \]

\[ iv) \ c^1_{t-1} \geq c^1_t \geq c^1_{t+1}: \]

\[ U(C^2) - U(C^1) = \frac{1}{\delta} (D(t)L_t - D(t + 1)L_{t+1})(u(c^2_t) - u(c^1_t)). \]

From i), ii), iii), and iv), it is obvious that \( U(C^2) - U(C^1) \geq 0 \) only if \( L_{t+1} \leq \frac{D(t)}{D(t+1)} \) and \( D(t+1) < D(t) \). Since \( L_t \geq 1 \), we conclude \( L_{t+1} \leq \frac{D(t)}{D(t+1)} \) for all \( t \). Conversely, suppose that \( L_{t+1} \leq \frac{D(t)}{D(t+1)} \) for every \( t \geq 0 \). We denote the sum of all terms involving \( u(c_t) \) for \( t \geq 0 \) by \( A_t u(c_t) \). Then, \( U \) is increasing if \( A_t \geq 0 \) for every \( t \geq 0 \). We consider the following cases:

\[ i) \ c_t - 1 < c_t < c_{t+1}: \ A_t = D(t) - D(t + 1). \]

\[ ii) \ c_{t-1} < c_t < c_{t+1}: \ A_t = D(t) - D(t + 1)L_{t+1}. \]

\[ iii) \ c_{t-1} > c_t < c_{t+1}: \ A_t = D(t)L_t - D(t + 1). \]

\[ iv) \ c_{t-1} > c_t > c_{t+1}: \ A_t = D(t)L_t - D(t + 1)L_{t+1}. \]

Since \( 1 \leq L_t \leq \frac{D(t)}{D(t+1)} \), we deduce that \( A_t \geq 0 \) for every \( t \geq 0 \). This completes the proof of (1).

For (2), the utility function is concave if and only if \( A_t \) in the proof of part (1) is nonnegative. This is true if and only if the utility function is increasing as the proof of part (1) shows.

\[ \square \]

B. Technical Conditions for Admissible Policies

We define admissible set \( \Pi \) of \( (c_t, \pi_t) \): (i) \( c_t \) is \( F_t \)-adapted, non-negative, right-continuous with left limits (RCLL), has bounded variation, and is integrable over any finite time interval, i.e., \( \int_0^t c_t dt < \infty \) for all \( t \geq 0 \) almost surely, (ii) \( \pi_t \) is \( F_t \)-measurable adapted and square integrable, i.e., \( \int_0^t ||\pi_t||^2 dt < \infty \) for all \( t \geq 0 \) almost surely, and (iii) \( c_t \) satisfies \( \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( \max(0, -u(c_t)) dt + \alpha du^*_t + \beta du^-_t \right) \right] < \infty \).

C. Proof of Proposition 2.1

Since \( u(c_t) = u(c_{t^-}) + \int_0^t du(c_s) \), we have

\[ \mathbb{E} \left[ \int_0^T e^{-\delta t} u(c_t) dt \right] = \mathbb{E} \left[ \int_0^T e^{-\delta t} \left( u(c_{t^-}) + \int_0^t (du^*_s - du^-_s) \right) dt \right] \]

\[ = 1 - e^{-\delta T} u(c_{t^-}) + \mathbb{E} \left[ \int_0^T e^{-\delta t} \frac{1 - e^{-\delta(T-t)}}{\delta} (du^*_s - du^-_s) \right]. \]

(27)
Therefore,

$$U = \mathbb{E} \left[ \int_0^T e^{-\delta t} \left( u(c_t)dt - adu_t^+ - \beta du_t^- \right) \right]$$

$$= \frac{1 - e^{-\delta T}}{\delta} u(c_0^-) + \mathbb{E} \left[ \int_0^T e^{-\delta t} \left( 1 - e^{-\delta (T-t)}(du_t^+ - du_t^-) \right) \right]$$

$$= \frac{1 - e^{-\delta T}}{\delta} u(c_0^-) + \frac{1}{\delta} \mathbb{E} \left[ \int_0^T e^{-\delta t}(1 - e^{-\delta (T-t)} - \delta u_t^+)(du_t^+ + Ldu_t^-) \right].$$

Let $T \to \infty$ and we obtain (9).

D Analysis of Solution: A Sketch

Here we provide a sketch of the analysis. A detailed proof is given in the Supplemental Material (Section S.1.1). As described in Section 3.2.2, the agent’s optimal decision is described by three regions in the state space as explained in the following proposition.

**Proposition D.1.** The optimal consumption policy of Problem 3 is described by three regions: the inaction region (NR), the increasing region (IR), and the decreasing region (DR) of the state space $\mathcal{D} = \{(y,c)|y > 0, c > 0\}$.

$$\text{IR} = \{(y,c) \in \mathcal{D} | J_c(y,c) = \alpha u'(c)\},$$

$$\text{NR} = \{(y,c) \in \mathcal{D} | -\beta u'(c) < J_c(y,c) < \alpha u'(c)\},$$

$$\text{DR} = \{(y,c) \in \mathcal{D} | J_c(y,c) = -\beta u'(c)\}. \quad (29)$$

Thus, the optimal consumption policy is determined by the ratio of the marginal valuation of consumption implied by the dual value function, $J_c(y,c)$, to the marginal utility of consumption, $u'(c)$.

Differentiating the HJB equation (18) with respect to $c$, we obtain the following ordinary differential equation (ODE):

$$\frac{\delta^2 y^2}{2} \frac{\partial^2 J_c(y,c)}{\partial y^2} + (\delta - r)y \frac{\partial J_c(y,c)}{\partial y} + u'(c) - y - \delta J_c(y,c) = 0. \quad (30)$$

Dividing (30) by $u'(c)$, we have an ODE with respect to $z$:

$$\frac{\delta^2 z^2}{2} \frac{d^2 H}{dz^2} + (\delta - r)z \frac{dH}{dz} + 1 - z - \delta H = 0. \quad (31)$$

Here $H(z)$ is the normalized marginal valuation of consumption defined by

$$H(z) = \frac{J_c(y,c)}{u'(c)} \quad \text{and} \quad z = \frac{y}{u'(c)},$$

where $z$ is the ratio of the marginal utility of wealth, $y$, and the marginal utility of consumption, $u'(c)$ and is called the marginal utility ratio. In the absence of adjustment costs, the two marginal values are the same, but they are not necessarily equalized in their presence. More precisely, the marginal utility ratio process $z_t = y_t^{1/(c_{t-1})} - 1$ is time-varying. Proposition D.1 implies that consumption is not adjusted when $-\beta < H(z_t) < \alpha$, and is adjusted upward when $H(z_t) = \alpha$, and downward when $H(z_t) = -\beta$. In other words, there is an interval $(z_{\alpha}, z_{\beta})$ such that consumption is adjusted upward if $z_t < z_{\alpha}$ and adjusted downward if $z_t > z_{\beta}$ and inaction is optimal if $z_{\alpha} < z < z_{\beta}$. See Figure 2 for the graphical illustration of the regions and adjustment of consumption. In sum, we have

$$H(z) = \alpha \quad \text{if} \ z \leq z_{\alpha} \quad \text{and} \ \ H(z) = -\beta \quad \text{if} \ z \geq z_{\beta}, \quad (32)$$

and the super-contact condition implies

$$H'(z) = 0, \quad \text{if} \ z \leq z_{\alpha} \text{ or if } z \geq z_{\beta}. \quad (33)$$
Now we solve ODE (31) with boundary conditions (32) and (33). Let \( m_1 \) and \( m_2 \) be the positive and negative roots of the following quadratic equation:

\[
\frac{\theta^2}{2} m^2 + (\delta - r - \frac{\theta^2}{2})m - \delta = 0.
\]  

(34)

Then, a general solution to (31) takes the following form:

\[
H(z) = D_1 \left( \frac{z}{z_\alpha} \right)^{m_1} + D_2 \left( \frac{z}{z_\alpha} \right)^{m_2} + \frac{1 - \frac{z}{\theta}}{r}, \quad \text{for } z_\alpha < z < z_\beta,
\]

where \( D_1, D_2 \) are some constants. From (32) and (33), we have

\[
H(z_\alpha) = \alpha, \quad H'(z_\alpha) = 0, \quad H(z_\beta) = \beta, \quad H'(z_\beta) = 0.
\]  

(35)

Note (35) is a system of four equations in four unknowns, \( D_1, D_2, z_\alpha, \) and \( z_\beta \). A simple calculation shows that the solutions are given by

\[
D_1 = \frac{(\alpha - \frac{1}{\theta}) m_2 + (m_2 - 1) \frac{z_\beta}{r}}{m_2 - m_1}, \quad D_2 = \frac{(\alpha - \frac{1}{\theta}) m_1 + (m_1 - 1) \frac{z_\alpha}{r}}{m_1 - m_2},
\]  

(36)

\[
z_\alpha = (1 - \delta \alpha) \frac{m_1 - 1}{m_1} \frac{L w^{m_1} - 1}{w^{m_1} - 1}, \quad z_\beta = (1 + \delta \beta) \frac{m_1 - 1}{m_1} \frac{w^{m_1} - 1}{w^{m_1} - w},
\]  

(37)

where \( L \) is ILA and \( w \in (0, \frac{1}{\theta}) \) is a unique root to the equation

\[
(m_1 - 1) m_2 (1 - w^{1-m_2}) (L w^{m_1} - 1) - m_1 (m_1 - 1) (w^{m_1} - w) (L - w^{-m_2}) = 0.
\]  

(38)

A direct calculation using (36) and (37) shows \( H'(z) < 0 \) for \( z \in (z_\alpha, z_\beta) \).

Finally the dual value function \( J \) is constructed from \( H \) by taking the integration of \( J_c(y, c) = u'(c) H(\frac{y}{w(c)}) \) (see the detail in the Supplemental Material). We provide the dual value function in the following proposition.

**Proposition D.2.** The dual value function \( J(y, c) \) of Problem 3 is a convex function of \( y \) and given by

\[
J(y, c) = \begin{cases}
\frac{D_1 y c}{(1 - \gamma + \gamma m_1) z_\alpha} \left( \frac{y}{e - \gamma z_\alpha} \right)^{m_1-1} + \frac{D_2 y c}{(1 - \gamma + \gamma m_2) z_\alpha} \left( \frac{y}{e - \gamma z_\alpha} \right)^{m_2-1}, & \text{for } (y, c) \in \mathbb{IR}, \\
\frac{1}{\delta} u(c) - \frac{y c}{r}, & \text{for } (y, c) \in \overline{\mathbb{NR}}, \\
\frac{1}{\delta} u(c) - \frac{y c}{r} + \alpha \left( u(c) - u(I(\frac{y}{z_\alpha})) \right), & \text{for } (y, c) \in \mathbb{IR}, \\
\frac{1}{\delta} u(c) - \frac{y c}{r} - \beta \left( u(c) - u(I(\frac{y}{z_\beta})) \right), & \text{for } (y, c) \in \mathbb{DR},
\end{cases}
\]

(39)

and the regions \( \mathbb{IR}, \overline{\mathbb{NR}}, \) and \( \mathbb{DR} \) are rewritten by

\[
\mathbb{IR} = \{(y, c) \in \mathbb{D} \mid y \leq u'(c) z_\alpha \},
\]

\[
\overline{\mathbb{NR}} = \{(y, c) \in \mathbb{D} \mid u'(c) z_\alpha \leq y \leq u'(c) z_\beta \},
\]

\[
\mathbb{DR} = \{(y, c) \in \mathbb{D} \mid u'(c) z_\beta \leq y \}.
\]

where \( I(z) \equiv (u')^{-1}(z) = z^{-1}, \overline{\mathbb{NR}} \) is the closure of \( \mathbb{NR} \) in \( \mathbb{R} \).

**E** Proof of Proposition 3.1: Sketch

We now proceed to find the value function of the primal problem (Problem 2) from the dual value function obtained in Proposition D.2. For an admissible consumption process \( \mathcal{C} = (c_t)_{t=0}^{\infty} \) satisfying the budget constraint (13), define

\[
\mathcal{V}(X, \mathcal{C}) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) dt - c_t \delta c_t - \delta d\xi_t \right) \right], \quad J(y, \mathcal{C}) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} (u(c_t) - y e^{\delta t} \xi(c_t)) dt \right].
\]  

(40)
Since $c_t$ satisfies (13), we have the following inequality: for every $y > 0$,

$$V(x, c) \leq V(x, c) + y \left( x - E \left[ \int_0^\infty \xi_t c_t dt \right] \right) = J(y, c) + yx.$$  \hspace{1em} (41)

This implies that

$$V(x, c) \leq \inf_{y \geq 0} [J(y, c) + yx].$$  \hspace{1em} (42)

Hence, the value function $V(x, c)$ of Problem 2 satisfies the following:

$$V(x, c) \leq \max_{(c_t) \in \Pi(c)} \min_{y \geq 0} [J(y, c) + yx] \leq \min_{y \geq 0} \left[ \max_{(c_t) \in \Pi(c)} J(y, c) + yx \right] = \min_{y \geq 0} [J(y, c) + yx],$$  \hspace{1em} (43)

where $\Pi(c)$ denotes the set of admissible consumption plans satisfying (13). The second inequality is valid because of the usual inequality that the maximum of minima of a function is smaller than or equal to the minimum of its maxima. Indeed the two inequalities in (43) become equal (see the detail in Section S.1.2 of the Supplemental Material). Moreover,

$$\frac{\partial^2 J}{\partial y^2} = c \left( \frac{D_1m_1(m_1 - 1)}{(1 - \gamma + \gamma m_1)z_n} \left( \frac{y^{m_1 - 1}}{z_n} \right) + \frac{D_2m_2(m_2 - 1)}{(1 - \gamma + \gamma m_2)z_n} \left( \frac{y^{m_2 - 1}}{z_n} \right) \right).$$  \hspace{1em} (44)

Because $D_1 > 0$, $D_2 < 0$, $1 - \gamma + \gamma m_1 > 0$, and $1 - \gamma + \gamma m_2 < 0$, we have $J_{yy} > 0$ for $y > 0$ and thus $J(y, c)$ is strictly convex in $y$. Finally the value function $V(x, c)$ is strictly concave in $x$ because $V_{xx} = -1/J_{yy} < 0$.

### F Proof of Proposition 4.1

**Proof of (a)** Proposition 3.1 implies that there exists a unique solution $y^*$ for the minimization problem (21). By using the first-order condition and Proposition D.2, we have $X = -J_y(y, c) = cX(y^*/c^{-\gamma})$, where $X(y)$ is defined by

$$X(y) = \frac{1}{r} - \left( \frac{D_1m_1}{(1 - \gamma + \gamma m_1)z_n} \left( \frac{y^{m_1 - 1}}{z_n} \right) + \frac{D_2m_2}{(1 - \gamma + \gamma m_2)z_n} \left( \frac{y^{m_2 - 1}}{z_n} \right) \right).$$  \hspace{1em} (45)

Since Problem 3 is time-consistent, $y^* = y^* e^{\delta(s-t)}H_s/H_t$ is the minimizer for the duality relationship starting at $t \geq 0$. Thus, for wealth $X_t$ at time $t$, we have $X_t = c_{t-}X(y^*_t/(c_{t-})^{-\gamma})$. During the time in which $y^*_t$ is inside the NR, the optimal consumption $c^*_t$ is constant, i.e., $c^*_t = c_{t-}$ and thus the wealth $X_s$ for $s \geq t$ is given by $X_s = c_{t-}X(y^*_t/(c_{t-})^{-\gamma})$.

Let us define $\tilde{x}$, $\tilde{\bar{x}}$, $\tilde{z}$ and $\tilde{c}$ as follows:

$$\tilde{x} = X(z_s), \quad \tilde{z} = 1/\tilde{x}, \quad \text{and} \quad \tilde{c} = 1/\tilde{z}.$$

It follows from $D_1 > 0$, $D_2 < 0$, $1 - \gamma + \gamma m_1 > 0$, and $1 - \gamma + \gamma m_2 < 0$ in (45) that $X(y)$ is strictly decreasing in $y$. This implies that the consumption stays constant if and only if $\tilde{c}_{t-} < X_s < \tilde{x}_{t-}$ or $\tilde{z} < c_{t-}/X_s < \tilde{c}$. This completes the proof.

**Proof of (b)** Note that $\log y_t/c_{t-}^{-\gamma}$ is a regulated Brownian motion with drift $(\delta - r)$ and volatility $-\theta$ on $[\log z_n, \log z_\beta]$. By Proposition 5.5 in Harrison (1985) or Proposition 10.8 in Stokey (2009), $\log y_t/c_{t-}^{-\gamma}$ has a stationary distribution with the density function $p(x) = \zeta e^{2x}/(z_\beta^2 - z_n^2)$, where $\zeta = 2(\delta - r)/\theta^2$.

From (a), we have $X_t/c_t = X(y_t/c_t^{-\gamma})$. Then, for $x \in [\tilde{c}, \tilde{c}]$,

$$\lim_{t \to \infty} P \left( \frac{c_t}{X_t} \leq x \right) = \lim_{t \to \infty} P \left( \frac{y_t}{c_t^{-\gamma}} \leq X^{-1} \left( \frac{1}{x} \right) \right) = \int_{\log z_n}^{\log X^{-1}(\frac{1}{x})} p(u)du = \int_{\log z_n}^{\log X^{-1}(\frac{1}{x})} \left[ \frac{p(\log X^{-1}(\frac{1}{x})) e^{-2u}}{x^2 X'(X^{-1}(\frac{1}{x})) X^{-1}(\frac{1}{x})} \right] du.$$

Thus, $c_t/X_t$ has a stationary distribution with the density function

$$q(x) = \frac{p(\log X^{-1}(\frac{1}{x}))}{x^2 X'(X^{-1}(\frac{1}{x})) X^{-1}(\frac{1}{x})}, \quad x \in [\tilde{c}, \tilde{c}].$$
G Proof of Proposition 5.1

Without loss of generality, assume \( \bar{c} \leq c_{t-}/X_t \leq \check{c}. \) This implies \( z_{\bar{c}} \leq z_t^* \leq z_{\check{c}} \) with \( z_t^* = y_t^*(c_{t-})^{-\gamma} \). Applying Itô’s lemma to equation (10), we get

\[
dX_t = -[(\delta - r)y_tJ_{yy}(y_t,c_{t-}) + \frac{1}{2}\theta^2 y_t^2 J_{yy}(y_t,c_{t-})]dt + \theta y_tJ_{yy}(y_t,c_{t-})dB_t, \tag{47}
\]

since \( dy_t = (\delta - r)y_tdt - \theta y_tdB_t \). By comparing equation (47) with (10),

\[
\pi_t = \frac{\theta}{\sigma} y_tJ_{yy}(y_t,c_{t-}) = \frac{\mu - r}{\sigma^2} y_tJ_{yy}(y_t,c_{t-}). \tag{48}
\]

By using the explicit form of \( J \) in Proposition D.2, we obtain

\[
\pi_t^* = \frac{\theta}{\sigma} c_{t-} \left( \frac{D_1m_1(m_1-1)}{1 - \gamma + \gamma m_1} \frac{z_{\check{c}}}{(c_{t-})^{-\gamma} z_{\check{c}}} m_{1-1}^1 + \frac{D_2m_2(m_2-1)}{1 - \gamma + \gamma m_2} \frac{z_{\check{c}}}{(c_{t-})^{-\gamma} z_{\check{c}}} m_{2-1}^2 \right),
\]

which is the same as (23) after properly matching the coefficients in each position.

H Proof of Theorem 5.1

(Proof of (a)) First, we have \( X_t/c_{t-} - (\gamma \sigma/\theta) \pi_t^*/c_{t-} = -H'(z_t^*) \), where \( H(\cdot) \) is defined in Appendix D. Define function \( G \) by

\[
G(z_t^*) \triangleq \frac{\gamma \sigma}{\sigma} \pi_t^*/X_t = 1 + \frac{\sigma}{X_t} H'(z_t^*) = 1 + \frac{H'(z_t^*)}{X(z_t^*)}, \tag{49}
\]

where \( X(\cdot) \) is defined in (45). By Proposition S.1.1 in the Supplemental Material (Section S.1.1), we know that \( H'(z_t^*) \leq 0 \) in \([z_{\bar{c}}, z_{\check{c}}]\) and this leads to \( G(z_t^*) \leq 1 \). Hence, the risky share is smaller than \( \frac{\mu - r}{\sigma \gamma} \) inside the inaction interval.

(Proof of (b)). We have \( G(z_{\bar{c}}) = G(z_{\check{c}}) = 1 \) since \( H'(z_{\bar{c}}) = H'(z_{\check{c}}) = 0 \). This means that the risky share is equal to \( \frac{\mu - r}{\gamma \sigma} \) only if \( X_t/c_{t-} = \bar{c} \) or \( X_t/c_{t-} = \check{c} \).

(Proof of (c)). We will show that there exists a unique \( \hat{z} \in (z_{\bar{c}}, z_{\check{c}}) \) such that \( G(\cdot) \) is a strictly decreasing function on \((z_{\bar{c}}, \hat{z})\) and strictly decreasing function on \((\hat{z}, z_{\check{c}})\). Take the derivative on \( G \) and we have

\[
G'(z) = \frac{H''(z)X(z) - H''(y)X'(z)}{(X(z))^2}, \tag{50}
\]

For notational convenience, we drop subscripts \( t \) and \( * \) for time and optimality. Let us define the numerator of \( G'(z) \) as \( \tilde{G}(z) \), i.e., \( \tilde{G}(y) = H'''(z)X(z) - H''(z)X'(z) \).

We have \( \tilde{G}(z_{\bar{c}}) < 0 \) and \( \tilde{G}(z_{\check{c}}) > 0 \) since \( H''(z_{\bar{c}}) < 0 \), \( H''(z_{\check{c}}) > 0 \) (see the Supplemental Material (Section S.1.1)). We have

\[
X'(z) = \frac{D_1m_1(m_1-1)}{1 - \gamma + \gamma m_1 \frac{z}{z_{\check{c}}}} \left( \frac{z}{z_{\check{c}}} \right)^{m_{1-2}} + \frac{D_2m_2(m_2-1)}{1 - \gamma + \gamma m_2 \frac{z}{z_{\check{c}}}} \left( \frac{z}{z_{\check{c}}} \right)^{m_{2-2}},
\]

\[
H''(z) = \frac{D_1m_1(m_1-1)}{1 - \gamma + \gamma m_1 \frac{z}{z_{\check{c}}}} \left( \frac{z}{z_{\check{c}}} \right)^{m_{1-2}} + \frac{D_2m_2(m_2-1)}{1 - \gamma + \gamma m_2 \frac{z}{z_{\check{c}}}} \left( \frac{z}{z_{\check{c}}} \right)^{m_{2-2}}.
\]

Hence,

\[
G(z) = \gamma z^{m_{2-2}/z} \left( \frac{D_1m_1(m_1-1)^2}{r(1 - \gamma + \gamma m_1)} \left( \frac{z}{z_{\check{c}}} \right)^{m_{1-2}} - \frac{D_1D_2m_2(m_1-1)^2}{(1 - \gamma + \gamma m_1)(1 - \gamma + \gamma m_2) z_{\check{c}}} \left( \frac{z}{z_{\check{c}}} \right)^{m_{1-2}} + \frac{D_2m_2(m_2-1)^2}{r(1 - \gamma + \gamma m_2)} \right). \]

Define \( \tilde{G}(y) \equiv \gamma y^{m_{2-2}/z} \tilde{G}(y) \). Then, \( \tilde{G}(y) \) is strictly increasing in \( y \) since \( m_1 > 1, m_2 < 0, D_1 > 0, D_2 < 0, 1 - \gamma + \gamma m_1 > 0 \), and \( 1 - \gamma + \gamma m_2 < 0 \). Note that \( \tilde{G}(z_{\bar{c}}) < 0 \), \( \tilde{G}(z_{\check{c}}) > 0 \). Thus, \( \tilde{G}(z_{\bar{c}}) < 0 \), \( \tilde{G}(z_{\check{c}}) > 0 \). Therefore, there exists a unique \( \hat{z} \in (z_{\bar{c}}, z_{\check{c}}) \) such that \( \tilde{G}(\hat{z}) = 0 \). This implies that \( G'(z) < 0 \) for \( z \in (z_{\bar{c}}, \hat{z}) \) and \( G'(z) > 0 \) for \( y \in (\hat{z}, z_{\check{c}}) \).

In summary, the risky share is strictly decreasing for \( c \in (\hat{c}, \check{c}) \) and strictly increasing for \( c \in (\bar{c}, \hat{c}) \). Moreover, the risky share is \( \frac{\mu - r}{\gamma \sigma} \) if \( c/X \) is \( \bar{c} \) or \( \check{c} \). Here, \( \hat{c} = 1/X(\hat{z}) \) and \( \check{z} \in (z_{\bar{c}}, z_{\check{c}}) \) is a unique solution to \( G(z) = 0 \).
I Proof of Proposition 5.2

(Step 1) Let $\bar{x}(\alpha, \beta)$ and $\bar{z}(\alpha, \beta)$ be the two boundaries $\bar{x}$ and $\bar{z}$ defined in Appendix F for given $(\alpha, \beta)$. Then, the following statements are true.

(a) If $\alpha_1 > \alpha_2$ and $\beta_1 = \beta_2$, $\bar{x}(\alpha_1, \beta_1) \geq \bar{x}(\alpha_2, \beta_2)$ and $\bar{z}(\alpha_1, \beta_1) \leq \bar{z}(\alpha_2, \beta_2)$.

(b) If $\beta_1 > \beta_2$ and $\alpha_1 = \alpha_2$, $\bar{x}(\alpha_1, \beta_1) \geq \bar{x}(\alpha_2, \beta_2)$ and $\bar{z}(\alpha_1, \beta_1) \leq \bar{z}(\alpha_2, \beta_2)$.

Proof: From the proof of Proposition S.1.1 in the Supplemental Material, $z_\alpha$ and $z_\beta$ satisfy

$$m_1 \int_{z_\alpha}^{x_m} (1 - \alpha \delta) \xi^{-m_1 - 1} d\xi - \delta (\alpha + \beta) z_m^{-m_1} = 0, \quad m_2 \int_{z_\alpha}^{x_m} (1 + \beta \delta) \xi^{-m_2 - 1} d\xi - \delta (\alpha + \beta) z_m^{-m_2} = 0 \quad (51)$$

with $0 < z_\alpha < (1 - \alpha \delta)$, $(1 + \beta \delta) < z_\beta < \infty$. By differentiating the coupled equations in (51) with respect to $\alpha$, we obtain

$$((1 + \beta \delta) - z_\beta) z_m^{-m_1 + 1} \frac{dz_m}{d\alpha} - ((1 - \alpha \delta) - z_\alpha) z_m^{-m_1} \frac{dz_m}{d\alpha} = \frac{\delta}{m_1} \alpha^{-m_1},$$

$$((1 + \beta \delta) - z_\beta) z_m^{-m_2 + 1} \frac{dz_m}{d\alpha} - ((1 - \alpha \delta) - z_\alpha) z_m^{-m_2} \frac{dz_m}{d\alpha} = 0. \quad (52)$$

It follows from (52) that

$$\frac{1}{1 - \alpha \delta - z_\alpha} \left( \frac{z_\beta}{z_\alpha} \right)^{m_2 + 1} - \frac{z_\beta}{z_\alpha} \right) \frac{dz_m}{d\alpha} = \frac{\delta z_\beta}{m_1} \frac{z_\beta}{z_\alpha}^{-m_1},$$

$$\frac{1}{1 + \beta \delta - z_\alpha} \left( \frac{z_\alpha}{z_\beta} \right)^{m_1 + 1} - \frac{z_\alpha}{z_\beta} \right) \frac{dz_m}{d\alpha} = \frac{\delta z_\alpha}{m_1} \frac{z_\alpha}{z_\beta}^{-m_1}. \quad (53)$$

Thus, $\frac{dz_m}{d\alpha} < 0$ and $\frac{dz_\beta}{d\alpha} > 0$. Therefore, the following relationship is established:

$$\alpha_1 > \alpha_2 \quad \text{and} \quad \beta_1 = \beta_2 \quad \Rightarrow \quad \alpha_1 \leq \alpha_2 \quad \text{and} \quad z_{\beta_1} \geq z_{\beta_2}. \quad (54)$$

Recall the two boundaries $\bar{x}$ and $\bar{z}$ defined in (46). Let $(\bar{x}_i, \bar{z}_i) \quad (i = 1, 2)$ be the two boundaries $(\bar{x}, \bar{z})$ in (46) corresponding to $(\alpha_i, \beta_i)$. From (36),

$$\bar{x} = \frac{1}{\gamma} \left( \frac{1}{1 - \gamma + \gamma m_1} \frac{m_1 - 1}{m_1} - \frac{m_1 - 1}{m_1 - m_2} \frac{m_1 m_2}{m_1 - m_2} \right) \frac{\gamma}{1 - \gamma + \gamma m_1 \gamma m_2} \frac{m_1 m_2}{z_{\alpha_1}} \frac{(1 - \delta \alpha)}{\gamma} \quad (55)$$

Hence, $\bar{x}$ increases as $z_{\alpha}$ decreases. Since $\alpha_1 > \alpha_2$, the argument (54) implies $\bar{z}_{\alpha_1} \leq \bar{z}_{\alpha_2}$ and thus we obtain that $\bar{x}_1 \geq \bar{x}_2$. Similarly,

$$\bar{z} = \bar{x} \left( \frac{1}{1 - \gamma + \gamma m_1} \frac{m_1 - 1}{m_1} - \frac{m_1 - 1}{m_1 - m_2} \frac{m_1 m_2}{m_1 - m_2} \right) \frac{\gamma}{1 - \gamma + \gamma m_1 \gamma m_2} \frac{m_1 m_2}{z_{\alpha_1}} \frac{(1 - \delta \beta)}{\gamma} \quad (56)$$

Hence, $\bar{z}$ increases as $z_{\beta}$ decreases. Since $\alpha_1 > \alpha_2$, (54) implies $z_{\beta_1} \geq z_{\beta_2}$ and thus we obtain that $\bar{z}_1 \leq \bar{z}_2$. This completes the proof of part (a) in (Step 1). The proof of part (b) is omitted since it is essentially same as that of part (a).

(Step 2) $\bar{x}(L)$ decreases with $L$ and $\bar{z}(L)$ increases with $L$.

Proof: Let $L_1 > L_2 > 1$ be given. There exist $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ such that

$L_1 = \frac{(1 + \delta \beta_1)}{(1 - \delta \alpha_1)}$ and $L_2 = \frac{(1 + \delta \beta_2)}{(1 - \delta \alpha_2)}$. Let us temporarily denote by $\tilde{\beta}_1 = \frac{L_1 - 1}{\delta}$ and $\tilde{\beta}_2 = \frac{L_2 - 1}{\delta}$. Clearly, $\tilde{\beta}_1 > \tilde{\beta}_2$. Theorem 3.2 implies $\bar{x}(L_1) = \bar{x}(0, \tilde{\beta}_1) = \bar{x}(0, \tilde{\beta}_2)$ and $\bar{x}(L_2) = \bar{x}(0, \beta_2) = \bar{x}(0, \beta_2)$. By Step 1, we have $\bar{x}(0, \tilde{\beta}_1) \geq \bar{x}(0, \beta_2)$ and $\bar{x}(0, \tilde{\beta}_2) \geq \bar{x}(0, \beta_2)$ and thus $\bar{x}(L_1) \geq \bar{x}(L_2)$ and $\bar{x}(L_1) \geq \bar{x}(L_2)$.

Finally, since $\bar{c} = \bar{x}/\bar{x}$ and $\bar{z} = 1/\bar{x}$, it follows from Steps 1 and 2 that $\bar{c}$ increases with $L$ and $\bar{z}$ decreases with $L$. This completes the proof.
Proof of Proposition 5.3

(Step 1) The RCRRA increases as $L$ increases.

Proof: For given $L_1 > L_2$, there exist two pairs $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ such that $L_1 = \frac{1 + \delta \beta_1}{\delta}$ and $L_2 = \frac{1 + \delta \beta_2}{\delta}$. By Proposition 3.2, we can assume $\alpha_1 = \alpha_2 = 0$ and $\beta_1 > \beta_2$. That is, it is sufficient to consider only the case where $\alpha$ is zero.

By an argument similar to that in Subsection I we have $z_{\alpha_1} \leq z_{\alpha_2}$ and $z_{\beta_2} \leq z_{\beta_1}$, where $(z_{\alpha_i}, z_{\beta_i})$ $(i = 1, 2)$ are the free boundaries of $H(z; \alpha_i, \beta_i)$ corresponding to $(\alpha_i, \beta_i)$ defined in Appendix D. This fact means that $z_{\alpha}$ decreases as $L$ increases.

$$\frac{\partial}{\partial z_{\alpha}} \left( \frac{D_1}{z_{\alpha_1}^\nu} \right) = \frac{m_1 m_2}{m_2 - m_1} \frac{z_{\alpha} - m_1^{-1}}{z_{\alpha} - m_1^{-1}} (1 - z_{\alpha}) > 0, \quad \frac{\partial}{\partial z_{\alpha}} \left( \frac{D_2}{z_{\alpha_2}^\nu} \right) = \frac{m_1 m_2}{m_1 - m_2} \frac{z_{\alpha} - m_2^{-1}}{z_{\alpha} - m_2^{-1}} (1 - z_{\alpha}) < 0.$$

Therefore, $\frac{D_1}{z_{\alpha_1}^\nu} (> 0)$ decreases and $\frac{D_2}{z_{\alpha_2}^\nu} (< 0)$ increases as $L$ increases since $z_{\alpha}$ decreases as $L$ increases. Also, $m_1 \frac{D_1}{z_{\alpha_1}^\nu}$ $(> 0)$ and $m_2 \frac{D_2}{z_{\alpha_2}^\nu}$ $(> 0)$ decrease as $L$ increases.

On the other hand, we know that for given $x$ and $c_{0-}$ there exists a unique $z^* \in (z_{\alpha}, z_{\beta})$ such that $x/c_{0-} = \mathcal{X}(z^*)$. Moreover,

$$\frac{\sigma}{\partial z_{\alpha_2}} \left( \frac{D_1}{z_{\alpha_1}^\nu} \right) = \frac{m_1 m_2 (m_1 - 1)}{(1 - \gamma + \gamma m_1) z_{\alpha_1} m_1} (z^*)^{m_1 - 1} + \frac{m_2 (m_2 - 1) D_2}{(1 - \gamma + \gamma m_2) z_{\alpha_2} m_2} (z^*)^{m_2 - 1}. \quad (57)$$

There are two cases: (i) $z^*$ decreases as $L$ increases, (ii) $z^*$ increases as $L$ increases.

(i) Rewrite (57) as

$$\frac{\sigma}{\partial z_{\alpha_2}} \left( \frac{D_1}{z_{\alpha_1}^\nu} \right) = (m_1 - 1) \left( \frac{1}{r} - \frac{x}{c_{0-}} \right) + \frac{(m_1 - m_2) m_1 D_1}{(1 - \gamma + \gamma m_1) z_{\alpha_1} m_1} (z^*)^{m_1 - 1}. \quad (58)$$

The (term A) in (58) is independent of $L$. Since $\frac{m_1 D_1}{z_{\alpha_1}^\nu}$ $(> 0)$ and $z^*$ decrease as $L$ increases, the (term B) in (58) decreases as $L$ increases. Thus, $\frac{\sigma}{\partial z_{\alpha_2}}$ decreases as $L$ increases.

(ii) In this case, $(z^*)^{m_2 - 1}$ decreases as $L$ increases. Moreover,

$$\frac{\sigma}{\partial z_{\alpha_2}} \left( \frac{D_2}{z_{\alpha_2}^\nu} \right) = (m_2 - 1) \left( \frac{1}{r} - \frac{x}{c_{0-}} \right) + \frac{(m_2 - m_1) m_2 D_2}{(1 - \gamma + \gamma m_2) z_{\alpha_2} m_2} (z^*)^{m_2 - 1}. \quad (59)$$

Since $\frac{m_2 D_2}{z_{\alpha_2}^\nu}$ $(> 0)$ decreases as $L$ increases and $(m_2 - m_1)/(1 - \gamma + \gamma m_2) > 0$, we deduce that the (term D) in (59) decreases as $L$ increases. The (term C) in (59) is also independent of $L$ and thus $\frac{\sigma}{\partial z_{\alpha_2}}$ decreases as $L$ increases.

From (i) and (ii), we conclude that the RCRRA increases as $L$ increases.

(Step 2) As $L$ goes to infinity, the maximum value of RCRRA goes to infinity.

Proof: Since $w \in (0, \frac{1}{r})$ is a unique solution of the algebraic equation (38), we deduce that as $L$ goes to infinity, $w$ goes to zero. Thus,

$$z_{\alpha}^{\infty} = \lim_{L \to \infty} z_{\alpha} = \lim_{w \to 0} \frac{(1 - \delta \alpha) m_1 - 1}{m_1} \frac{L w m_1 - 1}{w m_1 - 1} = (1 - \delta \alpha) m_1 - 1,$$

$$z_{\beta}^{\infty} = \lim_{L \to \infty} z_{\beta} = \lim_{w \to 0} \frac{(1 + \delta \beta) m_1 - 1}{m_1} \frac{w m_1 - 1}{w m_1 - 1} = \infty. \quad (60)$$

This leads to $\lim_{L \to \infty} D_1 = 0$ and $\lim_{L \to \infty} D_2 = \frac{(1 - \delta \alpha)}{\delta} \frac{1}{(m_2 - 1)}$. From the definition of the two boundaries $\bar{z}$ and $\bar{z}$ in (46):

$$\bar{z}^{\infty} \equiv \lim_{L \to \infty} \bar{z} = \frac{\gamma (m_2 - 1)}{r - 1 + \gamma m_2} \quad \text{and} \quad \bar{z}^{\infty} \equiv \lim_{L \to \infty} \bar{z} = \frac{1}{r}.$$
As the wealth-consumption ratio goes to the boundary $\bar{\zeta}$, the portfolio to consumption ratio goes to zero, i.e.,

$$\lim_{L \to \infty} \frac{\theta}{\sigma} \left( \frac{D_1 m_1 (m_1 - 1)}{(1 - \gamma + \gamma m_1) z_0} \left( \frac{z_0}{z_0} \right)^{m_1 - 1} + \frac{D_2 m_2 (m_2 - 1)}{(1 - \gamma + \gamma m_2) z_0} \left( \frac{z_0}{z_0} \right)^{m_2 - 1} \right) = 0,$$

which implies that the RCRRA goes to infinity as $L \to \infty$.

K Simulation and Regression Analysis in Section 5.2

(Step 1) Simulation of Wealth and Portfolio for $N$-individuals:

- We divide the interval $[0, T]$ into $12 \times T$ subintervals with end points $t_j = 1, 2, ..., 12 \times T$, where $T$ is a positive integer.
- Suppose that the means $m_\alpha, m_\beta$ and variances $v_\alpha, v_\beta$ for the distributions of adjustment costs $\alpha$ and $\beta$ are given. For each $i, i = 1, 2, ..., N$ we generate log-normally distributed random variables $\alpha_i$ and $\beta_i$ whose the mean and variance are $(m_\alpha, v_\alpha)$ and $(m_\beta, v_\beta)$, respectively.
- We set $c_0 = 1$ for each individual and generate individual $i$'s initial wealth $x_0$ randomly according to the uniform distribution over $(\bar{x}_i, \bar{x}_i)$.
- Generate a $12 \times T$ random vector $\omega$ that follows a standard normal distribution. Using this vector $\omega$, we generate the process of the risky asset returns $\Delta S_t/S_t$ and the dual process $y^*_t$ in Proposition 5.1 for all the $N$ individuals.
- By Appendix F and Proposition 5.1, we can simulate the optimal wealth and portfolio processes of $N$ individuals.

(Step 2) Regression Analysis:

- Let $(X^1, \Pi^1), (X^2, \Pi^2), ..., (X^N, \Pi^N)$ be the simulated wealth/portfolio processes for $N$ individuals obtained in (Step 1). (Note that $X^i$ and $\Pi^i$ are $(12 \times T + 1)$ random vectors for $i = 1, 2, ..., N$).
- For given $T$ and $k$, there are $(T - k + 1)$ numbers of $\Delta_k$.
  For $i = 1, 2, ..., N$ and $j = 1, 2, ..., (T - k + 1)$ let
  
  $$DR(i, j) \equiv \Delta_k \log \frac{\Pi^i_{j+k}}{X^i_{j+k}} = \log \frac{\Pi^i(12 \times (j + k - 1) + 1)}{X^i(12 \times (j + k - 1) + 1)} - \log \frac{\Pi^i(12 \times (j - 1) + 1)}{X^i(12 \times (j - 1) + 1)},$$
  
  $$X(i, j) \equiv \Delta_k \log X^i_{j+k} = \log X^i(12 \times (j + k - 1) + 1) - \log X^i(12 \times (j - 1) + 1).$$

- We regress Eq. (26) with OLS using the simulated $DR$ and $X$.

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20The log-normal distribution implies that there are households having fairly large values of $\alpha$’s even though their density in the population is very small. We drop those households whose $\alpha$ values violates Assumption 1.